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ON THE BOUNDEDNESS IN GENERALIZED WEIGHTED GRAND LEBESGUE SPACES OF SOME INTEGRAL OPERATORS ASSOCIATED TO THE SCHRÖDINGER OPERATOR

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Abstract. In the present note, in generalized weighted grand Lebesgue spaces on R^d, $d \geq 3$, we consider the boundedness of diffusion semi-group maximal functions, Riesz transforms and their adjoints, as well as the Littlewood-Paley quadratic functions related to the Schrödinger differential operator $-\Delta + V$, where the potential V satisfies a reverse Hölder inequality with an exponent, greather than $d/2$. The class of weights, more general than that of Muchenhoupt' one, is used.

1. Introduction

Our note deals with the mapping properties of certain integral operators associated with the Schrödinger differential operator

$$\mathcal{L} = -\Delta + V(x), \quad x \in R^d, \quad d \geq 3,$$

where Δ is the Laplasian and $V(x)$ is non-negative, non-identicaly zero and for some $q > \frac{d}{2}$ satisfies the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int\limits_B v^q(y)dy\right)^{1/q} \leq \frac{c}{|B|}\int\limits_B v(y)dy,$$

for every ball $B \subset R^d$.

We consider the boundedness problems relating to the Schrödinger integral operators in some nonstandard Banach function space.

The mapping properties in L^p of several types Schrödinger–Riesz transforms have been studied in a pioneer work by Z. W. Shen [9], in which he introduced the following critical radius function associated with the potential V:

$$\rho(x) = \sup\left\{r > 0 : \ r^{\frac{1}{d-q}} \int\limits_{B(x,r)} \leq 1\right\}, \quad x \in R^d.$$

This notion has played an essential role in the extensive study of the boundedness of Schrödinger integral operators in weighted L^p spaces with weights, larger, in general, than Muchenhoupt's ones.

Here, we present the definition of generalized weighted grand Lebesgue spaces $L_v^{p),\phi}(R^n, w)$.

Let $1 < p < \infty$, ϕ be a positive non-decreasing function on $(0, p-1)$ satisfying $\phi(0+) = 0$. The generalized weighted grand Lebesgue space $L_v^{p),\phi}(R^n, w)$ is defined as the set of all everywhere finite measurable functions for which

$$\|f\|_{L_v^{p),\phi}(R^n,w)} = \sup_{0<\varepsilon<p-1}\left(\phi(\varepsilon)\int\limits_{R^n} |f(x)|^{p-\varepsilon} w(x) v^\varepsilon(x)dx\right)^{\frac{1}{p-\varepsilon}} < \infty,$$

where $wv^\varepsilon \in L^1_{\text{loc}}(R^n)$ for all ε, $0 < \varepsilon < p-1$.

Further, we follow the definitions given in [3].

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Definition 1. Let $1 < p < \infty$. A weight function $w \in A_p^{\text{loc}}$ if there exists a constant $c > 0$ such that

$$\Big(\int\limits_B w(y)dy\Big)^{1/p}\Big(\int\limits_B w^{-\frac{1}{p-1}}(y)dy\Big) \le c|B|$$

for every ball $B = B(x,r)$, where $0 \le r \le \rho(x)$, $x \in R^d$.

Definition 2. Given $p > 1$, the class

$$A_p^\rho := \bigcup_{\theta\ge 0} A_p^{\rho,\theta},$$

where $A^{\rho,\theta}$ is defined as the weights w such that

$$\Big(\int\limits_B w(y)dy\Big)^{1/p}\Big(\int\limits_B w^{-\frac{1}{p-1}}(y)dy\Big)^{1/p'} \le c|B|\left(1+\frac{r}{\rho(x)}\right)^\theta$$

for all balls $B(x,r)$.

The following proper inclusions

$$A_p \subset A_p^\rho \subset A_p^{\text{loc}}$$

are valid.

In the case for $\rho \equiv 1$, the function $w(x) = 1 + |x|^\gamma$, $\gamma > d(p-1)$ belongs to A_p^ρ, but it is not in A_p.

Here, we establish the weighted inequalities in $L_v^{p),\phi}(R^n, w)$ for the following Schrödinger operators:

i) Maximal operator of the diffusion semi-group

$$\mathcal{M}^* f(x) = \sup_{t>0} e^{-t\mathcal{L}} f(x);$$

ii) $\mathcal{L}$- Riesz transform

$$R = \nabla \mathcal{L}^{-\frac{1}{2}},$$

and its adjoint

$$R^* = \mathcal{L}^{-\frac{1}{2}}\nabla;$$

iii) $\mathcal{L}$-Littlewood-Paley function

$$g(f)(x) = \Big(\int\limits_0^\infty \Big|\frac{d}{dt}e^{-t\mathcal{L}}(f)(x)\Big|^2 tdt\Big)^{\frac{1}{2}}.$$

Let T stand for any of the above operators.

Now we present one of the main results of our note.

Theorem 1. *Let $1 < p < \infty$, $w \in A_p^\rho$ and let $v \in L^p(R^n, w)$, $v^\gamma \in A_p^\rho$ for some $\gamma > 0$. Then the operator T is bounded in $L_v^{p),\phi}(R^n, w)$.*

By T_{loc} we denote the ρ-localization of T:

$$T_\rho f(x) = T\left(f\chi_{B(x,\rho(x))}(x)\right).$$

Theorem 2. *Let $1 < p < \infty$, $w \in A_p^{\rho,\text{loc}}$ and let $v \in L^p(R^n, w)$, $v^\gamma \subset A_p^{\rho,\text{loc}}$ for some $\gamma > 0$. Then the operator T_{loc} is bounded in $L_v^{p),\phi}(R^n, w)$.*

The boundedness problems for the classical versions of the above-mentioned integral operators when $\mathcal{L} = -\Delta$ in weighted grand Lebesgue spaces in the framework of Muckenhoupt's A_p classes were studied in [4–7] (see also the monograph [8, Chapter 7], and references therein).

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