Transactions of A. Razmadze
Mathematical Institute
Vol. **174** (2020), issue 2, 257–259

A NOTE ON THE MULTIPLE FRACTIONAL INTEGRALS DEFINED ON THE PRODUCT OF NONHOMOGENEOUS MEASURE SPACES

VAKHTANG KOKILASHVILI[1] AND TSIRA TSANAVA[1,2]

Abstract. In this note we present a trace type inequality in the mixed-norm Lebesgue spaces for multiple fractional integrals defined on an arbitrary measure quasi-metric space.

1. Introduction

Let (X,d,μ) be of nonhomogeneous type, i.e., a topological space endowed with a locally finite complete measure μ and quasi-metric $d : X \longrightarrow R$ satisfying the following conditions:
(i) $d(x,x)=0$, for all $x\in X$;
(ii) $d(x,y)>0$, for all $x\neq y$, $x,y\in X$;
(iii) there exists a positive constant a_0 such that $d(x,y)\le a_0 d(y,x)$ for every $x,y\in X$;
(iv) there exists a positive constant a_0 such that

$$d(x,y)\le a_1\left(d(x,z)+d(z,y)\right) \quad \text{for every} \quad x,y,z\in X;$$

(v) for every neighbourhood V of the point $x\in X$ there exists $r>0$ such that the ball $B(x,r)=\{y\in X:\ d(x,y)<r\}$ is contained in V;
(vi) the ball $B(x,r)$ is measurable for every $x\in X$ and for arbitrary $r>0$.
Let

$$I^{\gamma}f(x)=\int\limits_X (d(x,y))^{\gamma-1}f(y)d\mu, \quad 0<\gamma<1.$$

In [3] (see also [2, Chapter 6]), the following statement is proved.

Theorem A. *Let $1<p<q<\infty$ and $0<\gamma<1$. The operator I^{γ} acts boundedly from $L^p_{\mu}(X)$ to $L^q_{\mu}(X)$ if and only if there exists a constant $c>0$ such that*

$$\mu B(x,r)\le cr^{\beta}, \quad \beta=\frac{pq(1-\gamma)}{pq+p-q}$$

for an arbitrary ball $B(x,r)$.

Let now X_j,d_j,μ_j $(j=1,2,\dots,n)$ be the measure quasi-metric spaces. Assume that $\overrightarrow{p}=(p_1,\dots,p_n)$, $1<p_j<\infty$ $(j=1,2,\dots,n)$ and $\overrightarrow{\mu}=(\mu_1,\mu_2,\mu\dots,\mu_n)$. For the measure $f:\prod\limits_{j=1}^{n}X_j\longrightarrow R^1$ we set the mixed-norm Lebesgue spaces $L^{\overrightarrow{p}}_{\overrightarrow{\mu}}\Big(\prod\limits_{j=1}^{n}X_j,\prod\limits_{j=1}^{n}\mu_j\Big)$ with the norm

$$\|f\|_{L^{\overrightarrow{p}}_{\overrightarrow{\mu}}}=\left(\int\limits_{X_1}\cdots\left(\int\limits_{X_n-1}\left(\int\limits_{X_n}|f(x_1,\dots,x_n)|^{p_n}d\mu_n\right)^{\frac{p_{n-1}}{p_n}}d\mu_{n-1}\right)^{\frac{p_{n-2}}{p_{n-1}}}\dots d\mu_1\right)^{\frac{1}{p_1}}.$$

The mixed-norm Lebesgue spaces were introduced and studied in [1].

2020 *Mathematics Subject Classification.* 26A33, 43A15, 42B35.
Key words and phrases. Multiple fractional integrals; Mixed-norm Lebesgue space; Trace inequality; Nonhomogeneous measure space.

Consider the multiple fractional integral defined on the product space $X = X_1 \times \cdots \times X_n$:

$$I^\gamma f(x) = \int\limits_X \frac{f(y_1,\dots,y_n)d\mu_1\cdots d\mu_n}{\prod\limits_{j=1}^n \big(d_j(x_j,y_j)\big)^{1-\gamma j}}, \quad \gamma = (\gamma_1,\dots,\gamma_n).$$

The following statement is true.

Theorem 1. *Let $1 < p_j < q_j < \infty$ $(j = 1,2,\dots,n)$. The operator I^γ is bounded from $L^{\overrightarrow{p}}_{\overrightarrow{\mu}}$ to $L^{\overrightarrow{q}}_{\overrightarrow{\mu}}$ if and only if there exists a positive constant c such that*

$$\mu_j B_j(x_j,r_j) \le c r_j^{\frac{p_j q_j(1-\gamma_j)}{p_j q_j + p_j - q_j}}, \quad j = 1,2,\dots,n \tag{1}$$

for arbitrary balls B_j from X_j.

