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THE PUNCH PROBLEM OF THE PLANE THEORY OF VISCOELASTICITY WITH A FRICTION

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Abstract. The paper considers the problem of pressure of a rigid punch onto a viscoelastic half-plane in the presence of friction. The problems of the linear theory of viscoelasticity attracted the attention of many scientists first of all due to the fact that building and composite materials (concrete, plastic polymers, wood, human fabric, etc.) exhibit significant viscoelastic properties and, thus, calculations of constructions for strength, with regard for the viscoelastic properties, are now becoming increasingly important. Thanks to this fact, various methods of calculating the above-mentioned problems were proposed, one of which is the Kelvin–Voigt differential model on which the present paper is based.

Using the methods of a complex analysis elaborated in the plane theory of elasticity by N. I. Muskhelishvili and his followers, the unknown complex potentials, characterizing viscoelastic equilibrium of a half-plane, are constructed effectively and the tangential and normal stresses under the punch are defined.

Introduction

The theory of viscoelasticity originated in the works by Boltzmann [3] and developed in his works by Volterra [10] finds applications not only in mechanics of deformable solid bodies, but also in other branches of mathematical physics. Viscoelasticity combines the properties of materials to be viscous or elastic during deformation. In addition, elastic bodies and viscous liquids, as is known, differ significantly in their properties under the deformation; the former after removal of applied loads return to their undeformed state and the latter (for example, incompressible liquids) are deprived of this property. Moreover, stresses in an elastic body are connected directly with strains, but in viscous liquids (with some exception) they are connected with deformation velocities (for details, see [2,4,5,8,9]. For viscoelastic materials, the ordinary equilibrium equations, the boundary conditions and compatibility equations written in terms of stresses remain valid for purely elastic bodies under the condition that the constants E and σ obtained in the equations are replaced by the functions $E(t)$ and $\sigma(t)$. Moreover, unlike purely elastic materials (steel, aluminium, quartz) whose behavior does not deviate much from the linear elasticity, such materials as synthetic polymers, wood, metals, human fabric, etc., exhibit under high temperatures significant viscoelastic properties.

Of great importance in the development of the theory of viscoelasticity are synthetic materials worked out at the end of the twentieth century and also their widespread applications in various fields.

Subsequently, various models of material properties evaluation for viscoelasticity have been elaborated (see [1]).

In the theory of linear viscoelasicity, Hook's law can be represented either by the Volterra equation (integral model), or by the dependence where there occur both the deformations and their derivatives in time (differential model).

In the present work the use is made of the Kelvin–Voigt differential model in which Hook's law is of the form [8]

$$\begin{aligned} X_x &= \lambda\theta + 2\mu e_{xx} + \lambda^*\dot{\theta} + 2\mu^*\dot{e}_{xx}, \\ Y_y &= \lambda\theta + 2\mu e_{yy} + \lambda^*\dot{\theta} + 2\mu^*\dot{e}_{yy}, \end{aligned} \tag{1}$$

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$$X_y = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + \mu^*\left(\frac{\partial \dot v}{\partial x} + \frac{\partial \dot u}{\partial y}\right),$$

where $\vartheta = e_{xx} + e_{yy} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$, X_x, Y_y, X_y, u, v, e_{xx}, e_{yy}, e_{xy} are the functions of the variables x, y, t. Under t we will always mean the time parameter and the dots in the expressions $\dot\theta, \dots, \dot u$ (unlike dashes) will denote time derivatives t; λ, μ are elastic and λ^*, μ^* are viscoelasticity constants.

We cite here the certain well-known Kolosov–Muskhelishvili's formulas which can, as is known, be attributed to any solid bodies (see [6])

$$\begin{gathered} X_x + Y_y = 4\mathrm{Re}[\Phi(z,t)] = 4\mathrm{Re}[\varphi'(z,t)], \\ Y_y - X_x + 2iX_y = 2[\overline{z}\varphi''(z,t) + \psi'(z,t)] = 2[\overline{z}\Phi'(z,t) + \Psi(z,t)]. \end{gathered} \tag{2}$$

In the sequel, we will also use the formula following from formulas (2),

$$Y_y - iX_y = \Phi(z,t) + \overline{\Phi(z,t)} + z\overline{\Phi'(z,t)} + \overline{\Psi(z,t)}. \tag{3}$$

