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THE UNIFORM SUBSETS OF THE EUCLIDEAN PLANE

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Abstract. We consider some measurability properties of the uniform subsets of the Euclidean plane $\mathbf{R^2}$. Furthermore, it is shown that there exists an uniform subset of the plane which is simultaneously a Hamel basis of the plane.

Many years ago, Luzin posed the so-called graph problem, in particular, he asked whether there exists a function

$$\phi : \mathbf{R} \to \mathbf{R}$$

such that the whole plane $\mathbf{R}^2$ may be covered by countable isometric copies of the graph of ϕ.

Let us define the standard terminology which was introduced by Luzin (see, e.g., [7,9]).

Let e be an arbitrary nonzero vector in the Euclidean plane and ω be the first infinite cardinal number (i.e., $\omega = card(N)$).

- A set $A \subset \mathbf{R^2}$ is called uniform in direction e if $card(l \cap A) \leq 1$ for any straight line $l \subset \mathbf{R^2}$, parallel to e.
- A set $B \subset \mathbf{R^2}$ is called finite in direction e if $card(l \cap B) < \omega$ for any straight line $l \subset \mathbf{R^2}$, parallel to e.
- A set $C \subset \mathbf{R^2}$ is called countable in direction e if $card(l \cap C) \leq \omega$ for any straight line $l \subset \mathbf{R^2}$, parallel to e.

After this definition we can reduce the equivalent formulation of Luzin's problem:

There exists a countable family of uniform sets, whose union is identical to $\mathbf{R^2}$.

The Luzin's problem has found interesting applications for the mathematicians, in particular, this topic has a close connection with Sierpinski,s partition of the plane $\mathbf{R^2}$. Furthermore, under the assumption of the Continuum Hypothesis (**CH**) Sierpinski has solved positively the question (see, e.g., [8,9]).

Sierpinski's Theorem. *Assuming Continuum Hypothesis in* $\mathbf{R^2}$, *there exist two subsets* A *and* B *such that*

- *The set* A *is uniform with respect to the axis* $\mathbf{R} \times 0$*;*
- *The set* B *is uniform with respect to the axis* $0 \times \mathbf{R}$*;*
- *There exists a countable family* $\{h_n : n > \omega\}$ *of translations of* $\mathbf{R}^2$*, for which we have*

$$\cup\{h_n(A \cup B) : n < \omega\} = \mathbf{R}^2.$$

Note that the converse assertion holds true. In particular, the existence of the sets A and B satisfying the above-mentioned properties, implies the validity of **CH**.

The theorem of Sierpinski and the problem of Luzin have found interesting connections with the measure extension problem. The study of the measurability properties of uniform sets is an essential topic of our research. In the measure theory, the standard concept of measurability of sets and functions with respect to a fixed measure μ on a base (ground) set E is well known. We now introduce the concept of measurability of sets and functions not with respect to a fixed measure , but with respect to certain classes of measures, which are defined on different σ-algebras of subsets of the base space E (see [4,5]).

Let E be a set and let M be a class of measures on E (in general, we do not assume that measures belonging to M are defined on one and the same σ-algebra of subset of E).

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- We say that a function $f : E \to \mathbf{R}$ is absolutely (or universally) measurable with respect to M if f is measurable with respect to all measures from M.
- We say that a function $f : E \to \mathbf{R}$ is relatively measurable with respect to M if there exists at least one measure μ from M such that f is μ-measurable.
- We say that a function $f : E \to \mathbf{R}$ is absolutely nonmeasurable with respect to M if there exists no measure μ from M such that f is μ-measurable.

Accordingly, we say that a set $X \subset E$ is relatively measurable (absolutely measurable, absolutely nonmeasurable) with respect to M if its characteristic function χ_X is relatively measurable (absolutely measurable, absolutely nonmeasurable) with respect to M.

Example 1. There exists μ Π_2-invariant extension of the Lebesgue measure λ_2 such that all uniform sets in direction of the Oy-axis are measurable with respect to μ.

Example 2. There exist the A uniform set in direction of the Oy-axis and the B uniform set in direction of the Ox-axis such that $A \cup B$ is absolutely nonmeasurable with respect to the class of all Π_2-invariant extensions of the two-dimensional Lebesgue measure.

