Transactions of A. Razmadze
Mathematical Institute
Vol. **174** (2020), issue 2, 251–255

ON THE BOUNDEDNESS OF MULTIPLE CAUCHY SINGULAR AND FRACTIONAL INTEGRALS DEFINED ON THE PRODUCT OF RECTIFIABLE CURVES

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Abstract. The present paper deals with the boundedness criteria of multiple Cauchy singular integrals and multiple fractional integrals defined on the product of rectifiable curves in weighted Lebesgue spaces.

1. Introduction

This research is stimulated by the R. Coifman and Y. Meyer's well-known lectures [5] and the paper by St. Semme [18] discussing various problems of non-harmonic Fourier Analysis, among them the problem of boundedness of integral operators generated by the Cauchy singular integrals defined on the sets of intricate geometry. A special interest to this problem is shown by its wide possible applications to the boundary value problems of analytic and harmonic functions, boundary integral equations, PDEs of Mathematical Physics, Mechanics of continuum media, etc.

A complete description of those rectifiable curves, for which Cauchy singular integral operator is bounded in $L^p(\Gamma)$ $(1 < p < \infty)$ has been done by G. David [6]. A modern weight theory for the Cauchy singular integrals in the framework of Muckenhoupt weights is constructed in [4] and [12]. In [13], the boundedness criteria in weighted L^p $(1 < p < \infty)$ spaces was established for multiple Cauchy singular integrals defined on the product of two smooth curves. The necessary and sufficient condition both for curves and weights ensuring the boundedness of the Cauchy singular integral operator in some non-standard Banach function spaces, namely, in weighted grand Lebesgue spaces, can be found in [16] (see also [11, 16, 17]).

The mapping properties of a conjugate function of several variables and the related problems of Fourier trigonometric series were investigated by K. Sokol–Sokolovskii [19]. Further exploration of the problems of multi-dimensional Fourier series and conjugate functions is developed in the papers due to A. Zygmund [23], L. V. Zhizhiashvili (see, e. g., [20–22]), C. L. Fefferman [8], P. L. Lizorkin [18]. To the comprehensive study of multiple singular integrals on the product spaces in weighted setting are devoted the papers by R. Fefferman [10] and E. M. Stein [9]. For the surveys of multiple Fourier series and related integral operators we refer to [1, 2] and [23].

2. Function Spaces. Integral Operators

Let $\Gamma = \{t \in \mathbb{C} : \ t = t(s), \ 0 \le s \le l < \infty\}$ be a simple rectifiable curve with the arc-length measure ν. In the sequel, we set

$$D(t, r) = \Gamma \cap B(t, r), \quad 0 < r < d, \quad d = \operatorname{diam} \Gamma,$$

where

$$B(t, r) = \{z \in \mathbb{C} : \ |z - t| < r\}, \quad t \in \Gamma.$$

Γ is called Carleson (regular) curve if

$$\sup_{\substack{t \in \Gamma \\ 0 < r < \operatorname{diam} \Gamma}} \frac{\nu D(t, r)}{r} < \infty.$$

2020 *Mathematics Subject Classification.* Primary: 42A50, 44A30, 42B28, 42B35.

Key words and phrases. Singular integrals; Carleson curve; Weights; Multiple Cauchy singular integrals; Multiple fractional integrals.

An almost everywhere positive integrable on Γ function is called a weight. By $L^p_w(\Gamma)$, $(1<p<\infty)$ we denote the set of all measurable functions $f:\Gamma\longrightarrow\mathbb{C}$ for which the norm

$$\|f\|_{L^p_w(\Gamma)}=\left(\int\limits_{\Gamma}|f(t)|^p w(t)d\nu\right)^{\frac{1}{p}}$$

is finite.

A weight function w is said to be of Muckenhoupt type class if

$$\sup_{\substack{t\in\Gamma\\0<r<d}}\left(\frac{1}{r}\int\limits_{D(t,r)}w(\tau)d\nu\right)\left(\frac{1}{r}\int\limits_{D(t,r)}w^{1-p'}(\tau)d\nu\right)^{p-1}<\infty,\quad p'=\frac{p}{p-1}.$$

Now let $\Gamma=\Gamma_1\times\Gamma_2$ be the product of two simple rectifiable curves of finite lengths, endowed by a product measure $\nu=\nu_1\times\nu_2$.

Let $1<p<\infty$ and a weight function w be given on Γ. By $L^p_w(\Gamma)$ we denote the set of all ν-measurable functions $f:\Gamma\longrightarrow\mathbb{C}$ for which the norm

$$\|f\|_{L^p_w(\Gamma)}=\left(\int\limits_{\Gamma}|f(t,\tau)|^p w(t,\tau)d\nu\right)^{1/p}$$

is finite.

