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THE METHOD OF PROBABILISTIC SOLUTION FOR DETERMINATION OF ELECTRIC AND THERMAL STATIONARY FIELDS IN CONIC AND PRISMATIC DOMAINS

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Abstract. In this paper, for determination of the electric and thermal stationary fields the Dirichlet ordinary and generalized harmonic problems are considered. The term "generalized" indicates that a boundary function has a finite number of first kind discontinuity curves. For numerical solution of boundary problems the method of probabilistic solution (MPS) is applied, which in its turn is based on a modeling of the Wiener process. The suggested algorithm does not require an approximation of a boundary function, which is main of its important properties. For examining and to illustrate the effectiveness and simplicity of the proposed method four numerical examples are considered on finding the electric and thermal fields. In the role of domains are taken: finite right circular cone and truncated cone; a rectangular parallelepiped. Numerical results are presented.

1. Introduction

Let D be a finite domain in the Euclidian space R^3, bounded by one closed piecewise smooth surface S (i.e., $S = \bigcup\limits_{j=1}^{p} S^j$), where each part S^j is a smooth surface. Besides, we assume: equations of the parts S^j are given; for the surface S it is easy to show that a point $x = (x_1, x_2, x_3) \in R^3$ lies in $\overline{D}$ or not.

It is known (see, e.g., [1,2,6,12,14–17]) that in practical stationary problems (for example, for the determination of the temperature of the thermal field or the potential of the electric field, and so on) there are cases when it is necessary to consider the Dirichlet ordinary (or generalized) harmonic problems: A (or B).

Problem A. Find a function $u(x) \equiv u(x_1, x_2, x_3) \in C^2(D) \bigcap C(\overline{D})$ satisfying the conditions:

$$\Delta u(x) = 0, \quad x \in D,$$
$$u(y) = h(y), \quad y \in S,$$

where $\Delta = \sum\limits_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator and $h(y) \equiv h(y_1, y_2, y_3)$ is a continuous function on S.

It is known (see, e.g., [12, 16, 17]) that Problem A is correct, i.e., its solution exists, is unique and depends on data continuously. It should be noted that in general the difficulties and respectively the laboriousness of solving problems sharply increase along with the dimension of the problems considered. Therefore, as a rule, one fails to develop standard methods for solving a wide class of multidimensional problems with the same high accuracy as in the one-dimensional case. For example, the exact solution of Problem A for a circle is written by one-dimensional Poisson's integral and in the case of kernel by two-dimensional Poisson's integral. The given simple example shows the difficulty in determining of the solution with the high accuracy of the Dirichlet ordinary harmonic problem when the dimension increases. In this paper, besides the fact that numerical solution of problems of type A by MPS is interesting and important (see, e.g., [3, 4, 18]), it has an additional role in this paper (see section 3).

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Problem B. Function $g(y)$ is given on the boundary S of the domain D and is continuous everywhere, except a finite number of curves $l_1, l_2, \dots, l_n$ which represent discontinuity curves of the first kind for the function $g(y)$. It is required to find a function $u(x) \equiv u(x_1, x_2, x_3) \in C^2(D) \bigcap C(\overline{D} \setminus \bigcup\limits_{k=1}^{n} l_k)$ satisfying the conditions:

$$\Delta u(x) = 0, \qquad x \in D, \tag{1.1}$$

$$u(y) = g(y), \qquad y \in S, \quad y \overline{\in} l_k \subset S \ \ (k = \overline{1,n}), \tag{1.2}$$

$$|u(y)| < c, \qquad\qquad y \in \overline{D}, \tag{1.3}$$

where c is a real constant.

It is shown (see [5,20]) that Problem (1.1), (1.2), (1.3) has a unique solution depending continuously on the data, and for a generalized solution $u(x)$ the generalized extremum principal is valid:

$$\min_{x \in S} u(x) < \underset{x \in D}{u(x)} < \max_{x \in S} u(x), \tag{1.4}$$

where for $x \in S$ it is assumed that $x \overline{\in} l_k$ $(k = \overline{1,n})$.

It is evident that actually, the surface S is divided into open parts S_i $(i = \overline{1,m})$ by curves l_k $(k = \overline{1,n})$ or $S = (\bigcup\limits_{i=1}^{m} S_i) \bigcup (\bigcup\limits_{k=1}^{n} l_k)$, where for the concrete case, between m and n from the following conditions: $n = m$, $n < m$, $n > m$ take place one of. On the basis of noted, the boundary function $g(y)$ has the following form

$$g(y) = \begin{cases} g_1(y), & y \in S_1, \\ g_2(y), & y \in S_2, \\ \dots\dots\dots\dots & \\ g_m(y), & y \in S_m, \end{cases} \tag{1.5}$$

where the functions $g_i(y), y \in S_i$ are continuous on the parts S_i of S, respectively.

Note (see [20]) that the additional requirement (1.3) of boundedness concerns actually only the neighborhoods of discontinuity curves of the function $g(y)$ and it plays an important role in the extremum principle (1.4).

On the basis of (1.3), in general, the values of $u(y)$ are not defined on the curves l_k. For example, if Problem B concerns the determination of the thermal (or the electric) field, then $u(y) = 0$ when $y \in l_k$, respectively, in this case, in physical sense the curves l_k are non-conductors (or dielectrics).

Remark 1. If inside the surface S there is a vacuum then we have the ordinary and generalized problems with respect to closed shells.

In general, it is known (see [6, 7, 20]) that the methods used to obtain an approximate solution to ordinary boundary problems are less suitable (or not suitable at all) for solving boundary problems of type B. In particular, the convergence of the approximate process is very slow in the neighborhood of boundary singularities and, consequently, the accuracy of the approximate solution of the generalized problem is very low.

