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WEIGHTED NORM ESTIMATES FOR ONE-SIDED MULTILINEAR INTEGRAL OPERATORS

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Abstract. In this note one-sided and two-weight inequalities for one-sided multilinear fractional integrals are derived. One-weight estimates are based on Welland's type pointwise estimates which are also presented. Integral operators studied in this note involve one-sided multi(sub)linear fractional maximal operators, multilinear Riemann-Liouville and Weyl integral transforms.

In this note one– and two-weight norm inequalities for one-sided multilinear fractional integrals are presented. One-weight estimates are based on Welland's type pointwise inequalities which are also derived. Integral operators involve one-sided multisublinear fractional maximal operators, multilinear Riemann–Liouville and Weyl integral transforms.

Let $f_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \dots, m$, be measurable functions and let

$$\overrightarrow{f} := (f_1, \dots, f_m).$$

Throughout the note, it will be assumed that p is a constant satisfying the condition

$$\frac{1}{p} = \sum_{i=1}^{m} \frac{1}{p_i}, \tag{1}$$

where $1 < p_i < \infty$, $i = 1, \dots, m$.

Multilinear fractional integrals were introduced and studied in the papers by L. Grafakos [4], C. Kenig and E. Stein [7], L. Grafakos and N. Kalton [5]. In particular, these works deal with the operator

$$B_\gamma(f, g)(x) = \int_{\mathbb{R}^n} \frac{f(x+t)g(x-t)}{|t|^{n-\gamma}}\, dt, \quad x \in \mathbb{R}^n,$$

where γ is a constant parameter satisfying the condition $0 < \gamma < n$.

In the above-mentioned papers it was proved that if $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{n}$, where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, then B_γ is bounded from $L^{p_1} \times L^{p_2}$ to L^q.

As a tool to understand B_γ, the operator

$$\mathcal{I}_\gamma(\overrightarrow{f})(x) = \int_{(\mathbb{R}^n)^m} \frac{f_1(y_1)\cdots f_m(y_m)}{(|x-y_1| + \cdots + |x-y_m|)^{mn-\gamma}}\, d\overrightarrow{y},$$

where $x \in \mathbb{R}^n$, γ is a constant satisfying the condition $0 < \gamma < nm$, $\overrightarrow{f} := (f_1, \dots, f_m)$, $\overrightarrow{y} := (y_1, \dots, y_m)$, was studied, as well. The corresponding multisublinear maximal operator is given by (see [11]) the formula

$$\mathcal{M}_\gamma(\overrightarrow{f})(x) = \sup_{Q \ni x} \prod_{i=1}^{m} \frac{1}{|Q|^{1-\frac{\gamma}{mn}}} \int_Q |f_i(y_i)| dy_i,$$

where the supremum is taken over all cubes Q containing x. It can be immediately checked that

$$\mathcal{I}_\gamma(\overrightarrow{f})(x) \ge c_{n,\gamma} \mathcal{M}_\gamma(\overrightarrow{f})(x),$$

where $f_i \ge 0$, $i = 1, \dots, m$ and c is a positive constant, depends only on n and γ. If $m = 1$, then $\mathcal{I}_\gamma$ will be denoted by I_γ.

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Let $0 < r < \infty$ and let w be a weight function (i.e., w be an a.e. positive function) on $\mathbb{R}^n$. We denote by $L^r_w(\mathbb{R}^n)$ the class of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^r_w(\mathbb{R}^n)} := \left(\int\limits_{\mathbb{R}^n} |f(x)|^r w(x) dx \right)^{1/r} < \infty.$$

In 1974, Muckenhoupt and Wheeden [12] showed that the weighted Sobolev-type inequality

$$||I_\gamma(f)||_{L^s_{w^s}(\mathbb{R}^n)} \leq C ||f||_{L^r_{w^r}(\mathbb{R}^n)},$$

where $1 < r < \infty$, $0 < \gamma < 1/r$, $1/s = 1/r - \gamma/n$, holds if and only if $w \in A_{r,s}$. A locally integrable non-negative function (weight) w on $\mathbb{R}^n$ is said to belong to $A_{r,s}$ $(1 < r, s < \infty)$ if and only if

$$\sup_Q \left(\frac{1}{|Q|} \int\limits_Q w^s(x) dx \right)^{1/s} \left(\frac{1}{|Q|} \int\limits_Q w^{-r'}(x) dx \right)^{1/r'} < \infty, \quad \frac{1}{r} + \frac{1}{r'} = 1,$$

where the supremum is taken over all n-dimensional cubes Q with sides, parallel to the coordinate axes.