We assume that the resultant vector (X,Y) of outer forces applied to the punch is finite, and stresses and rotation vanish at infinity, thus for large $|z|$, we have

$$\Phi(z,t) = -\frac{X+iY}{2\pi z} + o\left(\frac{1}{z}\right); \quad \Psi(z,t) = \frac{X-iY}{2\pi z} + o\left(\frac{1}{z}\right). \tag{4}$$

It can be easily seen that from the correlations (1) and (2), for the function $\vartheta(z,t) = e_{xx} + e_{yy}$ we obtain the following differential equation

$$\dot\vartheta(z,t) + k\vartheta(z,t) = \frac{2}{\lambda^* + \mu^*}\mathrm{Re}[\varphi'(z,t)], \quad \Big(k = \frac{\lambda+\mu}{\lambda^*+\mu^*}\Big),$$

whose solution is of the form (assuming $\vartheta(z;0) = 0$)

$$\vartheta(z,t) = \frac{2}{\lambda^*+\mu^*}\int_0^t \mathrm{Re}[\varphi'(z,\tau)]e^{k(\tau-t)}d\tau. \tag{5}$$

Similarly, from the same correlations (1) and (2), for the function $\gamma(z,t) = e_{xx} - e_{yy}$ we have

$$\dot\gamma(z,t) + m\gamma(z,t) = -\frac{1}{\mu^*}\mathrm{Re}[\overline{z}\varphi''(z,t) + \psi'(z,t)], \quad \Big(m = \frac{\mu}{\mu^*}\Big),$$

whose solution under zero initial conditions has the form

$$\gamma(z,t) = -\frac{1}{\mu^*}\int_0^t \mathrm{Re}\,[\overline{z}\varphi''(z,\tau) + \psi'(z,\tau)]e^{m(\tau-t)}d\tau. \tag{6}$$

From (5) and (6), we get

$$\begin{aligned} 2\mu^* e_{xx} &= \int_0^t \mathrm{Re}\left[\varkappa^*\varphi'(z,\tau)e^{k(\tau-t)} - \left(\overline{z}\varphi''(z,\tau) + \psi'(z,\tau)\right)e^{m(\tau-t)}\right]d\tau, \\ 2\mu^* e_{yy} &= \int_0^t \mathrm{Re}\left[\varkappa^*\varphi'(z,\tau)e^{k(\tau-t)} + \left(\overline{z}\varphi''(z,\tau) + \psi'(z,\tau)\right)e^{m(\tau-t)}\right]d\tau, \end{aligned} \tag{7}$$

where

$$\varkappa^* = \frac{2\mu^*}{\lambda^*+\mu^*}.$$

Taking into account equalities $dx = dz$, $dx = d\overline{z}$, $dy = -idz$, $dy = id\overline{z}$,, from (7), by integration with respect to x and y, respectively, we obtain the formula

$$2\mu^*(u+iv) = \int_0^t \left[\varkappa^*\varphi(z,\tau)e^{k(\tau-t)} + \Big(\varphi(z,\tau) - z\overline{\varphi'(z,\tau)} - \overline{\psi(z,\tau)}\Big)e^{m(\tau-t)}\right]d\tau + 2\mu^*(u_0+iv_0), \tag{8}$$

where $u_0 = u(z,0)$, $v_0 = v(z,0)$.

Formula (8) is an analogue of Kolosov–Muskhelishvili's formula for the second basic problem of the plane theory of elasticity ([6]) for viscoelastic isotropic body.

From formula (8), by differentiation with respect to x, we obtain

$$2\mu^* v'(x,y,t) = \mathrm{Im}\Big[\int\limits_0^t \varkappa^* e^{k(\tau-t)}\Phi(z,\tau)d\tau\Big]$$

$$+\mathrm{Im}\Big[\int\limits_0^t e^{m(\tau-t)}\Big(\Phi(z,\tau)-\overline{\Phi(z,\tau)}-z\overline{\Phi'(z,\tau)}-\overline{\Psi(z,\tau)}\Big)d\tau\Big]+2\mu^* {v'}_0(x,y,0). \tag{9}$$