The choice and construction of computational schemes (algorithms) mainly depend on problem class, its dimension, geometry and location of singularities on the boundary, e.g., Dirichlet generalized plane problems for harmonic functions with concrete location of discontinuity points in the cases of simply connected domains are considered in [1, 2, 6, 8, 15], and general cases for finite and infinite domains are studied in [9–11, 13, 14, 19].

In the case of 3D harmonic generalized problems, due to their higher dimension, the difficulties become more significant. In particular, there does not exist a standard scheme which can be applied to a wide class of domains. In the classical literature, simplified, or so called "solvable" generalized problems (problems whose "exact" solutions can be constructed by series, whose terms are represented by special functions) are considered, and for their solution the classical method of separation of variables is mainly applied and therefore the accuracy of the solution is rather low. In the mentioned problems, the boundary functions (conditions) are mainly constants, and in the general case, the analytic form of the "exact" solution is so difficult in the sense of numerical implementation, that it only has theoretical significance (see, e.g., [1, 2, 6, 12, 15]).

As a consequence of the above, from our viewpoint, the construction of high accuracy and effectively realizable computational schemes for approximate solution of 3D Dirichlet generalized harmonic problems (whose application is possible to a wide class of domains) have both theoretical and practical importance.

It should be noted that in literature (see, e.g., [1, 2, 6, 12, 15]), while solving Dirichlet generalized harmonic problems, the existence of discontinuity curves often is neglected. This fact and application of classical methods to solving problems of type B are reasons of the inaccuracies. Therefore, for numerical solution of generalized harmonic problems we should apply such methods which do not require approximation of a boundary function and in which the existence of discontinuity curves is not ignored. The suggested algorithm is one of such methods.

2. The Method of Probabilistic Solution

In this section the essence of the suggested algorithm for numerical solving problems of type A and B is given, and its detail description is in [21]. The main theorem in realization of the MPS is the following one (see, e.g., [5])

Theorem 1. *If a finite domain $D \in R^3$ is bounded by piecewise smooth surface S and $g(y)$ is continuous (or discontinuous) bounded function on S, then the solution of the Dirichlet ordinary (or generalized) boundary problem for the Laplace equation at the fixed point $x \in D$ has the form*

$$u(x) = E_x g(x(\tau)). \tag{2.1}$$

In (2.1): $E_x g(x(\tau))$ is the mathematical expectation of the values of the boundary function $g(y)$ at the random intersection points of the trajectory of the Wiener process and the boundary S; τ is the random moment of first exit of the Wiener process $x(t) = (x_1(t), x_2(t), x_3(t))$ from the domain D. It is assumed that the starting point of the Wiener process is always $x(t_0) = (x_1(t_0), x_2(t_0), x_3(t_0)) \in D$, where the value of the desired function is being determined. If the number N of the random intersection points $y^i = (y_1^i, y_2^i, y_3^i) \in S$ $(i = 1, 2, \dots, N)$ is sufficiently large, then according to the law of large numbers, from (2.1) we have

$$u(x) \approx u_N(x) = \frac{1}{N} \sum_{i=1}^{N} g(y^i) \tag{2.2}$$

or $u(x) = \lim u_N(x)$ for $N \to \infty$, in probability. Thus, in the presence of the Wiener process the approximate value of the probabilistic solution to Problems A and B at a point $x \in D$ are calculated by formula (2.2).

In order, to simulate of the Wiener process we use the following recursion relations (see, e.g., [21]):

$$\begin{aligned} x_1(t_k) &= x_1(t_{k-1}) + \gamma_1(t_k)/nq, \\ x_2(t_k) &= x_2(t_{k-1}) + \gamma_2(t_k)/nq, \\ x_3(t_k) &= x_3(t_{k-1}) + \gamma_3(t_k)/nq, \\ (k &= 1, 2, \dots), \quad x(t_0) = x, \end{aligned} \tag{2.3}$$

according of which the coordinates of the point $x(t_k) = (x_1(t_k),\ x_2(t_k),\ x_3(t_k))$ are being determined. In (2.3): $\gamma_1(t_k), \gamma_2(t_k), \gamma_3(t_k)$ are three normally distributed independent random numbers for the k-th step, with zero means and variances one; nq is a number of quantification (nq) such that $1/nq = \sqrt{t_k - t_{k-1}}$ and when $nq \to \infty$, then the discrete process approaches to the continuous Wiener process. In the implementation, the random process is simulated at each step of the walk and continues until it crosses the boundary.

In the considered case computations and generation of random numbers are done in MATLAB.

3. Numerical Examples

In this section, problems of type A and B are solved for one and the same domain. The reason of this is the following: since there exist exact test problems for type A, and there are none for type B, therefore, Problem A has an additional role in this paper. Namely, verification of a scheme needed for numerical solution of Problem B and corresponding calculating program is carried out with the help of Problem A, which consists in following.

Function

$$u(x^0, x) = \frac{1}{|x - x^0|}, \quad x \in D, \quad x^0 = (x_1^0, x_2^0, x_3^0) \overline{\in} \overline{D}, \tag{3.1}$$

is taken in the role of the exact test solution for Problem A under boundary condition $h(y) = \frac{1}{|y-x^0|}$, $y \in S$, where $|x - x^0|$ denotes the distance between the points x and x^0. After this, function $h(y)$ is taken in the role of functions $g_i(y)$ $(i = \overline{1,m})$ in Problem B and consequently in calculating program. Evidently, in this case curves l_k represent removable discontinuity curves for function $g_(y)$, therefore instead of problem of type B we have problem of type A. For the obtained problem, verification of the scheme needed for numerical solution of Problem B and corresponding calculating program (comparison of the obtained results with exact solution) is carried out first of all, and then Problem B is being solved under boundary conditions (1.5).

In the case when Problems A and B concern electrostatic field, for full investigation of the field it is necessary to find both potential and strength of the field. It is known [6, 15] that the strength $E(x) = (E_1(x), E_2(x), E_3(x))$ of electrostatic field is defined as follows:

$$E(x) = -\left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}\right), \quad x \in D, \tag{3.2}$$

where $u(x)$ is potential of electrostatic field. It is known that the vector $E(x)$ is directed where the potential of the electric field is less.