We say that a vector of weights $\overrightarrow{w} = (w_1, \dots, w_m)$ satisfies the $A_{\overrightarrow{p},q}$ condition ($\overrightarrow{p} = (p_1, \dots, p_m)$) if

$$\sup_Q \left(\frac{1}{|Q|} \int\limits_Q \left(\prod_{i=1}^m w_i(x) \right)^q dx \right)^{1/q} \prod_{i=1}^m \left(\frac{1}{|Q|} \int\limits_Q {w_i}^{-p_i'}(x) dx \right)^{1/p_i'} < \infty.$$

Theorem A ([11]). *Suppose that $0 < \gamma < nm$ and $1 < p_1, \dots, p_m < \infty$ are exponents with $1/m < p < n/\gamma$ and q is the exponent defined by $1/q = 1/p - \gamma/n$. Then the inequality*

$$\left(\int\limits_{\mathbb{R}^n} \left(\left| \mathcal{I}_\gamma(\overrightarrow{f})(x) \right| \left(\prod_{i=1}^m w_i(x) \right) \right)^q dx \right)^{1/q} \leq C \prod_{i=1}^m \left(\int\limits_{\mathbb{R}^n} \left(|f_i(x)| w_i(x) \right)^{p_i} dx \right)^{1/p_i}$$

holds for every $\overrightarrow{f} \in L^{p_1}(w_1^{p_1}) \times \cdots \times L^{p_m}(w_m^{p_m})$ if and only if $\overrightarrow{w}$ satisfies the $A_{\overrightarrow{p},q}$ condition.

In [14], the authors derived the following different type one-weighted result.

Theorem B. *Let $0 < \gamma < nm$, suppose that $f_i \in L^{p_i}_{w^{p_i}}(\mathbb{R}^n)$ with $1 < p_i < mn/\gamma$ $(i = 1, \dots, m)$ and $w \in \bigcap\limits_{i=1}^m A_{p_i, q_i}$ i.e.,*

$$\prod_{i=1}^m \sup_Q \left(\frac{1}{|Q|} \int\limits_Q w^{q_i}(x) dx \right)^{1/q_i} \left(\frac{1}{|Q|} \int\limits_Q w^{-p_i'}(x) dx \right)^{1/p_i'} < \infty,$$

where $\frac{1}{q_i} = \frac{1}{p_i} - \frac{\gamma}{mn}$. We set $\frac{1}{q} = \sum_{i=1}^m \frac{1}{q_i}$. Then there is a constant $C > 0$, independent of f_i such that

$$||\mathcal{I}_\gamma(\overrightarrow{f})||_{L^q_{w^q}(\mathbb{R}^n)} \leq C \prod_{i=1}^m ||f_i||_{L^{p_i}_{w^{p_i}}(\mathbb{R}^n)}.$$

The one-weight problem for multisublinear maximal functions and multilinear singular integrals was studied in [8] under the $A_{\overrightarrow{p}}$ condition. Various types of Fefferman–Stein multisublinear inequalities for fractional maximal functions were established in [13] and [6].

We introduce the following one-sided multisublinear fractional maximal functions:

$$\mathcal{M}^-_\alpha(\overrightarrow{f})(x) = \sup_{h>0} \prod_{i=1}^m \frac{1}{h^{1-\alpha/m}} \int\limits_{x-h}^{x} |f_i(y_i)| dy_i, \quad 0 < \alpha < m,$$

$$\mathcal{M}^+_\alpha(\overrightarrow{f})(x) = \sup_{h>0} \prod_{i=1}^m \frac{1}{h^{1-\alpha/m}} \int\limits_{x}^{x+h} |f_i(y_i)| dy_i, \quad 0 < \alpha < m,$$

which play an important role in the study of multilinear variants of the Riemann-Liouville and Weyl integral transforms

$$\mathcal{R}_\alpha(\overrightarrow{f})(x) = \int_{-\infty}^{x} \cdots \int_{-\infty}^{x} \frac{f_1(y_1)\cdots f_m(y_m)}{((x-y_1)+\cdots+(x-y_m))^{m-\alpha}} d\overrightarrow{y}, \quad 0<\alpha<m, \quad x\in\mathbb{R},$$

$$\mathcal{W}_\alpha(\overrightarrow{f})(x) = \int_{x}^{\infty} \cdots \int_{x}^{\infty} \frac{f_1(y_1)\cdots f_m(y_m)}{((y_1-x)+\cdots+(y_m-x))^{m-\alpha}} d\overrightarrow{y}, \quad 0<\alpha<m, \quad x\in\mathbb{R},$$

respectively.

