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WEIGHTED MULTILINEAR HARDY AND RELLICH INEQUALITIES

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Abstract. Multilinear variants of weighted Rellich inequalities are derived on the real line. Weighted estimates for multilinear Hardy operators are also discussed.

1. Introduction and Preliminaries

A considerable effort has been made in recent years to establish the (weighted) boundedness of integral operators in Lebesgue spaces. Such problems have been studied extensively in Harmonic Analysis, especially in the last two decades (see, e.g., the monograph [9] and references therein). Our aim is to establish an $m-$ linear weighted Rellich inequality

$$\Big\| \prod_{j=1}^{m} u_j \Big\|_{L^p_{w(\delta(\cdot))}(I)} \le C \prod_{j=1}^{m} \Big\| u_j'' \Big\|_{L^{p_j}(I)}, \quad I := (a,b), \quad -\infty \le a < b \le +\infty, \tag{1}$$

with a certain positive constant c, independent of $u_k \in C_0^\infty(I)$, $k = 1, \dots, m$, where $\delta(x)$ is the distance function on I given by the formula

$$\delta(x) = \min\{x-a, b-x\}, \tag{2}$$

and p is defined as follows:

$$\frac{1}{p} := \sum_{k=1}^{m} \frac{1}{p_k}, \quad 1 < p_k < \infty, \quad k = 1, \dots, m. \tag{3}$$

Throughout the paper, we assume that m is a positive integer, and p is determined by (3). Note that in this case $0 < p < \infty$.

Let v be an a.e. positive function (i.e., a weight) on the interval $I := (a,b)$, $-\infty \le a < b \le \infty$. We denote by $L^r_v(I)$ (or by $L^r_v(a,b)$), $0 < r < \infty$, the Lebesgue space defined by the norm for $r \ge 1$ (quasi-norm if $0 < r < 1$):

$$\|g\|_{L^r_v(I)} = \bigg(\int_a^b |g(x)|^r v(x) dx \bigg)^{1/r}.$$

If $v \equiv$ *const*, then $L^r_v(I)$ will be denoted by $L^r(I)$ (or by $L^r(a,b)$).

We establish (1) by deriving appropriate multilinear weighted Hardy inequalities

$$\Big\| \prod_{j=1}^{m} \int_a^x f_j(t) dt \Big\|_{L^p_v(a,b)} \le c \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(a,b)}, \tag{4}$$

$$\Big\| \prod_{j=1}^{m} \int_x^b f_j(t) dt \Big\|_{L^p_v(a,b)} \le c \prod_{j=1}^{m} \|f_j\|_{L^{p}(a,b)}. \tag{5}$$

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It should be noted that the necessary and sufficient conditions governing the two-weight bilinear Hardy inequality

$$\left(\int_a^b \left(\int_a^b f\right)^q \left(\int_a^x g\right)^q w(x)dx\right)^{1/q} \le C\left(\int_a^b f^{p_1}w_1\right)^{1/p_1}\left(\int_a^b g^{p_2}w_1\right)^{1/p_2},$$

for non-negative f and g were found in [10] under different conditions on weights for various ranges of p_1, p_2 and q, with $q>1$.

The Rellich inequality in the linear setting has first appeared in [14]. The papers [2, 4–8] (see also the monograph [1]) were devoted to this problem, generally speaking, in a higher-dimensional setting.

Here we formulate the following statements which are inherited from [5].

Theorem A (the case $n=1$)**.** *Suppose that $-\infty < a < b \le \infty$ and let $r \in (1,\infty)$; put $\delta(t) = \min\{t-a, b-t\}$. Then for all $u \in C_0^2(a,b)$,*

$$\int_a^b \frac{|u(t)|^r}{\delta(t)^{2r}}dt \le \left(\frac{r}{2r-1}\right)^r \left(\frac{r}{r-1}\right)^r \int_a^b |u''(t)|^r\,dt.$$

Theorem B (the higher-dimensional case)**.** *Let Ω be a non-empty, proper open subset of $\mathbb{R}^n$ and let $r\in(1,\infty)$; suppose that $u \in C_0^2(\Omega)$. If $r=2$, then*

$$\int_\Omega \frac{|u(x)|^2}{\delta_{M,4}(x)^4}dx \le \frac{16}{9}\int_\Omega |\Delta u(x)|^2\,dx,$$

while if $r\in(1,\infty)\backslash\{2\}$, then for some explicit constant $K(r,n)$,

$$\int_\Omega \frac{|u(x)|^r}{\delta_{M,2r}(x)^{2r}}dx \le K(r,n)\int_\Omega |\Delta u(x)|^r\,dx.$$

Here, $\delta_{M,4}$ and $\delta_{M,2r}$ are the mean distance functions obtained by averaging, in a certain sense, the distance to the boundary of Ω in all possible directions.

2. Results

We have proved the following statements.

Theorem 2.1. *Let $-\infty < a < b < \infty$, $I := (a,b)$, and let w be a weight function on the interval $\big(0,(b-a)\big)$. If*

$$\widetilde{D}_{a,b} := \sup_{0<\tau<b-a}\left(\int_\tau^b w(x)x^{mp}dx\right)^{1/p}\tau^{m-1/p} < \infty,$$

then for all $u_j \in C_0^2(I)$, $j=1,\dots,m$, inequality (1) *holds with the constant C given by the formula*

$$C = 4^{1/p}\widetilde{D}_{a,b}\left[1+2^{mp-2}\prod_{i=1}^m (p_i')^p\right]^{1/p}. \tag{6}$$

The next statement deals with the cases $b=\infty$ and $a=-\infty$, respectively.