Since in our case Problems A and B are solved by a numerical method, therefore, for the test problem, coordinates of vector $E(x)$ are defined by formula (3.2), and in the case of numerical solution by the central difference formula

$$f'(t) \approx \frac{f(t+h) - f(t-h)}{2h} \tag{3.3}$$

is used, whose accuracy is $O(h^2)$

Thus on the basis of (3.2) and (3.3) for definition components of the vectors $E(x)$ and $E^N(x)$ we have:

$$E_k(x) = -\frac{\partial u(x)}{\partial x_k} = \frac{x_k - x_k^0}{|x - x^0|^3}, \quad (k = 1, 2, 3); \tag{3.4}$$

$$\begin{aligned} E_k^N(x) = -\frac{\partial u_N(x)}{\partial x_k} &\approx -[u_N((x_1 + h)\delta_{1k}, (x_2 + h)\delta_{2k}, (x_3 + h)\delta_{3k}) \\ &-u_N((x_1 - h)\delta_{1k}, (x_2 - h)\delta_{2k}, (x_3 - h)\delta_{3k})]/(2h), \end{aligned} \tag{3.5}$$

where δ_{ik} is Kronecker symbols.

In the present paper the MPS is applied to four examples. In tables, N is the number of the implementation of the Wiener process for the given points $x^i = (x_1^i, x_2^i, x_3^i) \in D$, and nq is the number of the quantification. For simplicity, in the considered examples the values of nq and N are the same. In tables for problems of type A we present the maximum absolute errors Δ^i at the points $x^i \in D$ of $u_N(x)$, in the MPS approximation, for $nq = 200$ and various values of N, and under notations of type $(E \pm k)$, $10^{\pm k}$ are meant. In particular, $\Delta^i = |u_N(x^i) - u(x^0, x^i)|$, where $u_N(x^i)$ is the approximate solution of Problem A at the point x^i, which is defined by formula (2.2), and the exact solution $u(x^0, x^i)$ of the test problem is given by (3.1). In tables, for problems of type B, the probabilistic solution $u_N(x)$ is presented at the points x^i, defined by (2.2).

Remark 2. The Problems A and B for ellipsoidal, spherical, cylindrical domains and for the kernel layer are considered in [21].

Example 3.1. *In the first example it is required to determine the electrostatic field in the domain D. In the role of D is taken interior of the finite right circular cone S_c :*

$$(x_1)^2 + (x_2)^2 - \left(\frac{r}{h}\right)^2 (h - x_3)^2 = 0, \qquad 0 \le x_3 \le h, \tag{3.6}$$

where h is a height of the cone, r is a radius of its base S_1, and $x(x_1, x_2, x_3)$ is a current point of the conic surface S_c (the full surface S of D is $S = S_c \bigcup S_1$).

In numerical experiments for the considered example, is taken: 1) $h = 2$, $r = 1$; 2) in the test Problem A the boundary function $h(y) = 1/|y - x^0|$, $y \in S$, $x^0 = (0, 0, -5)$; 3) in Problem B the boundary function $g(y) \equiv g(y_1, y_2, y_3)$ has the form

$$g(y) = \begin{cases} 2, & y \in S_1 = \{y \in S \; |0 \le (y_1)^2 + (y_2)^2 < 1, y_3 = 0\}, \\ 1.5, & y \in S_2 = \{y \in S_c \; |0 < y_3 < 0.5\}, \\ 1, & y \in S_3 = \{y \in S_c \; |0.5 < y_3 < 1\}, \\ 0.5, & y \in S_4 = \{y \in S_c \; |1 < y_3 \le 2\}, \\ 0, & y \in l_k \; (k = 1, 2, 3). \end{cases} \tag{3.7}$$

It is evident that in the considered case l_1 is the circle of the base S_1; l_2 and l_3 are the circles, which are obtained by intersection of the planes $x_3 = 0.5$, $x_3 = 1$ and the surface S_c. Besides, in the physical sense the circles l_k are non-conductors.

In order to determine the intersection points $y^i = (y_1^i, y_2^i, y_3^i)$ $(i = \overline{1, N})$ of the trajectory of the Wiener process and the surface S, we operate in the following way. During the implementation of the Wiener process, for each current point $x(t_k)$, defined from (2.3), its location with respect to S is checked, i.e., for the point $x(t_k)$ the value

$$d = (x_1(t_k))^2 + (x_2(t_k))^2 - \left(\frac{r}{h}\right)^2 (h - x_3(t_k))^2$$

is calculated and the following conditions: 1) $d = 0$ and $0 < x_3(t_k) < h$; 2) $d < 0$ and $0 < x_3(t_k) < h$; 3) $d < 0$ or $d > 0$ and $x_3(t_k) < 0$; 4) $d > 0$ and $0 < x_3(t_k) < h$ are checked. In the first case $x(t_k) \in S_c$ and $y^i = x(t_k)$. In the second case $x(t_k) \in D$ and the process continuous until it crosses the boundary of D. In the cases (3) and (4) $x(t_k) \overline{\in} \overline{D}$.

