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TWO-POINT BOUNDARY VALUE PROBLEMS FOR SINGULAR TWO-DIMENSIONAL LINEAR DIFFERENTIAL SYSTEMS

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Abstract. For two-dimensional systems of ordinary linear differential equations with singular coefficients, unimprovable in a certain sense conditions are established guaranteeing, respectively, the Fredholmicity and unique solvability of the Dirichlet and the Nicoletti problems.

On an open interval $]a,b[$, we consider the two-dimensional linear differential system

$$u_i' = p_i(t)u_{3-i} + q_i(t) \quad (i = 1, 2) \tag{1}$$

with the Dirichlet boundary conditions

$$u_1(a+) = 0, \quad u_1(b-) = 0, \tag{2_1}$$

and the Nicoletti boundary conditions

$$u_1(a+) = 0, \quad u_2(b-) = 0, \tag{2_2}$$

where p_1 and $q_1 :]a,b[\to \mathbb{R}$ are Lebesgue integrable functions, while the functions p_2 and $q_2 :]a,b[\to \mathbb{R}$ are Lebesgue integrable on every closed interval contained in $]a,b[$.

We are mainly interested in the case where the functions p_2 and q_2 have nonintegrable singularities at the points a and b, i.e. the case, where

$$\int_a^b \big(|p_2(t)| + |q_2(t)|\big)dt = +\infty.$$

System (1) is singular in that sense.

In the case, where $p_1(t) \equiv 1$ and $q_1(t) \equiv 0$, i.e., when system (1) is equivalent to a second order linear differential equation, the singular problems $(1),(2_1)$ and $(1),(2_2)$ are investigated in sufficient detail (see, [1–6] and the references therein). In the general case the above mentioned problems are still not well studied. The present paper is devoted exactly to this case.

Theorems 1_1 and 1_2 below contain conditions guaranteeing, respectively, the Fredholmicity of the singular problems $(1),(2_1)$ and $(1),(2_2)$. Based on these theorems we have established unimprovable in a certain sense conditions for the unique solvability of these problems (see, Theorems 2_1 and 2_2, and their corollaries). They are generalizations of some results by T. Kiguradze [4] concerning the unique solvability of the Dirichlet and the Nicoletti problems for singular second order linear differential equations.

We use the following notation.

$$[x]_+ = \frac{|x| + x}{2}, \quad [x]_- = \frac{|x| - x}{2};$$

$u(t_0+)$ and $u(t_0-)$ are the right and the left limits, respectively, of the function u at the point t_0;

$L([a,b])$ is the space of Lebesgue integrable on $[a,b]$ real functions;

$L_{loc}(]a,b[)$ and $L_{loc}(]a,b])$ are the spaces of real functions which are Lebesgue integrable on every closed interval contained in $]a,b[$ and $]a,b]$, respectively;

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If $p \in L([a,b])$, then

$$I_a(p)(t)=\int_a^t p(s)ds, \quad I_{a,b}(p)(t)=\int_a^t p(s)ds\int_t^b p(s)ds \quad \text{for } a\le t\le b.$$

A vector-function $(u_1,u_2):]a,b[\to\mathbb{R}^2$ is said to be **a solution of system** (1) if its components are absolutely continuous on every closed interval contained in $]a,b[$ and satisfy system (1) almost everywhere on $]a,b[$.

A solution of system (1) satisfying the boundary conditions (2_1) (the boundary conditions (2_2)) is said to be **a solution of problem** $(1),(2_1)$ **(of problem** $(1),(2_2)$**)**.

We investigate problem $(1),(2_1)$ in the case where the functions p_i and q_i $(i=1,2)$ satisfy the conditions

$$p_1\in L([a,b]),\quad q_1\in L([a,b]),\quad p_2\in L_{loc}(]a,b[),\quad q_2\in L_{loc}(]a,b[), \tag{3}$$

$$p_1(t)\ge 0 \ \text{ for } a<t<b,\quad \delta=\int_a^b p_1(t)dt>0. \tag{4}$$

Along with system (1) we consider the corresponding homogeneous system

$$u_i'=p_i(t)u_{3-i}\ \ (i=1,2). \tag{1_0}$$

The following theorem is valid.