Theorem 1 says that if the condition (1) fails, then I^γ is unbounded from $L^{\overrightarrow{p}}_{\overrightarrow{\mu}}$ to $L^{\overrightarrow{q}}_{\overrightarrow{\mu}}$. Nevertheless, there exists a weight $\overrightarrow{v} : X \longrightarrow R^1$ such that I^γ is bounded from $L^{\overrightarrow{p}}_{\overrightarrow{\mu}}$ to $L^{\overrightarrow{q}}_{\overrightarrow{\mu}}(\overrightarrow{v})$.

Let us introduce the functions

$$\Omega(x_j) = \sup_{r_j>0} \frac{\mu_j B(x_j,r_j)}{r_j^{\beta_j}},$$

where

$$\beta_j = \frac{p_j q_j(1-\gamma_j)}{p_j q_j + p_j - q_j}. \tag{2}$$

The following statement holds.

Theorem 2. *Let $1 < p_j < q_j < \infty$ $(j = 1,2,\dots,n)$. Then there exists a positive constant $c > 0$ such that for an arbitrary $f \in L^{\overrightarrow{p}}_{\overrightarrow{\mu}}(X)$ we have*

$$\Big\| I^\gamma f(x_1,\dots,x_n) \prod_{j=1}^n \Omega_j^{\frac{\gamma_j-1}{p_j}}(x_j) \Big\|_{L^{\overrightarrow{q}}_{\overrightarrow{\mu}}} \le c\|f\|_{L^{\overrightarrow{p}}_{\overrightarrow{\mu}}}.$$

Let now $\Gamma_i = \{t \in \mathbb{C} :\ t = t(s), 0 \le s \le l\}$ be arbitrary rectifiable simple curves with arc-length measures ν_i $(i = 1,2,\dots,n)$.

Suppose

$$D_j(t_j,r_j) = \Gamma_j \bigcap B_i(t_j,r_j),$$

where

$$B_i(t_i,r_i) = \{z_i \in \mathbb{C} :\ |z_j - t_j| < r_j\}, \quad t_j \in \Gamma_j.$$

Let

$$\Omega_j(t_j) = \sup_{r_j>0} \frac{\nu_j D(t_j,r_j)}{r_j^{\beta_j}},$$

where β_j are defined by (2).

Then for the operator

$$I^\gamma_\Gamma f(t_1,t_2,\dots,t_n) = \int\limits_\Gamma \frac{f(\tau_1,\tau_2,\dots,\tau_n)d\nu_1\dots d\nu_n}{\prod\limits_{j=1}^n |t_j-\tau_j|^{1-\gamma_j}}, \quad \Gamma = \Gamma_1 \times \cdots \times \Gamma_n$$

we have the following assertion.

Theorem 3. *Let $1 < p_j < q_j < \infty$. Then there exists a positive constant c such that for an arbitrary $f \in L^{\overrightarrow{p}}_{\overrightarrow{\nu}}(\Gamma)$ we have*

$$\|I^\gamma_\Gamma f(t_1,t_2,\dots,t_n)\cdot \Omega_j^{\frac{\gamma_j-1}{p_j}}(t_j)\|_{L^{\overrightarrow{q}}_{\overrightarrow{\nu}}(\Gamma)} \le c\|f\|_{L^{\overrightarrow{q}}_{\overrightarrow{\nu}}}.$$

Acknowledgement

This work was supported by the Shota Rustaveli National Scientific Foundation grant (contract number: FR-18-2499).

References

1. A. Benedek, R. Panzone, The space L^P, with mixed norm. *Duke Math. J.* **28** (1961), 301–324.
2. D. E. Edmunds, V. Kokilashvili, A. Meskhi, *Bounded and Compact Integral Operators.* Mathematics and its Applications, 543. Kluwer Academic Publishers, Dordrecht, 2002.
3. V. Kokilashvili, A. Meskhi, Fractional integrals on measure spaces. *Fract. Calc. Appl. Anal.* **4** (2001), no. 1, 1–24.

(Received 08.04.2020)

[1]A. Razmadze Mathematical Institute of I. Javakhishvili of Tbilisi State University, 6 Tamarashvili Str., Tbilisi 0177, Georgia

[2]Department of Mathematics, Georgian Technical University, 77 Kostava Str., Tbilisi 0171, Georgia

E-mail address: vakhtang.kokilashvili@tsu.ge
E-mail address: ts.tsanava@gtu.ge