Let $x(t_{k-1}) \in D$ for the moment $t = t_{k-1}$ and $x(t_k) \overline{\in} \overline{D}$ for the moment $t = t_k$. In the mentioned case we have only two variants: 3) or 4). In the case 3) we find the intersection point $y = (y_1, y_2, 0)$ of the plane $x_3 = 0$ and a line l passing through the points $x(t_{k-1})$ and $x(t_k)$. If $0 \le (y_1)^2 + (y_2)^2 < r^2$ then $y^i = (y_1, y_2, 0)$. In the case 4), for approximate determination of the point y^i, a parametric equation of a line L passing through the points $x(t_{k-1})$ and $x(t_k)$ is firstly obtained, which has the following form

$$\begin{cases} x_1 = x_1^{k-1} + (x_1^k - x_1^{k-1})\theta, \\ x_2 = x_2^{k-1} + (x_2^k - x_2^{k-1})\theta, \\ x_3 = x_3^{k-1} + (x_3^k - x_3^{k-1})\theta, \end{cases} \tag{3.8}$$

where (x_1, x_2, x_3) is the current point of L and θ is a parameter $(-\infty < \theta < \infty)$, and $x_i^{k-1} \equiv x_i(t_{k-1})$, $x_i^k \equiv x_i(t_k)$ $(i = 1, 2, 3)$. After this, for definition of the intersection points x^* and x^{**} of the line L and the surface S equation (3.6) is solved with respect to θ.

It is easy to see that for parameter θ we obtain an equation

$$A\theta^2 + 2B\theta + C = 0 \tag{3.9}$$

whose discriminant $d^* = B^2 - AC > 0$.

Since the discriminant of (3.9) is positive, the points x^* and x^{**} are defined respectively on the basis of (3.8) for solutions of (3.9) θ_1 and θ_2. In the role of the points y^i the one (from x^* and x^{**}) for which $|x(t_k) - x|$ is minimal is chosen.

In Table 3.1A the errors Δ^i of the approximate solution $u_N(x)$ of the test problem at the points $x^i \in D$ $(i = \overline{1, 5})$ are presented. On the basis of (3.4) and (3.5) we calculated exact and approximate strengths of the electric field (or $E_3(x)$ and $E_3^N(x)$) on the axis Ox_3 at the points x^i $(i = 1, 2, 3)$

TABLE 3.1A. Results for Problem A (in Example 3.1)

x^i	$(0,0,0.5)$	$(0,0,1)$	$(0,0,1.8)$	$(0.2,0.2,0.5)$	$(-0.2,-0.2,0.5)$
N	Δ^1	Δ^2	Δ^3	Δ^4	Δ^5
$5E+3$	$0.30E-3$	$0.39E-3$	$0.74E-5$	$0.52E-4$	$0.17E-4$
$1E+4$	$0.63E-4$	$0.14E-3$	$0.13E-4$	$0.11E-3$	$0.15E-3$
$5E+4$	$0.98E-4$	$0.72E-4$	$0.40E-4$	$0.25E-4$	$0.72E-4$
$1E+5$	$0.24E-4$	$0.49E-4$	$0.18E-4$	$0.49E-4$	$0.21E-4$
$5E+5$	$0.66E-5$	$0.26E-4$	$0.31E-4$	$0.18E-4$	$0.36E-4$
$1E+6$	$0.32E-5$	$0.42E-4$	$0.31E-4$	$0.15E-4$	$0.27E-4$

for $N = 10^6, nq = 200, h = 0.03$. We obtained the following results: $E_3(0,0,0.5) = 0.033111$; $E_3(0,0,1) = 0.027778$; $E_3(0,0,1.8) = 0.021626$; $E_3^N(0,0,0.5) = 0.033061$; $E_3^N(0,0,1) = 0.027781$; $E_3^N(0,0,1.8) = 0.021651$;

It is evident that the results obtained for $E_3^N(x^i)$ are in good agreement with the values of $E_3(x^i)$ $(i = 1,2,3)$.

TABLE 3.1B. Results for Problem B (in Example 3.1)

x^i	$(0,0,0.5)$	$(0,0,1)$	$(0,0,1.8)$	$(0.2,0.2,0.5)$	$(-0.2,-0.2,0.5)$
N	$u_N(x^1)$	$u_N(x^2)$	$u_N(x^3)$	$u_N(x^4)$	$u_N(x^5)$
$5E+3$	1.32910	0.77620	0.50010	1.31410	1.31470
$1E+4$	1.33335	0.77370	0.50010	1.32060	1.31690
$5E+4$	1.33374	0.77282	0.50016	1.31599	1.32765
$1E+5$	1.33447	0.77314	0.50035	1.32166	1.32319
$5E+5$	1.33318	0.77234	0.50023	1.32131	1.32318
$1E+6$	1.33356	0.77211	0.50023	1.32223	1.32185

In Table 3.1B the values of the approximate solution $u_N(x)$ to Problem B at the same points $x^i(i = \overline{1,5})$ are given. The boundary function (3.7) is symmetric with respect to the axis Ox_3, respectively, the obtained results for x^4 and x^5 are symmetric with respect to the axis Ox_3 and have sufficient accuracy for many practical problems.

For illustration, we calculated the electrostatic field strength by (3.5) on the axis Ox_3 at the same points $x^i(i = 1,2,3)$ for $N = 10^6, nq = 200, h = 0.03$. We obtained the following results: $E_3^N(0,0,0.5) = 1.234714$; $E_3^N(0,0,1) = 0.989775$; $E_3^N(0,0,1.8) = 0.00426$. The obtained results are in good agreement with the real physical picture.

Example 3.2. *In this example, the problem on the temperature distribution is considered. In the role of domain D the interior of a truncated right circular cone S_c is taken:*

$$(x_1)^2 + (x_2)^2 - \Big(\frac{R-r}{h}\Big)^2 \Big(\frac{Rh}{R-r} - x_3\Big)^2 = 0,\ 0 \le x_3 \le h,$$

where h is the height, R the radius of the lower base, r is the radius of the upper base, and $x(x_1,x_2,x_3)$ is a current point of the conic surface S_c. The boundary of D is $S = S_1 \bigcup S_c \bigcup S_2$, where $S_1 = \{y \in S\ |0 \le d < R, y_3 = 0\}$ and $S_2 = \{y \in S\ |0 \le d < r, y_3 = h\}$, and $d = sqrt((y_1)^2 + (y_2)^2)$.