Theorem 1_1. *Let along with* (3) *and* (4) *the conditions*

$$\int_a^b I_{a,b}(p_1)(t)[p_2(t)]_-dt<+\infty$$

and

$$\int_a^b I_{a,b}(p_1)(t)\Big(I_{a,b}(|q_1|)(t)[p_2(t)]_+ + |q_2(t)|\Big)dt<+\infty \tag{5}$$

be satisfied. Then for the unique solvability of problem $(1),(2_1)$ *it is necessary and sufficient that the corresponding homogeneous problem* $(1_0),(2_1)$ *to have only the trivial solution.*

Theorem 2_1. *Let there exist a constant* $\lambda\ge 1$ *and a measurable function* $p:]a,b[\to[0,+\infty[$ *such that along with* (3)–(5) *the conditions*

$$[p_2(t)]_-=p(t)p_1^{1-\frac{1}{\lambda}}(t)\quad \textit{for } a<t<b,$$

and

$$\int_a^b I_{a,b}(p_1)(t)p^\lambda(t)dt\le\Big(\frac{\pi}{\delta}\Big)^{2\lambda-2}\delta \tag{6}$$

are satisfied. Then problem $(1),(2_1)$ *has one and only one solution.*

Corollary 1_1. *If along with* (3)–(5) *the condition*

$$\int_a^b I_{a,b}(p_1)(t)[p_2(t)]_-dt\le\delta \tag{7}$$

holds, then problem $(1),(2_1)$ *has one and only one solution.*

Corollary 2_1. *If along with* (3)–(5) *the conditions*

$$p_2(t) \geq -\Big(\frac{\pi}{\delta}\Big)^2 p_1(t) \quad for \;\; a < t < b, \tag{8}$$

$$\text{mes}\Big\{t \in]a,b[\,:\, p_2(t) > -\Big(\frac{\pi}{\delta}\Big)^2 p_1(t)\Big\} > 0 \tag{9}$$

hold, then problem $(1),(2_1)$ *has one and only one solution.*

Example 1_1. If

$$0 \leq p_1(t) \leq \exp\left(-\frac{b-a}{(t-a)(b-t)}\right) \;\text{ for }\; a<t<b, \quad \delta = \int_a^b p_1(t)dt > 0,$$

$$-\frac{\delta}{(b-a)(t-a)^2(b-t)^2}\exp\left(\frac{b-a}{(t-a)(b-t)}\right) \leq p_2(t) \leq 0 \;\text{ for }\; a<t<b,$$

$$|q_2(t)| \leq \frac{\ell}{(t-a)^\mu (b-t)^\mu}\exp\left(\frac{b-a}{(t-a)(b-t)}\right) \;\text{ for }\; a<t<b,$$

where $\ell > 0$, $\mu < 3$, then all the conditions of Corollary 1_1 are fulfilled, and therefore problem $(1),(2_1)$ has a unique solution.

The above example shows that the functions p_2 and q_2 in the conditions of Theorems 1_1 and 2_1 may have singularities of arbitrary order at the points a and b.

Remark 1_1. Inequalities (6) and (7) in Theorem 2_1 and Corollary 1_1 are unimprovable and they cannot be replaced, respectively, by the conditions

$$\int_a^b I_{a,b}(p_1)(t)p^\lambda(t)dt \leq \Big(\frac{\pi}{\delta}\Big)^{2\lambda-2}\delta + \varepsilon$$

and

$$\int_a^b I_{a,b}(p_1)(t)[p_2(t)]_-\,dt \leq \delta + \varepsilon,$$

no matter how small $\varepsilon > 0$ would be.

Remark 2_1. Inequalities (8) and (9) in Corollary 2_1 are unimprovable as well since if along with (4) the conditions

$$p_2(t) \equiv -\Big(\frac{\pi}{\delta}\Big)^2 p_1(t), \quad q_i(t) \equiv 0 \;\; (i=1,2)$$

hold, then problem $(1),(2_1)$ has an infinite set of solutions.