Statement of the Problem. Let a viscoelastic body occupy a lower half-plane S^-. By L we denote the boundary of that domain (i.e., the Ox-axis) and assume that a portion $L'=[-1;1]$ comes in contact with the punch of prescribed base shape and the punch goes into the half-plane by a given force acting onto the punch and directed vertically downwards. We will also assume that the displacement of the punch is translatory in the direction, normal to the boundary, in the presence of friction. In this case, the boundary conditions can be written in the form

$$\begin{aligned} X_y^-(x,t) =&\alpha p(x,t), \qquad \alpha=\text{const}>0, \quad x\in L';\\ X_y^-(x,t) =&Y_y^-(x,t)=0, \quad x\in L''=L-L';\\ v^-(x,t) =&f(x,t)+c, \qquad x\in L', \quad (c=\text{const}), \end{aligned} \tag{10}$$

where $f(x,0)=f(x)$ is the given function defining the base shape of the punch before pressing into the half-plane. In (10), by $X_y^-(x,t),\dots,v^-(x,t)$ we have denoted the expressions $X_y^-(x,0,t),\dots,$ $v^-(x,0,t)$, and the same writing will be retained in the sequel.

The total tangential stress in the case under consideration has the form $T_0=\alpha N_0$, where $N_0=\int\limits_{-1}^{1}N(x,t)dx$ $N(x,t)$ is a normal stress at the point $x\in L'$, and hence, the resultant vector of outer forces acting onto the punch (which are assumed to be prescribed) is of the kind $(X;Y)=(\alpha N_0;-N_0)$.

Relying on (3), formula (9) is written as follows:

$$\mathrm{Im}\Big[\varkappa^*\int\limits_0^t e^{k(\tau-t)}\Phi(z,\tau)d\tau+2\int\limits_0^t e^{m(\tau-t)}\Phi(z,\tau)d\tau\Big]$$

$$+\int\limits_0^t e^{m(\tau-t)}X_y(z,\tau)d\tau =2\mu^*[v'(x,y,t)-v'(x,y,0)]. \tag{11}$$

Passing in (11) to the limit as $z\to x$ $(z\in S^-)$, we obtain

$$\mathrm{Im}\Big[\varkappa^*\int\limits_0^t e^{k(\tau-t)}\Phi^-(x,\tau)d\tau+2\int\limits_0^t e^{m(\tau-t)}\Phi^-(x,\tau)d\tau\Big]$$

$$+\int\limits_0^t e^{m(\tau-t)}X_y^-(x,\tau)d\tau =f_1(x,t), \tag{12}$$

where

$$f_1(x,t)=2\mu^*[f'(x,t)-f'(x)].$$

Differentiating (12) with respect to t and adding the obtained equality with (12), multiplied by m, we have

$$\mathrm{Im}\Big[(\text{m-k})\varkappa^*\int\limits_0^t e^{k\tau}\Phi^-(x,\tau)d\tau+(\varkappa^*+2)e^{kt}\Phi^-(x,t)\Big]+e^{kt}X_y^-(x,t)=f_2(x,t), \tag{13}$$

where

$$f_2(x,t) = e^{kt}\left[\dot{f}_1(x,t) + m f_1(x,t)\right].$$

After the differentiation with respect to t, it follows from (13) that

$$\operatorname{Im}\Big[(\varkappa^* m + 2k)\Phi^-(x,t) + (\varkappa^* + 2)\dot{\Phi}^-(x,t)\Big] + \dot{X}_y^-(x,t) + kX_y^-(x,t) = \dot{f}_2(x,t)e^{-kt}. \tag{14}$$

Following N. I. Muskhelishvili (see [6]), we extend the function $\Phi(z,t)$ to the upper half-plane (i.e., S^+) so as to continue analytically the values of $\Phi(z,t)$ into the lower half-plane through the unloaded sections (i.e., to L'')).

In our case, on the basis of the boundary conditions (10) and formula (3), we define $\Phi(z,t)$ in S^+ as follows:

$$\Phi(z,t) = -\Phi_*(z,t) - z{\Phi'}_*(z,t) - \Psi_*(z,t), \quad z \in S^+, \tag{15}$$

where $\Phi_*(z,t) = \overline{\Phi(\overline{z},t)}$; $\Psi_*(z,t) = \overline{\Psi(\overline{z},t)}$.