Let $M(\mathbf{R}^2)$ be a class of all nonzero σ-finite translation invariant measures on $\mathbf{R}^2$. A set $X \subset \mathbf{R}^2$ is called *negligible* with respect to $M(\mathbf{R}^2)$ if these two conditions are satisfied for X:

- there exists a measure $\nu \in M(\mathbf{R}^2)$ such that $X \in dom(\nu)$;
- for any measure $\mu \in M(\mathbf{R}^2)$, the relation $X \in dom(\mu)$ implies the equality $\mu(X) = 0$.

A set $X \subset \mathbf{R}^2$ is called *absolutely negligible* with respect to $M(\mathbf{R}^2)$ if for every measure $\mu \in M(\mathbf{R}^2)$, there exists a measure $\mu' \in M(\mathbf{R}^2)$ such that the relations

$$\mu' \ extends\ \mu, Y \in dom(\mu'), \mu'(Y) = 0$$

hold true.

Let us notice that any $\mathbf{R}^2$-absolutely negligible set is also $\mathbf{R}^2$-negligible, but the converse assertion fails to be valid.

Example 3. In 1914, Mazurkiewicz presented a transfinite construction to show that there exists a point subset M of the Euclidean plane $\mathbf{R^2}$ such that every straight line in the plane meets M at exactly two points. After his result it is natural to say that a set $Z \subset \mathbf{R^2}$ is a Mazurkiewicz subset of $\mathbf{R^2}$ if $card(Z \cap l) = 2$ for every straight line l lying in $\mathbf{R^2}$. The above definition immediately implies that for any nonzero vector $e \in \mathbf{R^2}$, the Mazurkiewicz set Z is finite in direction e. If $M \subset \mathbf{R}^2$ is finite in some direction l, then M is negligible with respect to the class $M(\mathbf{R}^2)$. In particular, every Mazurkiewicz set is negligible with respect to the same class of measures.

Example 4. By the definition, a Hamel basis for $\mathbf{R}$ is any of its bases construed as a vector space over $\mathbf{Q}$. It is a well-known fact that in the theory $\mathbf{ZF}+\mathbf{DC}$, where $\mathbf{DC}$ denotes the so-called Axiom of Dependent Choice, the existence of a Hamel basis implies the existence of a subset of $\mathbf{R}$, nonmeasurable in the Lebesgue sense. Moreover, every Hamel basis of the space $\mathbf{R}^n$ is an absolutely negligible subset of $\mathbf{R}^n$.

We recall that a subset X of $\mathbf{R^n}$ is λ_n-*thick* (or λ_n massive) in $\mathbf{R^n}$ if for each λ_n-measurable set $Z \subseteq \mathbf{R^n}$ with $\lambda_n(Z) > 0$, we have

$$X \cap Z \neq \emptyset.$$

In other words, X is λ_n-*thick* in $\mathbf{R^n}$ if and only if the equality

$$(\lambda_n)_*(\mathbf{R^n} \setminus \mathbf{X}) = \mathbf{0}$$

is satisfied.

Example 5. In the $\mathbf{R^n}$ Euclidian space, there exists the set Y such that: (i) Y is finite in direction of any $e \in \mathbf{R^n}$ vector. (ii)There exists a countable family $\{h_n : n > \omega\}$ of translations of $\mathbf{R}^n$ for which the intersection of sets $(h_k(Y))_k \in N$ is λ_n-thick (massive) set in $\mathbf{R}^n$.

As is mentioned above, under **CH**, the Sierpinski theorem yields a positive solution to the Luzin problem, but in the frame of **ZFC**, the final result was obtained by Davis (see, e.g., [2]).

Davis Theorem. *There exist a function*

$$\phi : \mathbf{R} \to \mathbf{R}$$

and a countable family $(g_n)_{n<\omega}$ *of motions of the Euclidean plane* $\mathbf{R^2}$ *such that*

$$\cup\{g_n(\Gamma_\phi) : n < \omega\} = \mathbf{R^2},$$

where Γ_ϕ *denotes the graph of* ϕ*.*

Example 6. The graph of a function $\phi : \mathbf{R} \to \mathbf{R}$, which yields a positive solution of Luzins problem, is an absolutely nonmeasurable subset of $E = \mathbf{R^2}$ with respect to the class of all nonzero σ-finite measures on $\mathbf{R^2}$ that are invariant under the group of all isometries of $\mathbf{R^2}$.

It is shown in the above-presented example that there is a finite set which is simultaneously a Hamel basis. The proof of this statement can be find in [6]. This fact motivated us to prove the following theorem.

Theorem. *There exists an uniform subset of* $\mathbf{R^2}$ *which is a Hamel basis of* $\mathbf{R^2}$*.*

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