The problems A and B are solved when $h = 2$, $R = 1$, $r = 0.5$, $x^0 = (0, 0, -5)$, and the boundary function $g(y)$ has the form

$$g(y) = \begin{cases} 2, & y \in S_1, \\ 0, & y \in S_2, \\ 1.5, & y \in S_3, \\ 1, & y \in S_4, \\ 1.5, & y \in S_5, \\ 1, & y \in S_6, \\ 0, & y \in l_k \ (k = \overline{1,6}). \end{cases} \tag{3.10}$$

In (3.10): l_1, l_2 are the circles of the bases S_1 and S_2; $l_3,\ l_4,\ l_5,\ l_6$ are the generatices of the conic surface S_c, which pass through the points $(R, 0)$, $(0, R)$, $(-R, 0)$, $(0, -R)$, respectively; $S_3 = \{y \in S_c \mid r < d < R, y_1 > 0, y_2 > 0, 0 < y_3 < h\}$; $S_4 = \{y \in S_c \mid r < d < R, y_1 < 0, y_2 > 0, 0 < y_3 < h\}$; $S_5 = \{y \in S_c \mid r < d < R, y_1 < 0, y_2 < 0, 0 < y_3 < h\}$; $S_6 = \{y \in S_c \mid r < d < R, y_1 > 0, y_2 < 0, 0 < y_3 < h\}$. Besides, in this case the curves l_k and S_2 are non-conductors.

In the considered case, for determination of the intersection points y^i $(i = \overline{1, N})$ of the trajectory of the Wiener process and the surface S the same algorithm, described in Example 3.1 is applied.

In Table 3.2A the errors Δ^i of the approximate solution $u_N(x)$ of the test problem are presented at the points $x^i \in D$ $(i = \overline{1, 5})$.

Table 3.2A. Results for Problem A (in Example 3.2)

x^i	$(0, 0, 0.5)$	$(0, 0, 1)$	$(0, 0, 1.8)$	$(0.5, 0.5, 1)$	$(-0.5, -0.5, 1)$
N	Δ^1	Δ^2	Δ^3	Δ^4	Δ^5
$5E + 3$	$0.99E - 4$	$0.27E - 3$	$0.99E - 4$	$0.58E - 4$	$0.10E - 3$
$1E + 4$	$0.66E - 4$	$0.11E - 3$	$0.88E - 4$	$0.48E - 4$	$0.58E - 4$
$5E + 4$	$0.65E - 4$	$0.52E - 4$	$0.90E - 4$	$0.27E - 4$	$0.30E - 4$
$1E + 5$	$0.40E - 4$	$0.26E - 4$	$0.25E - 4$	$0.17E - 4$	$0.48E - 4$
$5E + 5$	$0.16E - 4$	$0.19E - 4$	$0.55E - 4$	$0.24E - 4$	$0.23E - 4$
$1E + 6$	$0.81E - 5$	$0.28E - 4$	$0.53E - 4$	$0.27E - 4$	$0.25E - 4$

The values of the approximate solution $u_N(x)$ of Problem B at the same points x^i are given in Table 3.2B. Since the boundary function (3.10) is symmetric with respect to the axis Ox_3, therefore, for control in the role of x^i $(i = 4, 5)$, the points which are symmetric with respect to the axis Ox_3 are taken. The obtained results have sufficient accuracy for many practical problems and are in good agreement with the real physical picture.

Table 3.2B. Results for Problem B (in Example 3.2)

x^i	$(0, 0, 0.5)$	$(0, 0, 1)$	$(0, 0, 1.8)$	$(0.5, 0.5, 1)$	$(-0.5, -0.5, 1)$
N	$u_N(x^1)$	$u_N(x^2)$	$u_N(x^3)$	$u_N(x^4)$	$u_N(x^5)$
$5E + 3$	1.53710	1.29085	0.56980	1.48650	1.48660
$1E + 4$	1.52895	1.29140	0.56875	1.48575	1.48580
$5E + 4$	1.52950	1.28705	0.57235	1.48548	1.48538
$1E + 5$	1.52615	1.28884	0.57426	1.48584	1.48558
$5E + 5$	1.52680	1.28701	0.57756	1.48578	1.48579
$1E + 6$	1.52670	1.28710	0.57615	1.48558	1.48566

Example 3.3. *Here in the role of domain D the interior of rectangular parallelepiped $MNKOM_1N_1K_1O_1$ is taken with the vertex at the origin $O(0, 0, 0)$ of Cartesian coordinate right-handed system and measurements a, b and c. It is evident that the boundary S of D is $S =$*

$(\bigcup_{j=1}^{6} S_j)\bigcup(\bigcup_{k=1}^{12} l_k)$, *where S_j are open faces and l_k are edges. In this example, the problem on the temperature distribution is considered.*

In order to determine the intersection points $y^i = (y_1^i, y_2^i, y_3^i)$ $(i = \overline{1, N})$ of the trajectory of the Wiener process and the surface S of mentioned parallelepiped the following way is used. During the implementation of the Wiener process, for each current point $x(t_k)$, defined by (2.3), its location with respect to S is checked, i.e., for the point $x(t_k)$ the following conditions

$$0 < x_1(t_k) < a, \quad 0 < x_2(t_k) < b, \quad 0 < x_3(t_k) < c$$

are checked. If the mentioned conditions are fulfilled then the process (2.3) continuous. If $x(t_k) \in S$ then $y^i = x(t_k)$.

Let $x(t) \in D$ for the moment $t = t_{k-1}$ and $x(t) \overline{\in} \overline{D}$ for the moment $t = t_k$. In this case, for approximate determination of the point y^i, a parametric equation of a line L passing through the points $x(t_{k-1})$ and $x(t_k)$ is firstly obtained in the form (3.8). After this, the intersection point x^* of the line L and that face, which is intersected by the trajectory of wiener process is found and respectively, in this case $y^i = x^*$.