Theorem 2.2. *Let $-\infty<a<\infty$. Suppose that $I:=(a,\infty)$. Let w be a weight function on $(0,\infty)$. If*

$$\widetilde{D} := \sup_{t>0}\left(\int_t^\infty w(x)x^{mp}dx\right)^{1/p} t^{m-1/p} < \infty, \tag{7}$$

then for all $u_j \in C_0^2(I)$, $j=1,\dots,m$, inequality (1) *holds, where*

$$C = 2^{m-1/p}\widetilde{D}\prod_{i=1}^m p_i'. \tag{8}$$

Theorem 2.3. *Let $-\infty < b < \infty$ and $I := (-\infty, b)$. Suppose that w is a positive function on $(0,\infty)$. If condition (7) is satisfied, then for all $u_j \in C_0^2(I)$, $j = 1, \ldots, m$, inequality (1) holds, where*

$$C = 2^{m-1/p} \widetilde{D} \prod_{i=1}^{m} p_i', \tag{9}$$

where $\widetilde{D}$ is defined by (7).

By applying Theorems 2.1, 2.2 and 2.3, we can easily deduce the following statements.

Corollary 2.4. *Let $-\infty < a < b < \infty$ and $I := (a, b)$. Then for all $u_j \in C_0^2(I)$, $j = 1, \ldots, m$, the inequalty*

$$\left(\int\limits_I \Big| \prod_{j=1}^{m} u_j(x) \Big|^p \delta(x)^{-2mp} dx \right)^{1/p} \le C \prod_{j=1}^{m} \|u_j''\|_{L^{p_j}(I)}, \tag{10}$$

holds, where

$$C = (2mp-1)^{-1/p} \left[1 + 2^{mp-2} \prod_{i=1}^{m} (p_i')^p \right]^{1/p}.$$

Corollary 2.5. *Let $-\infty < a < \infty$ and let $I := (a, \infty)$. Then inequality (10) holds for all $u_j \in C_0^2(I)$, $j = 1, \ldots, m$, where*

$$C = 2^{m-1/p}(2mp-1)^{-1/p} \prod_{j=1}^{m} p_j'.$$

Corollary 2.6. *Let $-\infty < b < \infty$. Suppose that $I := (-\infty, b)$. Then inequality (10) holds for all $u_j \in C_0^2(I)$, $j = 1, \ldots, m$, where C is defined by*

$$C = 2^{m-1/p}(2mp-1)^{-1/p} \prod_{j=1}^{m} p_j'.$$

To get the main results of this paper, we obtain the following statements about the weighted multilinear Hardy inequalities in which by H_a and H_b' are denoted the Hardy-type operators of the following form:

$$H_a f(x) = \frac{1}{x-a} \int\limits_a^x f(t)dt, \quad x \in (a,b), \quad -\infty < a < b \le \infty;$$

$$H_b' f(x) = \frac{1}{b-x} \int\limits_x^b f(t)dt; \quad x \in (a,b), \quad -\infty \le a < b < \infty.$$

Theorem 2.7. *Let $-\infty < a < b \le \infty$, v be a weight function on (a, b). Then inequality (4) with a positive constant c, independent of f_j, $f_j \in L^{p_j}(a, b)$, $j = 1, \ldots, m$, holds if and only if*

$$A_{a,b} := \sup_{a<t<b} \left(\int\limits_t^b v(x)dx \right)^{1/p} \Big(t - a \Big)^{m-1/p} < \infty.$$

Moreover, if c is the best possible constant in (4), then

$$A_{a,b} \le c \le C A_{a,b},$$

where

$$C = \begin{cases} \left(2 + 2^{mp-1} \prod_{i=1}^m \|H_a\|^p_{L^{p_i}(a,b)} \right)^{1/p}, & \text{if} \quad b < \infty, \\ 2^{m-1/p} \prod_{j=1}^m \|H_a\|_{L^{p_i}(a,\infty)}, & \text{if} \quad b = \infty. \end{cases}$$

Theorem 2.8. *Let $-\infty \le a < b < \infty$, v be a weight function on (a,b). Then inequality* (5) *with a positive constant c, independent of f_j, $f_j \in L^{p_j}(a,b)$, $j = 1,\dots,m$, holds if and only if*

$$B_{a,b} := \sup_{a<t<b} \Bigg(\int\limits_a^t v(x)dx \Bigg)^{1/p} \Big(b-t\Big)^{m-1/p} < \infty.$$

Moreover, if c is the best possible constant in (5), *then*

$$B_{a,b} \le c \le C B_{a,b},$$

where

$$C = \begin{cases} \Big(2 + 2^{mp-1} \prod_{i=1}^m \|H_b'\|^p_{L^{p_i}(a,b)}\Big)^{1/p}, & \text{if} \quad a > -\infty, \\ 2^{m-1/p} \prod_{j=1}^m \|H_b'\|_{L^{p_i}(-\infty,b)}, & \text{if} \quad a = -\infty. \end{cases}$$

Historically, in the linear case the two-weight problem for the Hardy operator was solved by B. Muckenhoupt [13] for the diagonal case, and by J. Bradly [3] and V. Kokilashvili [11] for the off-diagonal case (see also the monograph [12], Ch.1 and references therein).

Finally, we mention that the results of this note with proofs will appear separately.

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