Along with $L^p(\Gamma)$ we will treat also the weighted Lebesgue spaces with a mixed norm. For the simplicity of our presentation we consider a 2-multiple case.

Let $1<p_1<p_2<\infty$, $\overrightarrow{p}=(p_1,p_2)$, $\overrightarrow{\nu}=(\nu_1,\nu_2)$. By $L^{\overrightarrow{p}}_{\overrightarrow{\nu}}(\Gamma)$ we denote the set of all ν-measurable functions $f:\Gamma\longrightarrow\mathbb{C}$ for which the norm

$$\|f\|_{L^{\overrightarrow{p}}_{\overrightarrow{\nu}}(\Gamma)}=\left(\int\limits_{\Gamma_1}\left(\int\limits_{\Gamma_2}|f(t,\tau)|^{p_2}d\nu_2\right)^{\frac{p_1}{p_2}}d\nu_1\right)^{1/p_1}$$

is finite.

The notion and properties of the mixed-norm Lebesgue spaces were introduced in [3].

The goal of our paper is to discuss the boundedness problem for multiple maximal Cauchy singular integrals and multiple fractional integrals defined on the product $\Gamma=\Gamma_1\times\Gamma_2$.

Let $\Gamma_{\varepsilon\eta}(t,\tau)=(\Gamma_1\backslash D_1(t,\varepsilon))\times((\Gamma_2\backslash D_2(\tau,\eta))$.

The double maximal Cauchy singular integral is defined as

$$S^*_\Gamma f(t,\tau)=\sup_{\substack{0<\varepsilon_1<d_1\\0<\eta<d_2}}\left|\int\limits_{\Gamma_{\varepsilon\eta}}\frac{f(t_0,\tau_0)d\nu_1d\nu_2}{(t-t_0)(\tau-\tau_0)}\right|,\ \ d_i=\operatorname{diam}\Gamma_i.$$

The double fractional integral defined on Γ looks as

$$\mathbb{I}^\gamma f(t,\tau)=\int\limits_{\Gamma}\frac{f(t_0,\tau_0)d\nu_1d\nu_2}{|t-t_0|^{1-\gamma_1}|\tau-\tau_0|^{1-\gamma_2}},$$

$\gamma=(\gamma_1,\gamma_2)$, $0<\gamma_i<1$, $j=1,2$.

In the sequel, we will employ the following class of weight functions:

$$A_p(\Gamma)=\left\{w:\ \sup\frac{1}{r\varrho}\int\limits_{V_{r\rho}(t,\tau)}w(t_0,\tau_0)d\nu\left(\frac{1}{r\rho}\int\limits_{V_{r\rho}(t,\tau)}w^{1-p'}(t_0,\tau_0)d\nu\right)^{p-1}\right\}<\infty$$

where $V_{r\rho}(t,\tau)=D_1(t,\tau)\times D_2(\tau,\rho)$ and the supremum being taken over all $t\in\Gamma_1$, $\tau\in\Gamma_2$ and r, ρ, $0<r<d_1$ and $0<\rho<d_2$.

3. Main Results

Theorem 1. *Let $1 < p < \infty$. The operator S^*_Γ is bounded in $L^p_w(\Gamma)$ if and only if Γ_i are the Carleson curves and $w \in A_p(\Gamma)$.*

The proof of Theorem 1 is based on the following lemmas.

Lemma 2. *Let $1 < p < \infty$ and $w \in A_p(\Gamma)$. Let Γ_i $(i = 1, 2)$ be Carleson curves. Then for arbitrary $f \in L^p_w(\Gamma)$, almost all $(t, \tau) \in \Gamma$ and arbitrary ε and η, the following equality*

$$\int\limits_{\Gamma_1} \frac{d\nu_1}{t(s)-t(s_0)} \Bigg(\int\limits_{\Gamma_2\setminus D(\tau,\eta)} \frac{f(s_0,\sigma_0)d\nu_2}{\tau(\sigma)-\tau(\sigma_0)} \Bigg)$$
$$= \int\limits_{\Gamma_2\setminus D(\tau,\eta)} \frac{1}{\tau(\sigma)-\tau(\sigma_0)} \Bigg(\int\limits_{\Gamma_1} \frac{f(s_0,\sigma_0)d\nu_1}{t(s)-t(s_0)} \Bigg) d\nu_2$$

holds.