In numerical experiments, we took: 1) $a = 1$, $b = 2$, $c = 3$; 2) in the test Problem A, $x^0 = (0.5, 1, -5)$; 3) in Problem B the boundary function $g(y)$ has the following form

$$g(y) = \begin{cases} 3, & y \in S_1 = \{y \in S \mid y_1 = 0, 0 < y_2 < b, 0 < y_3 < c\}, \\ 1, & y \in S_2 = \{y \in S \mid y_1 = a, 0 < y_2 < b, 0 < y_3 < c\}, \\ 0.5, & y \in S_3 = \{y \in S \mid 0 < y_1 < a, y_2 = 0, 0 < y_3 < c\}, \\ 0.5, & y \in S_4 = \{y \in S \mid 0 < y_1 < a, y_2 = b, 0 < y_3 < c\}, \\ 0, & y \in S_5 = \{y \in S \mid 0 < y_1 < a, 0 < y_2 < b, y_3 = 0\}, \\ 2, & y \in S_6 = \{y \in S \mid 0 < y_1 < a, 0 < y_2 < b, y_3 = c\}, \\ 0, & y \in l_k \ (k = \overline{1, 12}), \end{cases} \tag{3.11}$$

where l_k and S_5 are dielectrics.

The errors Δ^i of the approximate solution $u_N(x)$ to test Problem A at the points $x^i \in D$ $(i = \overline{1, 5})$ are given in Table 3.3A.

TABLE 3.3A. Results for Problem A (in Example 3.3)

x^i	$(0.5, 1, 0.5)$	$(0.5, 1, 1)$	$(0.5, 1, 1.5)$	$(0.5, 1, 2)$	$(0.5, 1, 2.5)$
N	Δ^1	Δ^2	Δ^3	Δ^4	Δ^5
$5E+3$	$0.11E-3$	$0.28E-3$	$0.17E-3$	$0.14E-3$	$0.14E-3$
$1E+4$	$0.73E-4$	$0.28E-4$	$0.25E-3$	$0.77E-4$	$0.13E-4$
$5E+4$	$0.57E-4$	$0.66E-4$	$0.40E-4$	$0.48E-4$	$0.12E-4$
$1E+5$	$0.17E-4$	$0.20E-4$	$0.61E-4$	$0.29E-4$	$0.15E-4$
$5E+5$	$0.35E-4$	$0.32E-4$	$0.25E-4$	$0.15E-4$	$0.21E-4$
$1E+6$	$0.65E-5$	$0.43E-5$	$0.49E-5$	$0.19E-4$	$0.19E-4$

The values of the approximate solution $u_N(x)$ of Problem B at the points $x^i \in D$ $(i = 1, 2, 3)$ are given in Table 3.3B. Since the boundary function (3.11) is symmetric with respect to the plane $x_2 = 1$, therefore, for control in the role of x^i $(i = 4, 5)$, the points which are symmetric with respect to the plane $x_2 = 1$ are taken. The obtained results have sufficient accuracy for many practical problems and are in good agreement with the real physical picture (see Table 3.3B).

Example 3.4. *In this example, the problems A and B on the temperature distributionn are considered. In the role of D we took the same rectangular parallelepiped as in Example* 3.3. *In this case, Problem*

B under the boundary function $g(y)$ with specific form

$$g(y) = \begin{cases} v_1, & y \in S_1 = \{y \in S \mid y_1 = 0, 0 < y_2 < b, 0 < y_3 < c\}, \\ v_2, & y \in S_2 = \{y \in S \mid y_1 = a, 0 < y_2 < b, 0 < y_3 < c\}, \\ 0, & y \in (\bigcup_{j=3}^{6} S_j) \bigcup (\bigcup_{k=1}^{12} l_k) \end{cases} \tag{3.12}$$

is solved, where $S_j (j = \overline{1,6}), l_k (k = \overline{1,12})$ are the same as in Example 3.3, v_1 and v_2 are constants. It is evident that S_j $(j = \overline{3,6})$ and l_k $(k = \overline{1,12})$ are non-conductors.

Table 3.3B. Results for Problem B (in Example 3.3)

x^i	$(0.5, 1, 0.5)$	$(0.5, 1, 1.5)$	$(0.5, 1, 2.5)$	$(0.5, 0.5, 1.5)$	$(0.5, 1.5, 1.5)$
N	$u_N(x^1)$	$u_N(x^2)$	$u_N(x^3)$	$u_N(x^4)$	$u_N(x^5)$
$5E+3$	1.36592	1.77016	1.87272	1.51024	1.50580
$1E+4$	1.37656	1.79118	1.87680	1.48596	1.49162
$5E+4$	1.38256	1.79529	1.87079	1.49502	1.50417
$1E+5$	1.37245	1.79176	1.87154	1.51296	1.49943
$5E+5$	1.36930	1.78991	1.86844	1.50475	1.50233
$1E+6$	1.37228	1.78882	1.86735	1.50309	1.50362

In numerical experiments we took: $a = 1$, $b = 2$, $c = 3$, $v_1 = 3$, $v_2 = 1$ and $x^0 = (0.5, 1, -5)$. For determination of the intersection points y^i $(i = \overline{1,N})$ the same algorithm is applied, which is described in Example 3.3.

In Table 3.4A the errors Δ^i of the approximate solution $u_N(x)$ of the test problem A are presented at the points $x^i \in D$ $(i = \overline{1,5})$. The obtained results have sufficient accuracy for many practical problems.