Taking into account that $[\Phi_*(z,t)]_* = \Phi(z,t)$, $[\Psi_*(z,t)]_* = \Psi(z,t)$, from (15) we have

$$\Phi_*(z,t) = -\Phi(z,t) - z\Phi'(z,t) - \Psi(z,t). \tag{16}$$

The obtained in such a way piecewise holomorphic function we denote again by $\Phi(z,t)$, and then to find the function $\Psi(z,t)$ by $\Phi(z,t)$, from (16) we get

$$\Psi(z,t) = -\Phi(z,t) - \Phi_*(z,t) - z\Phi'(z,t) \tag{17}$$

Thus, the stress and displacement components are expressed in terms of one piecewise holomorphic function $\Phi(z,t)$.

Introducing the value (17) into (3), we have

$$Y_y - iX_y = \Phi(z,t) - \Phi(\overline{z},t) + (z-\overline{z})\overline{\Phi'(z,t)},$$

whence

$$Y_y^-(x,t) - i\,X_y^-(x,t) = \Phi^-(x,t) - \Phi^+(x,t), \quad x \in L'. \tag{18}$$

Owing to the fact that $X_y^- = -\alpha Y_y^-(x,t)$, from (18) we get

$$X_y^-(x,t) = \frac{\alpha}{1+i\alpha}\left[\Phi^+(x,t) - \Phi^-(x,t)\right] \tag{19}$$

Taking into account equalities $\overline{\Phi^-(x,t)} = \Phi_*^+(x,t)$ and $\overline{\Phi^+(x,t)} = \Phi_*^-(x,t)$ and bearing in mind that $X_y^-(x,t) = \overline{X_y^-(x,t)}$, from (19) we obtain

$$(1-i\alpha)\Phi^-(x,t) + (1+i\alpha)\Phi_*^-(x,t) = (1-i\alpha)\Phi^+(x,t) + (1+i\alpha)\Phi_*^+(x,t),$$

and thus we conclude that the vanishing at infinity function

$$(1-i\alpha)\Phi(z,t) + (1+i\alpha)\Phi_*(z,t)$$

is holomorphic on the whole plane and, consequently,

$$\Phi(z,t) = -\frac{1+i\alpha}{1-i\alpha}\Phi_*(z,t)$$

whence we obtain

$$\Phi^-(x,t) = -\frac{1+i\alpha}{1-i\alpha}\overline{\Phi^+(x,t)}; \quad \Phi^+(x,t) = -\frac{1+i\alpha}{1-i\alpha}\overline{\Phi^-(x,t)}. \tag{20}$$

On the basis of (20) and (19), we get

$$\begin{aligned} X_y^-(x,t) &= -\Big[\frac{\alpha}{1+i\alpha}\Phi^-(x,t) + \frac{\alpha}{1-i\alpha}\overline{\Phi^-(x,t)}\Big] \\ &= -\operatorname{Re}\Big[\frac{2\alpha}{1+i\alpha}\Phi^-(x,t)\Big] = -\operatorname{Im}\Big[\frac{2i\alpha}{1+i\alpha}\Phi^-(x,t)\Big]. \end{aligned} \tag{21}$$

From (2) and (13), with regard for the equality $X_y^- = -\alpha Y_y^-$, we have

$$\begin{gathered}\operatorname{Re}\left[\Phi^-(x,0)\right] = -\frac{X_y^-(x,0)}{4\alpha},\\ \operatorname{Im}\left[\Phi^-(x,0)\right] = -\frac{1}{\varkappa^*+2}\left[X_y^-(x,0) - f_2(x,0)\right].\end{gathered} \tag{22}$$

Thus, for $\Phi^-(x,0)$ we obtain the formula

$$\Phi^-(x,0) = -\frac{X_y^-(x,0)}{4\alpha} - \frac{i}{\varkappa^*+2}\left[X_y^-(x,0) - f_2(x,0)\right]. \tag{23}$$