If $m=1$, then the operators $\mathcal{R}_\alpha$, $\mathcal{W}_\alpha$, $\mathcal{M}_\alpha^-$ and $\mathcal{M}_\alpha^+$ will be denoted by R_α, W_α, M_α^- and M_α^+, respectively.

For the linear one-sided fractional integral operators the one-weight problem was solved in [1] (see also [2] Ch. 2 for related topics). In particular, the following statement holds.

Theorem C. *If $0\le\alpha<1$, $1<p<1/\alpha$ ($1/\alpha=\infty$, if $\alpha=0$), $1/q=1/p-\alpha$, $1/p+1/p'=1$. Then*

$$\Big[\int_{-\infty}^{\infty} |T(f)(x)u(x)|^q dx\Big]^{1/q} \le C\Big[\int_{0}^{\infty} |f(x)u(x)|^p dx\Big]^{1/p}$$

holds.

(a) *for $T=M_\alpha^-$ or $T=R_\alpha$ ($\alpha>0$) if and only if $u\in A_{p,q}^-$ i.e.,*

$$\Big[\frac{1}{h}\int_{a}^{a+h} u^q(x)dx\Big]^{1/q}\Big[\frac{1}{h}\int_{a-h}^{a} u^{-p'}(x)dx\Big]^{1/p'} \le C$$

for some constant C and all a, h with $a\in\mathbb{R}$, $h>0$;

(b) *for $T=M_\alpha^+$ or $T=W_\alpha$ ($\alpha>0$) if and only if $u\in A_{p,q}^+$ i.e.*

$$\Big[\frac{1}{h}\int_{a-h}^{a} u^q(x)dx\Big]^{1/q}\Big[\frac{1}{h}\int_{a}^{a+h} u^{-p'}(x)dx\Big]^{1/p'} \le C$$

for some constant C and all a, h with $a\in\mathbb{R}$, $h>0$.

For the two-weight theory for linear one-sided fractional integral operators under different types of conditions on weights we refer to the papers [3,9,10] (see also the monograph [2, ch. 2] and references cited therein).

Now we formulate the main statements of this note.

Welland-type Inequalities

Theorem 1. *Let $0<\alpha<m$ and $0<\epsilon<min\{\alpha,m-\alpha\}$. Then there exists a positive constant C depending only on m, α and ϵ such that the following pointwise inequality*

$$\Big|\mathcal{R}_\alpha(\overrightarrow{f})(x)\Big| \le C\Big[\Big(\mathcal{M}_{\alpha-\epsilon}^-(\overrightarrow{f})(x)\Big)\Big(\mathcal{M}_{\alpha+\epsilon}^-(\overrightarrow{f})(x)\Big)\Big]^{\frac{1}{2}}$$

holds for all $\overrightarrow{f}:=(f_1,\dots,f_m)$, where f_i, $i=1,\dots,m$, are bounded functions with a compact support.

The similar theorem can be written for the Weyl integral transform.

Theorem 2. *Let $0<\alpha<m$ and $0<\epsilon<\min\{\alpha,m-\alpha\}$. Then if $\overrightarrow{f}:=(f_1,\dots,f_m)$,*

$$\Big|\mathcal{W}_\alpha(\overrightarrow{f})(x)\Big| \le C\Big[\Big(\mathcal{M}_{\alpha-\epsilon}^+(\overrightarrow{f})(x)\Big)\Big(\mathcal{M}_{\alpha+\epsilon}^+(\overrightarrow{f})(x)\Big)\Big]^{\frac{1}{2}},$$

where f_i, $i=1,\dots,m$, are bounded functions with compact support and C depends only on m, α and ϵ.

One-weighted Inequalities

Theorem 3. *Let* $0 < \alpha < m$*, suppose that* $f_i \in L^{p_i}_{w^{p_i}}(\mathbb{R})$ *with* $1 < p_i < m/\alpha$ $(i = 1, \dots, m)$ *and* $w \in \bigcap_{i=1}^{m} A^-_{p_i,q_i}$ *i.e.,*

$$\prod_{i=1}^{m} \sup_{\substack{h>0\\ x\in\mathbb{R}}} \left(\frac{1}{h}\int_{x}^{x+h} w^{q_i}(t)dt\right)^{1/q_i} \left(\frac{1}{h}\int_{x-h}^{x} w^{-p_i'}(t)dt\right)^{1/p_i'} < \infty,$$

where $\frac{1}{q_i} = \frac{1}{p_i} - \frac{\alpha}{m}$*. We set* $\frac{1}{q} = \sum_{i=1}^{m} \frac{1}{q_i}$*. Then there is a constant* $C > 0$*, independent of* f_i *such that*

$$\|\mathcal{R}_\alpha(\overrightarrow{f})\|_{L^q_{w^q}(\mathbb{R})} \le C \prod_{i=1}^{m} \|f_i\|_{L^{p_i}_{w^{p_i}}(\mathbb{R})}.$$

Similar theorem for the Weyl integral transform holds.

