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MARAO, ABOUT HOPF FIBRATIONS

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Abstract. A marao is a cover of a vector space by a set of equidimensional subspaces with pairwise trivial intersections. Such structures give rise to fibrations of particular kind. Naturally occurring examples are described. In particular, it is explained how the classical Hopf fibrations can be uniformly obtained from maraos.

The stratification of the spheres is known, which Hopf noticed and published in 1931. It is a foliation of a three-dimensional sphere by circles, and this set of circles carries natural structure of a manifold diffeomorphic to a sphere of two dimensions. This situation is easy to describe, if we see a three-dimensional sphere as the set of rays of four-dimensional linear space. I must also draw attention to the analogy: a quotient space of linear space is the decomposition of the space into a set of affine subspaces, and Marao is the decomposition of the space into a set of linear subspaces.

Definition.

A set of linear subspaces of a linear space S is a marao if
– the intersection of each pair of two subspaces is zero
– the union of all subspaces is equal to the basic space S.

Notation: $M^k(S)$ is a marao of k-dimensional subspaces in the linear space S.

An example. The set of all one-dimensional subspaces of the linear space S is called projective space. This set meets the requirements of the definition and therefore it is the Marao $M^1(S)$.

The main example. Let $W \supset V$ be an extension of fields. Each W-linear space is also V-linear. The set of all W-linear subspaces of dimension one in a W-linear space S is a projective space. This set as the set of V-linear subspaces of the space S remains again a marao, but of larger dimension, $M^1(S)$ as seen from W and $M^k(S)$ as seen from V, $k = \dim_V W$.

From this example, we can show marao as a generalization of field extension.

The dimension of subspaces of the Marao does not exceed half of the dimension of the main space. The only exception is when Marao has a single element, Marao trivial, the only element of which is itself the linear space S. Marao M with only one subspace $S \in M$. Marao with dimension of subspaces half of the dimension of the main space will be called middle marao.

Let us have a marao $M^k(S)$. Let's consider the map from the set of all non-zero vectors in S to the marao $m : S^* \to M^k(S)$, $x \mapsto m(x) \ni x$, $x \in m(x) \in M^k(S)$. This map is part of the standard fibration over the Grassmanian, when viewing $M^k(S)$ as a subset of the Grassmanian of all k-dimensional linear subspaces of S. $S^* \to M$ and $S \to S/p$ are two orthogonal fibrations, for any $p \in M$: the fiber of one is a section in the other and vice versa.

Suppose that $M^k(S)$ is a marao in a linear space S of dimension n and the dimension of the members of the marao is k. Choose a point p of the marao and a complementary subspace A for p in S, of dimension $n-k$ and divide M into two subspaces M^1 and M^2. M^1 is the set of points of the marao which have zero intersection with A, while M^2 is the set of points having nonzero intersection with A, that is, elements of the marao which are completely in A or partially intersect A. Since a point of M^1 is completely (except zero, of course) outside A, it is represented as a graph of a linear map from p to A, hence M^1 can be identified with a subset of the space of all linear maps from p to A, $M^1 \subset \mathrm{Lin}(p, A)$. Consider the map $A^* \to M^k(S)$, $x \mapsto m(x) \in M^k(S)$. So the marao $M^k(S)$ is the union of the image of A^* under this map and of the subspace M^1 of the linear space $\mathrm{Lin}(p, A)$.

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The above simplifies for middle maraos since in that case for a complementary subspace can be used any point q of the marao different from p, and in this case the map $(q = A)^* \to M^k(S)$ sends every vector to the same point q. Hence in this case the marao is obtained from $M^1 \subset \mathrm{Lin}(p, A)$ by adding the single point q. Moreover, in this case fixing any vector e not in q, the map $x \to m(e + x)$ gives a bijection between the linear space q and the set $M^k(S)\backslash\{q\}$ of all elements of the marao except q. Thus, a middle marao is obtained from the linear space q by adding a single point.

To a vector u from S/p can be assigned a subspace of the marao: to the vector x of u we associate its containing element $m(x)$ of the marao, and the subset of all $m(x)$ for $x \in u$ is denoted by $m(u) \subset M$, $x \in m(x) \in m(u) \subset M^k(S)$.

There are many structures on the Grassmannian: the natural linear bundle, the fiber over the point p being itself p as linear space, and the tangent space of the Grassmanian at its point p is naturally isomorphic to $\mathrm{Lin}(p, S/p)$. The map $m : S^* \to M$ induces linear map on tangent spaces $S \to T_pM \subset \mathrm{Lin}(p, S/p)$ with the kernel p. The tangent space T_pM is therefore isomorphic to the quotient space S/p.

Hopf fibration Suppose given a linear space S and in S a Marao $M^n(S)$ (dimension of the subspaces n), and given as well a marao in each of its elements C (dimension of the subspaces k). We have a total space $M^k(S)$ of the fibration (the union of the small Maraos $M^k(C)$; it is a marao in S with dimension of the subspaces k) and the base Marao $M^n(S)$. Over each point p the fiber is equal to the Marao $M^k(p)$. Such a fibration can be named as Hopf fibration since the famous Hopf fibrations are main examples.

$$(\mathbb{C}^8 = \mathbb{R}^{16})^* \to S^{15} \to \mathbb{R}P^{15} = M^1(\mathbb{R}^{16}) \to \mathbb{C}P^7 = M^2(\mathbb{R}^{16}) \to M^4(\mathbb{R}^{16}) \to M^8\mathbb{R}^{16} = S^8$$

fibers: ray, two opposite directional rays or the sphere S^0, $M^1(\mathbb{R}^2)$ or the sphere S^1, $M^2(\mathbb{R}^4)$ or the sphere S^2, $M^4(\mathbb{R}^8)$ or the sphere S^4, from the second to the end the fiber is rays of the 8-dimensional linear space or the sphere S^7.

$$(\mathbb{C}^4 = \mathbb{R}^8)^* \to S^7 \to \mathbb{R}P^7 = M^1(\mathbb{R}^8) \to M^2\mathbb{R}^8 = (\mathbb{C}P^3) \to M^4(\mathbb{R}^8) = S^4$$

fibers: ray, two opposite directional rays or the sphere S^0, M_2^1 or the sphere S^1, M_4^2 or the sphere S^2, from the second to the end the fiber is rays of the 4-dimensional linear space or the sphere S^3.

$$(\mathbb{C}^2 = \mathbb{R}^4)^* \to S^3 \to \mathbb{R}P^3 = M_4^1 \to \mathbb{C}P^1 = M_4^2 = S^2$$

fibers: ray, two opposite directional rays or the sphere S^0, M_2^1 or the sphere S^1, from the second to the end the fiber is rays of the 4-dimensional linear space or the sphere S^3.

$$(\mathbb{C} = \mathbb{R}^2)^* \to S^1 \to \mathbb{R}P^1 = M_2^1 = S^1$$

fibers: ray, two opposite directional rays or the sphere S^0, two points or sphere S^0.

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