Table 3.4A. Results for Problem A (in Example 3.4)

x^i	$(0.9, 1, 1.5)$	$(0.8, 1, 1.5)$	$(0.5, 1, 1.5)$	$(0.2, 1, 1.5)$	$(0.1, 1, 1.5)$
N	Δ^1	Δ^2	Δ^3	Δ^4	Δ^5
$5E+3$	$0.17E-3$	$0.48E-4$	$0.19E-3$	$0.17E-3$	$0.53E-4$
$1E+4$	$0.40E-4$	$0.14E-4$	$0.69E-4$	$0.23E-3$	$0.69E-5$
$5E+4$	$0.52E-4$	$0.46E-4$	$0.14E-6$	$0.39E-4$	$0.25E-4$
$1E+5$	$0.23E-4$	$0.73E-5$	$0.18E-4$	$0.51E-4$	$0.51E-5$
$5E+5$	$0.72E-5$	$0.27E-4$	$0.19E-4$	$0.19E-5$	$0.27E-5$
$1E+6$	$0.69E-5$	$0.56E-5$	$0.19E-4$	$0.14E-4$	$0.38E-5$

The values of the approximate solution $u_N(x)$ of Problem B at the same points $x^i \in D$ are given in Table 3.4B. The obtained results have sufficient accuracy for many practical problems and are in good agreement with the real physical picture.

It should be noted that Example 3.4 is considered in [2], where it is solved by the method of separation of variables. It is shown that in conditions (3.12) the analytical solution to Problem B has the following form

$$u(x) \equiv u(x_1, x_2, x_3) = \frac{16}{\pi^2} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{f_1(x_1) f_2(x_2, x_3)}{(2p+1)(2q+1)}, \tag{3.13}$$

where

$$f_1(x_1) = \frac{v_1 sh(l(a - x_1)) + v_2 sh(l x_1)}{sh(la)},$$

$$f_2(x_2, x_3) = sin\frac{(2p+1)\pi x_2}{b} sin\frac{(2q+1)\pi x_3}{c},$$
$$l = \frac{\pi}{bc}\sqrt{(c(2p+1))^2 + (b(2q+1))^2},$$

and $sh(t)$ is hyperbolic sine.

TABLE 3.4B. Results for Problem B (in Example 3.4)

x^i	$(0.9, 1, 1.5)$	$(0.8, 1, 1.5)$	$(0.5, 1, , 1.5)$	$(0.2, 1, 1.5)$	$(0.1, 1, 1.5)$
N	$u_N(x^1)$	$u_N(x^2)$	$u_N(x^3)$	$u_N(x^4)$	$nu_N(x^5)$
$5E+3$	1.13260	1.26040	1.75080	2.46040	2.73280
$1E+4$	1.12920	1.27440	1.74840	2.44380	2.70770
$5E+4$	1.12640	1.25490	1.74864	2.44730	2.71620
$1E+5$	1.12798	1.25773	1.74749	2.45006	2.71432
$4E+5$	1.12858	1.26055	1.75092	2.44749	2.71528
$1E+6$	1.12824	1.25857	1.75128	2.44603	2.71731

It is easy to see that the series (3.13) converges rapidly for all points $x = (x_1, x_2, x_3) \in D$, when $p, q \to \infty$. In order to compare the results obtained by the MPS and the (3.13), the partial sum $u_m(x)$ of the series (3.13) for $p = \overline{0, m}$ and $q = \overline{0, m}$ at the points x^i $(i = \overline{1, 5})$ were calculadet (see Table 3.4B). Because of rapid convergence of the series (3.13) when $x \in D$, the calculations have shown that practically $u_m(0.9, 1, 1.5) = 1.12524$, $u_m(0.8, 1, 1.5) = 1.25747$, $u_m(0.5, 1, 1.5) = 1.75388$, $u_m(0.2, 1, 1.5) = 2.45277$, $u_m(0.1, 1, 1.5) = 2.72234$, when $m = 50, 100, 150$. These results are sufficiently close to results which are presented in Table 3.4B.

It is evident that for the solution $u(x)$ the boundary condition (3.12) is satisfied on $(\bigcup\limits_{j=3}^{6} S_j)\bigcup(\bigcup\limits_{k=1}^{12} l_k)$. If $x \in S_1 \bigcup S_2$, then the rate of convergence of (3.13) becomes worse, especially in the neighborhood of the discontinuity curves. In particular, the convergence is very slow and consequently, the accuracy in the satisfaction of boundary condition on $S_1 \bigcup S_2$ is very low (see Section 1). This is caused by the fact that, when $x \in S_1 \bigcup S_2$ and tends to the discontinuity curves (edges), then all the terms of the series (3.13) tend to zero.

TABLE 3.4C. Results for partial sum $u_m(x)$

i	x^i	$u_m(x), m = 50$	$u_m(x), m = 100$	$u_m(x), m = 150$
1	$(1, 1, 1.5)$	0.987309	0.993644	0.995760
2	$(1, 1.8, 1.5)$	0.973200	0.986554	0.991000
3	$(1, 1.9, 1.5)$	1.033759	0.976574	1.011400
4	$(1, 1.99, 1.5)$	0.867110	1.175235	1.021747
5	$(1, 1.999, 1.5)$	0.099228	0.198273	0.295694
6	$(1, 1, 2.99)$	0.623372	1.044791	1.176482
7	$(1, 1, 2.999)$	0.662021	0.132586	0.198485
8	$(1, 1.99, 2.99$	0.547480	1.235730)	1.207186
9	$(1, 1.999, 2.999)$	0.006653	0.026456	0.058941
10	$(1, 0.001, 0.5)$	0.100479	0.199549	0.295065

From Table 3.4C it is clear that accuracy of the solution $u(x)$ is very low in the neighborhood of the discontinuity curves, as expected.

Remark 3. if V_1 or V_2 is not constant then the analytic form of the solution is so difficult in the sense of numerical implementation, that it has only theoretical significance (see [1]).

In this work the problems of type B are specially solved when boundary functions $g_i(y)$ $(i = \overline{1,m})$ are constants. This was caused by our interest to find out how much the obtained results were in agreement with real physical picture. It is evident that solving Problem B under condition (1.5) is as easy as Problem A. In general, Problem B can be solved for all such locations of discontinuity curves, which give the possibility to establish the part of surface S where the intersection point is located.