Taking into account the fact that

$$\begin{aligned}\frac{2i\alpha}{1+i\alpha}\Phi^-(x,0) &= \frac{1}{1+\alpha^2}\left[2\alpha^2\operatorname{Re}\Phi^-(x,0) - 2\alpha\operatorname{Im}\Phi^-(x,0)\right]\\ &+\frac{i}{1+\alpha^2}\left[2\alpha\operatorname{Re}\Phi^-(x,0) + 2\alpha^2\operatorname{Im}\Phi^-(x,0)\right],\end{aligned}$$

from (21) follows

$$X_y^-(x,0) = -\frac{2\alpha}{1+\alpha^2}\left[\operatorname{Re}\Phi^-(x,0) + \alpha\operatorname{Im}\Phi^-(x,0)\right]. \tag{24}$$

Substituting into (24) the values $\operatorname{Re}\Phi^-(x,0)$ and $\operatorname{Im}\Phi^-(x,0)$ from (22), after not complicated calculations, we obtain

$$X_y^-(x,0) = -\frac{4\alpha^2 f_2(x,0)}{\varkappa^*(1+2\alpha^2)+2}. \tag{25}$$

After the appropriate calculations, it follows from (25) and (23) that

$$\Phi^-(x,0) = \frac{f_2(x,0)}{\varkappa^*(1+2\alpha^2)+2}[\alpha + i(1+2\alpha^2)]. \tag{26}$$

For the tangential and normal stresses under the punch we have

$$\begin{gathered}T(x,t) = X_y^-(x,t) = -2\alpha\operatorname{Im}\Big[\frac{i}{1+i\alpha}\Phi^-(x,t)\Big],\\ P(x,t) = Y_y^-(x,t) = -\frac{1}{\alpha}T(x,t) = 2\operatorname{Im}\Big[\frac{i}{1+i\alpha}\Phi^-(x,t)\Big],\end{gathered} \tag{27}$$

respectively.

Thus the problem reduces to finding of the function $\Phi^-(x,t)$. Relying on (21), from (14) we get

$$\operatorname{Im}\Big\{\Big[(\varkappa^*+2) - \frac{2i\alpha}{1+i\alpha}\Big]\dot{\Phi}^-(x,t) + \Big[\varkappa^* m + 2k - \frac{2ik\alpha}{1+i\alpha}\Big]\Phi^-(x,t)\Big\} = e^{-kt}\dot{f}_2(x,t).$$

We write the obtained equation in the form

$$\operatorname{Im}\left[(a+ib)\Phi^-(x,t) + (c+id)\Phi^-(x,t)\right] = f_3(x,t), \tag{28}$$

where

$$\begin{gathered}a = (\varkappa^*+2)(1+\alpha^2) - 2\alpha^2; \quad c = (\varkappa^* m + 2k)(1+\alpha^2);\\ b = -2\alpha; \qquad d = -2\alpha k;\\ f_3(x,t) = (1+\alpha^2)e^{-kt}\dot{f}_2(x,t)\end{gathered} \tag{29}$$

In view of (20), from (28), after simple transformations, we obtain

$$\begin{aligned}\Big[\dot{\Phi}^+(x,t) + \frac{c-id}{a-ib}\Phi^+(x,t)\Big] &= -\frac{(1+i\alpha)(a+ib)}{(1-i\alpha)(a-ib)}\Big[\dot{\Phi}^-(x,t) + \frac{c+id}{a+ib}\Phi^-(x,t)\Big]\\ &-\frac{2i(1+i\alpha)^2}{(a-ib)(1+\alpha^2)}f_3(x,t)\end{aligned} \tag{30}$$

Considering the piecewise holomorphic function $\Omega(z,t)$ defined by the formula

$$\Omega(z,t)=\begin{cases}\dot{\Phi}(z,t)+\frac{c-id}{a-ib}\Phi(z,t), & z\in S^+,\\ \dot{\Phi}(z,t)+\frac{c+id}{a+ib}\Phi(x,t), & z\in S^-,\end{cases}$$

from (30) we obtain the following boundary value problem of linear conjugation:

$$\Omega^+(x,t)=g\,\Omega^-(x,t)+F(x,t), \tag{31}$$

where

$$g=-\frac{(1+i\alpha)(a+ib)}{(1-i\alpha)(a-ib)}; \qquad F(x,t)=-\frac{2i(1+i\alpha)^2}{(a-ib)(1+\alpha^2)}f_3(x,t).$$