In contrast to problem $(1),(2_1)$, we investigate problem $(1),(2_2)$ in the case where instead of (3) the conditions

$$p_1 \in L([a,b]), \quad q_1 \in L([a,b]), \quad p_2 \in L_{loc}(]a,b]), \quad q_2 \in L_{loc}(]a,b]) \tag{10}$$

are satisfied.

Theorem 1_2. *Let along with* (4) *and* (10) *the conditions*

$$\int_a^b I_a(p_1)(t)[p_2(t)]_-\,dt < +\infty$$

and

$$\int_a^b I_a(p_1)(t)\big(I_a(|q_1|)(t)[p_2(t)]_+ + |q_2(t)|\big)\,dt < +\infty \tag{11}$$

be satisfied. Then for the unique solvability of problem $(1),(2_2)$ *it is necessary and sufficient that the corresponding homogeneous problem* $(1_0),(2_2)$ *to have only the trivial solution.*

Theorem 2_2. *Let there exist a constant* $\lambda \geq 1$ *and a measurable function* $p:]a,b[\to [0,+\infty[$ *such that along with* (4), (10), *and* (11) *the conditions*

$$[p_2(t)]_- = p(t) p_1^{1-\frac{1}{\lambda}}(t) \quad \text{for } a < t < b,$$

and

$$\int_a^b I_a(p_1)(t) p^\lambda(t) dt \leq \left(\frac{\pi}{2\delta}\right)^{2\lambda-2} \tag{12}$$

are satisfied. Then problem $(1),(2_2)$ *has one and only one solution.*

Corollary 1_2. *If along with* (4), (10), *and* (11) *the condition*

$$\int_a^b I_a(p_1)(t)[p_2(t)]_- dt \leq 1 \tag{13}$$

holds, then problem $(1),(2_2)$ *has one and only one solution.*

Corollary 2_2. *If along with* (4), (10), *and* (11) *the conditions*

$$p_2(t) \geq -\left(\frac{\pi}{2\delta}\right)^2 p_1(t) \quad \text{for } a < t < b, \tag{14}$$

$$\operatorname{mes}\left\{t \in]a,b[\,: p_2(t) > -\left(\frac{\pi}{2\delta}\right)^2 p_1(t)\right\} > 0 \tag{15}$$

hold, then problem $(1),(2_2)$ *has one and only one solution.*

Example 1_2. If

$$0 \leq p_1(t) \leq (t-a)^{-2} \exp\left(-\frac{1}{t-a}\right) \quad \text{for } a<t<b, \quad \delta = \int_a^b p_1(t)dt > 0,$$

$$-\frac{1}{b-a}\exp\left(\frac{1}{(t-a)(b-t)}\right) \leq p_2(t) \leq 0, \quad |q_2(t)| \leq \frac{\ell}{(t-a)^\mu}\exp\left(\frac{1}{t-a}\right) \quad \text{for } a<t<b,$$

where $\ell > 0$, $\mu < 1$, then all the conditions of Corollary 1_2 are fulfilled, and therefore problem $(1),(2_2)$ has a unique solution.

Remark 1_2. Inequalities (12) and (13) in Theorem 1_2 and Corollary 1_2 are unimprovable and they cannot be replaced, respectively, by the conditions

$$\int_a^b I_a(p_1)(t) p^\lambda(t) dt \leq \left(\frac{\pi}{2\delta}\right)^{2\lambda-2} + \varepsilon$$

and

$$\int_a^b I_a(p_1)(t)[p_2(t)]_- dt \leq 1 + \varepsilon,$$

no matter how small $\varepsilon > 0$ would be.

Remark 2_2. Inequalities (14) and (15) in Corollary 2_2 are unimprovable as well since if along with (4) the conditions

$$p_2(t) \equiv -\left(\frac{\pi}{2\delta}\right)^2 p_1(t), \quad q_i(t) \equiv 0 \ \ (i=1,2)$$

hold, then problem $(1),(2_2)$ has an infinite set of solutions.

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