The analysis of the results of numerical experiments show that the results obtained by the suggested algorithm are reliable and it is effective for numerical solution of problems of type A and B. In particular, the algorithm is sufficiently simple for numerical implementation.

Besides, it should be noted that the accuracy of probabilistic solution of problems A and B is not significantly increasing (except some cases, see tables) when $N \to \infty$. It is caused by the fact that nq (the number of the quantification) is fixed. If more accuracy is needed then calculations for sufficiently large values of nq and N (see [20]) must be realized. In this case, numerical realization on a PC takes much time. This difficulty can be avoided by applying the method of parallel calculation. For this suitable computing technique is needed. Respectively, significantly less time will be needed for numerical realization and besides the accuracy of the obtained results will improve.

4. Concluding Remarks

1. In this work have demonstrated that the method of probabilistic solution(MPS) is ideally suited for numerical solving of both ordinary and generalized(2D and 3D) Dirichlet problems for rather a wide class of domains, in the case of Laplace equation.

2. The MPS does not require an approximation of a boundary function, which is one of its important properties.

3. The MPS is a fast solver for the above noted problems. Besides, it is easy to programme, its computational cost is low, it characterized by an accuracy which is sufficient for many problems.

References

1. B. M. Budak, A. A. Samarski, A. N. Tikhonov, *A Collection of Problems in Mathematical Physics.* (Russian) Third edition. Nauka, Moscow, 1980.
2. H. S. Carslaw, J. C. Jaeger, *Conduction of Heat in Solids.* Oxford University Press, London, 1959.
3. A. Sh. Chaduneli, Z. A. Tabagari, M. Zakradze, On solving the internal three-dimensional Dirichlet problem for a harmonic function by the method of probabilistic solution. *Bull. Georgian Natl. Acad. Sci. (N.S.)* **2** (2008), no. 1, 25–28.
4. A. Sh. Chaduneli, M. V. Zakradze, Z. A. Tabagari, A method of probabilistic solution to the ordinary and generalized plane Dirichlet problem for the Laplace equation. Science and Computing. In: *Proc. Sixth TSTC Scientific Advisory Committee Seminar.* vol. 2, pp. 361–366, 2003.
5. E. B. Dynkin, A. A. Juškevič, *Theorems and Problems on Markov Processes.* (Russian) Nauka, Moscow, 1967.
6. G. A. Grinberg, *Selected Problems of the Mathematical Theory of Electric and Magnetic Phenomena.* (Russian) Izd. Akad. Nauk SSSR, 1948.
7. L. V. Kantorovich, V. I. Krylov, *Approximate Methods of Higher Analysis.* (Russian) Fifth corrected edition Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow-Leningrad 1962.
8. A. Karageorghis, Modified methods of fundamental solutions for harmonic and biharmonic problems with boundary singularities. *Numer. Methods Partial Differential Equations* **8** (1992), no. 1, 1–19.
9. N. Koblishvili, M. Zakradze, On solving the Dirichlet generalized problem for a harmonic function in the case of infinite plane with holes. *Proc. A. Razmadze Math. Inst.* **164** (2014), 71–82.
10. N. Koblishvili, Z. Tabagari, M. Zakradze, On reduction of the Dirichlet generalized boundary value problem to an ordinary problem for harmonic function. *Proc. A. Razmadze Math. Inst.* **132** (2003), 93–106.
11. N. Koblishvili, M. Kublashvili, Z. Sanikidze, M. Zakradze, On solving the Dirichlet generalized problem for a harmonic function in the case of an infinite plane with a crack-type cut. *Proc. A. Razmadze Math. Inst.* **168** (2015), 53–62.
12. N. S. Koshlyakov, E. B. Gliner, M. M. Smirnov, *Equations in Partial Derivatives of Mathematical Physics.* (Russian) Vysshaya Shkola, Moscow, 1970,
13. M. Kublashvili, Z. Sanikidze, M. Zakradze, A method of conformal mapping for solving the generalized Dirichlet problem of Laplace's equation. *Proc. A. Razmadze Math. Inst.* **160** (2012), 71–89.
14. M. A. Lavrentjev, B. V. Shabat, *Methods of the Theory of Functions of a Complex Variable.* (Russian) Nauka, Moscow, 1973.
15. W. R. Smythe, *Static and Dynamic Electricity.* second edition. New York, Toronto, London, 1950.
16. A. N. Tikhonov, A. A. Samarskii, *The Equations of Mathematical Physics.* (Russian) Fourth edition, corrected. Nauka, Moscow, 1972.

17. V. S. Vladimirov, *Equations of Mathematical Physics.* (Russian) Second edition, revised and augmented. Nauka, Moscow, 1971.
18. M. Zakradze, Z. Sanikidze, Z. Tabagari, On solving the external three-dimensional Dirichlet problem for a harmonic function by the probabilistic method. *Bull. Georgian Natl. Acad. Sci. (N.S.)* **4** (2010), no. 3, 19–23.
19. M. Zakradze, N. Koblishvili, A. Karageorghis, Y. Smyrlis, On solving the Dirichlet generalized problem for harmonic function by the method of fundamental solutions. *Semin. I. Vekua Inst. Appl. Math. Rep.* **34** (2008), 24–32, 124.
20. M. Zakradze, M. Kublashvili, Z. Sanikidze, N. Koblishvili, Investigation and numerical solution of some 3D internal Dirichlet generalized harmonic problems in finite domains. *Trans. A. Razmadze Math. Inst.* **171** (2017), no. 1, 103–110.
21. M. Zakradze, B. Mamporia, M. Kublashvili, N. Koblishvili, The method of probabilistic solution for 3D Dirichlet ordinary and generalized harmonic problems in finite domains bounded with one surface. *Trans. A. Razmadze Math. Inst.* **172** (2018), no. 3, part A, 453–465.

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