Lemma 3. *Let Γ be a Carleson curve, $1 < p < \infty$ and $w \in A_p(\Gamma)$. Then there exists a positive constant b such that for arbitrary $f \in L^p_w(\Gamma)$, arbitrary $\varepsilon > 0$, $t \in \Gamma$ and $\bar{t} \in D\left(t, \frac{\varepsilon}{4}\right)$ the inequality*

$$\left| \frac{f(s_0)d\nu}{t(s)-t(s_0)} \right| \le b \Bigg(\Bigg| \int\limits_{\Gamma} \frac{f(s_0)d\nu}{\bar{t}(s)-t(s_0)} \Bigg|$$
$$+ \Bigg| \int\limits_{\Gamma} \frac{f(s_0)\chi_{D(t,\varepsilon)}(s_0)d\nu}{\bar{t}(s)-t(s_0)} \Bigg| + M_\Gamma f(t) \Bigg)$$

holds.

Here

$$M_\Gamma f(t) = \sup_{0<r<d(\Gamma)} \frac{1}{r} \int\limits_{D(t,r)} |f(\tau)| d\nu.$$

Lemma 4. *Let Γ be a Carleson curve of finite length and let $\varphi \in L(\Gamma)$. Then for arbitrary $\varepsilon > 0$, for all $\bar{t} \in D\left(t, \frac{\varepsilon}{4}\right)$ and almost all $t \in \Gamma$ the inequality*

$$\Bigg| \int\limits_{\Gamma\setminus D(t,\varepsilon)} \left(\frac{1}{t(s)-t(s_0)} - \frac{1}{\bar{t}-t(s_0)} \right) \varphi(s_0) d\nu \Bigg| \le c M_\Gamma \varphi(t)$$

holds with a positive constant c, independent of φ and t.

Lemma 5. *Let $1 < p < \theta < \infty$ and let Γ be a simple Carleson curve. Suppose $w \in A_p(\Gamma)$. Assume that (Y, μ) is some measure space. Then for arbitrary measurable $f : \Gamma \times Y \longrightarrow \mathbb{C}$ the following inequality*

$$\Bigg(\int\limits_{\Gamma} \Bigg(\int\limits_{Y} M^\theta_\Gamma(f)(t,y) d\mu_y \Bigg)^{p/\theta} w(t) d\nu \Bigg)^{1/p}$$
$$\le c \Bigg(\int\limits_{\Gamma} \Bigg(\int\limits_{Y} |f(t,y)|^\theta d\mu_y \Bigg)^{p/\theta} w(t) d\nu \Bigg)^{\frac{1}{p}}$$

holds with a constant $c > 0$, independent of f.

Theorem 6. *Let $1 < p_i < \infty$ $(i = 1, 2)$. The operator S_Γ is bounded in $L^{\vec{p}}_{\vec{w}}(\Gamma)$ if and only if Γ_i $(i = 1, 2)$ are Carleson curves and $w_i \in A_{p_i}(\Gamma)$.*

In the sequel, we will discuss a description of those rectifiable curves for which the operator $\mathbb{I}^\gamma_\Gamma$ is bounded from $L^{\vec{p}}(\Gamma)$ to $L^{\vec{q}}(\Gamma)$, $1 < p_i < q_i < \infty$, $(i = 1, 2)$.

Theorem 7. *Let $1 < p_j < q_j < \infty$, $0 < \gamma_j < 1$ $(j = 1, 2)$. The operator $\mathbb{I}_\Gamma^\gamma$ is bounded from $L_{\vec{\nu}}^{\vec{p}}(\Gamma)$ to $L_{\vec{\nu}}^{\vec{q}}(\Gamma)$ if and only if*

$$\sup_{\substack{t_j r_j \\ j=1,2}} \nu_j\left(D_j(t_j, r_j)\right) r_j^{-\frac{p_j q_j (1-\gamma_j)}{p_j q_j + p_j - q_j}} < \infty.$$

From Theorem 7 follows

Theorem 8 (Sobolev type statement)**.** *Let $1 < p_j < \frac{1}{\gamma_j}$ and let $\frac{1}{q_j} = \frac{1}{p_j} - \gamma_j$. Then the operator $\mathbb{I}_\Gamma^\gamma$ is bounded from $L^{\vec{p}}(\Gamma)$ to $L^{\vec{q}}(\Gamma)$ if and only if Γ_j $(j = 1, 2)$ are Carleson curves.*

Acknowledgement

This work was supported by the Shota Rustaveli National Scientific Foundation grant (contract number: FR-18-2499).

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(Received 18.03.2020)

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