Theorem 4. *Let* $0 < \alpha < m$*, suppose that* $f_i \in L^{p_i}_{w^{p_i}}(\mathbb{R})$ *with* $1 < p_i < m/\alpha$ $(i = 1, \dots, m)$ *and* $w \in \bigcap_{i=1}^{m} A^+_{p_i,q_i}$ *i.e.,*

$$\prod_{i=1}^{m} \sup_{\substack{h>0\\ x\in\mathbb{R}}} \left(\frac{1}{h}\int_{x-h}^{x} w^{q_i}(t)dt\right)^{1/q_i} \left(\frac{1}{h}\int_{x}^{x+h} w^{-p_i'}(t)dt\right)^{1/p_i'} < \infty,$$

where $\frac{1}{q_i} = \frac{1}{p_i} - \frac{\alpha}{m}$*. We set* $\frac{1}{q} = \sum_{i=1}^{m} \frac{1}{q_i}$*. Then there is a constant* $C > 0$*, independent of* f_i *such that*

$$\|\mathcal{W}_\alpha(\overrightarrow{f})\|_{L^q_{w^q}(\mathbb{R})} \le C \prod_{i=1}^{m} \|f_i\|_{L^{p_i}_{w^{p_i}}(\mathbb{R})}.$$

Fefferman–Stein Two-weighted Inequalities

In the two-weighted setting, we proved the following Fefferman-Stein type inequalities:

Theorem 5. *Let* $0 < \alpha < m$ *and let* $1 < \min\{p_1, \dots, p_m\} \le \max\{p_1, \dots, p_m\} < \min\{q, m/\alpha\}$*. Suppose that* p *is defined by* (1)*. Let* v_i *be weights on* $\mathbb{R}$*,* $i = 1, \dots, m$*. We set* $v(x) = \prod_{i=1}^{m} v_i^{p/p_i}(x)$*. Then the inequalities*

$$\left\|\left(\mathcal{M}^-_\alpha(\overrightarrow{f})\right)v^{1/q}\right\|_{L^q(\mathbb{R})} \le C \prod_{i=1}^{m} \left\|f_i\big(\mathcal{M}^+_{\alpha,p_i,q} v_i\big)^{1/q}\right\|_{L^{p_i}(\mathbb{R})},$$

$$\left\|\left(\mathcal{M}^+_\alpha(\overrightarrow{f})\right)v^{1/q}\right\|_{L^q(\mathbb{R})} \le C \prod_{i=1}^{m} \left\|f_i\big(\mathcal{M}^-_{\alpha,p_i,q} v_i\big)^{1/q}\right\|_{L^{p_i}(\mathbb{R})}$$

hold, where C *is a constant, independent of* f_i*,* $i = 1, \dots, m$*, and*

$$\mathcal{M}^+_{\alpha,p_i,q} v_i(x) = \sup_{h>0} \left(\frac{1}{h^{(1-\alpha p_i/m)q/p}} \int_{x}^{x+h} v_i(y)dy\right)^{p/p_i},$$

$$\mathcal{M}^-_{\alpha,p_i,q} v_i(x) = \sup_{h>0} \left(\frac{1}{h^{(1-\alpha p_i/m)q/p}} \int_{x-h}^{x} v_i(y)dy\right)^{p/p_i}.$$

Corollary 1. *Let* α, p_i, q *and* m *satisfy the conditions of Theorem* 5*.*

If

$$\prod_{i=1}^{m} \sup_{I} \left(\frac{1}{|I|^{(1-\alpha p_i/m)q/p}} \int_{I} v_i(y)dy\right)^{p/p_i} < \infty,$$

then the following trace-type inequalities hold:

(i)

$$\left\|\mathcal{M}_\alpha^-(\overrightarrow{f})\right\|_{L_v^q(\mathbb{R})} \le C\prod_{i=1}^m \left\|f_i\right\|_{L^{p_i}(\mathbb{R})},$$

$$\left\|\mathcal{M}_\alpha^+(\overrightarrow{f})\right\|_{L_v^q(\mathbb{R})} \le C\prod_{i=1}^m \left\|f_i\right\|_{L^{p_i}(\mathbb{R})}.$$

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