Taking into account that on the basis of (29),

$$(1+i\alpha)(a+ib)=(\varkappa^*+2+i\alpha\varkappa^*)(1+\alpha^2),$$

we can write the constant g in the form

$$g=-\frac{1+i\beta_0}{1-i\beta_0}, \tag{32}$$

where

$$\beta_0=\frac{\alpha\varkappa^*}{\varkappa^*+2}.$$

Bearing in mind that $\alpha>0$, $\varkappa^*>0$ and introducing the constant δ defined by the conditions

$$\operatorname{tg}\,\pi\delta=\beta_0, \qquad 0\le\delta<\frac{1}{2}, \tag{33}$$

due to (32), the coefficient of problem (31) is written in the form

$$g=e^{2\pi i\gamma}, \tag{34}$$

where $\gamma=\frac{1}{2}+\delta$.

As a canonical function of problem (31) we can take the function

$$\chi(z)=(1+z)^{\frac{1}{2}+\delta}(1-z)^{\frac{1}{2}-\delta},$$

where under the right-hand side is meant the certain branch which is holomorphic outside of L', adopts on the upper side of the segment the positive values and takes at infinity the form

$$\chi(z)=(1+z)^{\frac{1}{2}+\delta}(1-z)^{\frac{1}{2}-\delta}=-ize^{\pi i\delta}+O(1). \tag{35}$$

Relying on the above reasoning, we obtain factorization of the coefficient of problem (31) in the form

$$g=\frac{\chi^-(x)}{\chi^+(x)}, \quad x\in L'. \tag{36}$$

Further, the vanishing at infinity solution of problem (31) of the class h_0 (for that class, see [7]) is of the form

$$\Omega(z,t)=\frac{1}{2\pi i\chi(z)}\int_{-1}^{1}\frac{\chi^+(\sigma)F(\sigma,t)}{\sigma-z}\,d\sigma+\frac{D_0}{\chi(z)}, \tag{37}$$

where D_0 is the constant defined from the conditions (4) and (35), having the form

$$D_0=\frac{(1+i\alpha)N_0}{2\pi}\,e^{\pi i\delta}.$$

Owing to (33), (34), (36) and (37), we have

$$\Omega^-(x,t)=\frac{e^{-2\pi i\delta}}{2}\left[F(x,t)-\frac{1}{\pi i\chi^+(x)}\int_{-1}^{1}\frac{\chi^+(\sigma)F(\sigma,t)}{\sigma-x}\,d\sigma\right]-\frac{D_0e^{-2\pi i\delta}}{\chi^+(x)}.$$

Having defined $\Omega^-(x,t)$, to find $\Phi^-(x,t)$, we obtain the following differential equation

$$\dot{\Phi}^-(x,t)+\lambda\Phi^-(x,t)=\Omega^-(x,t), \tag{38}$$

where

$$\lambda = \frac{c-id}{a-ib} = \lambda_1 + i\lambda_2; \quad \lambda_1 = \frac{[\varkappa^* m(1+\alpha^2)+2k][\varkappa^*(1+\alpha^2)+2]+4k\alpha^2}{[\varkappa^*(1+\alpha^2)+2]^2+4\alpha^2},$$
$$\lambda_2 = \frac{2\alpha\varkappa^*\,(m-k)\,(1+\alpha^2)}{[\varkappa^*(1+\alpha^2)+2]^2+4\alpha^2}. \tag{39}$$

The solution of equation (38) is represented by the formula

$$\Phi^-(x,t) = e^{-(\lambda_1+i\lambda_2)t}\left[\Phi^-(x,0) + \int\limits_0^t e^{(\lambda_1+i\lambda_2)\tau}\Omega^-(x,\tau)d\tau\right], \tag{40}$$

where $\Phi^-(x,0)$ is of the form (26).

On the basis of the above-obtained results, we can conclude that in our case (i.e., in the case of pressure of a rigid punch with friction) the tangential and normal stresses defined by formula (27) have, as is seen from (40), the character of damping oscillations with respect to time t. Also, taking into account (39), we can conclude that oscillations are absent in the following cases:

(1) for $\alpha = 0$ (i.e., without friction);

(2) for $m = k$ (i.e., the constants $\lambda, \dots, \mu^*$ are connected by the relation $\frac{\lambda}{\lambda^*} = \frac{\mu}{\mu^*}$).

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