Transactions of A. Razmadze
Mathematical Institute
Vol. **174** (2020), issue 2, 249–250

ON DOUBLE FOURIER SERIES WITH RESPECT TO THE CLASSICAL REARRANGEMENTS OF THE WALSH–PALEY SYSTEM

ROSTOM GETSADZE

Abstract. The following theorem is established: there exists a continuous function on $[0,1]^2$ with a certain smoothness, whose double Fourier-Walsh series diverges by rectangles on a set of positive measure. Similar theorem is true also for the double Walsh–Kaczmarz system.

1. Introduction

There are two classical rearrangements of the Walsh–Paley system: (a) the Walsh system and (b) the Walsh–Kaczmarz system. It is well-known (see [3, 4]) these systems are systems of convergence. The system of Rademacher functions $\{r_n(x)\}_{n=0}^{\infty}$ on $[0,1)$ is defined as follows. Set

$$r_0(x) = \begin{cases} 1 & \text{for } 0 \le x < \frac{1}{2}, \\ -1 & \text{for } \frac{1}{2} \le x < 1. \end{cases}$$

We extend the function $r_0(x)$ on $(-\infty, \infty)$ with period 1. For $n \ge 1$, we set

$$r_n(x) = r_0(2^n x).$$

For each $k \in N = \{0, 1, 2, \dots\}$, we introduce a function $\alpha_k : \ [0,1) \to \{0,1\}$ defined by the dyadic expansion of x

$$x = \sum_{k=0}^{\infty} \frac{\alpha_k(x)}{2^{k+1}}.$$

If x is a dyadic rational, then we suppose that its dyadic expansion contains infinitely many zeros.

The Walsh–Paley system of functions $\{W_n(x)\}_{n=0}^{\infty}$ on $[0,1)$ is defined as follows. Set $W_0(x) = 1$ for all $x \in [0,1)$. For $n \ge 1$, we consider the dyadic representation $n = 2^{m_1} + 2^{m_2} + \dots + 2^{m_q}$, $(n \ge 1$, $m_1 > m_2 \dots > m_q \ge 0)$ and set

$$W_n(x) = r_{m_1}(x) r_{m_2}(x) \dots r_{m_q}(x) \qquad x \in [0,1).$$

The modulus of continuity $\omega(F;\delta)$ of a continuous function F on $[0,1]^2$ is defined by

$$\omega(F;\delta) = \sup_{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \le \delta} \{|F(x_1,y_1) - F(x_2,y_2)|,\ (x_1,y_1),\ (x_2,y_2) \in [0,1]^2\}.$$

Recently [2], we have proved the following

Theorem 1. *There exists a continuous function F on $[0,1]^2$ such that*

$$\omega(F;\delta) = O\left(\frac{1}{\sqrt{\log_2 \frac{1}{\delta}}}\right), \qquad \delta \to 0+,$$

and the Fourier series of F with respect to the double Walsh–Paley system $\{W_m(x)W_n(y)\}_{m,n=0}^{\infty}$ diverges on a set of positive measure by rectangles.

2020 *Mathematics Subject Classification.* 42C10.

Key words and phrases. Walsh system; Rearrangements; Double Fourier series.

The Walsh system $\{\varphi_m(x)\}_{m=0}^{\infty}$ was introduced by Walsh (see, e.g., [4]) and defined as follows:

$$\varphi_0(x) = 1, \quad \varphi_1(x) = (-1)^{\alpha_0(x)}, \quad \varphi_{2^n}(x) = (-1)^{\alpha_{n-1}(x)+\alpha_n(x)},$$
$$\varphi_{2^n+k}(x) = \varphi_{2^n}(x)\varphi_k(x), \quad k = 0, 1, \dots, 2^n - 1; \quad n = 0, 1, \dots,$$

To define the Walsh–Kaczmarz system $\{h_m(x)\}_{m=1}^{\infty}$, we first introduce an auxiliary system of functions

$$\psi_{n,i}(x) = r_{n-j_1-1}(x)r_{n-j_2-1}(x)\dots r_{n-j_p-1}(x), \qquad x \in [0,1),$$

where $n, i \in N$, $2 \le i \le 2^n$, $n \ge 1$ and

$$i - 1 = 2^{j_1} + 2^{j_2} + \cdots + 2^{j_p},$$

with $j_1 > j_2 > \cdots > j_p \ge 0$, is the dyadic expansion of the integer $i - 1$.

For $i = 1$ and $n \ge 1$, we set

$$\psi_{n,1}(x) = 1, \qquad x \in [0,1).$$

The Walsh–Kaczmarz system $\{h_m(x)\}_{m=1}^{\infty}$ on $[0,1)$ is defined as follows:

$$h_1(x) = 1 \quad \text{and} \quad h_2(x) = r_0(x), \quad x \in [0,1).$$

For $m = 2^n + i$, $n \ge 1$, $1 \le i \le 2^n$, we set

$$h_m(x) = h_{2^n+i}(x) = \psi_{n,i}(x)r_n(x), \qquad x \in [0,1).$$

We establish the following two theorems.

Theorem 2. *There exists a continuous function G on $[0,1]^2$ such that*

$$\omega(G;\delta) = O\left(\frac{1}{\sqrt{\log_2 \frac{1}{\delta}}}\right), \quad \delta \to 0+,$$

and the Fourier series of G with respect to the double Walsh system $\{\varphi_m(x)\varphi_n(y)\}_{m,n=0}^{\infty}$ diverges on a set of positive measure by rectangles.

Theorem 3. *There exists a continuous function H on $[0,1]^2$ such that*

$$\omega(H;\delta) = O\left(\frac{1}{\sqrt{\log_2 \frac{1}{\delta}}}\right), \quad \delta \to 0+,$$

and the Fourier series of H with respect to the double Walsh–Kaczmarz system $\{h_m(x)h_n(y)\}_{m,n=1}^{\infty}$ diverges on a set of positive measure by rectangles.

A weaker result than Theorem 3 has been proved by us in [1].

Acknowledgement

The presented work was supported by grant No. DI-18-118 of the Shota Rustaveli National Science Foundation of Georgia.

References

1. R. Getsadze, On the divergence of multiple Fourier series with respect to the Walsh–Kaczmarcz system. (Russian) *Problems of Theory of Functions and Functional Analysis* **4** (1988), 36–42.
2. R. Getsadze, On the Divergence of Double Fourier–Walsh–Paley Series of Continuous Functions. *Acta Sci. Math. (Szeged)* **86** (2020), 287–302.
3. W. Sang Young, On rearrangements of Walsh–Fourier series and Hardy–Littlewood type maximal inequalities. *Bulletin of the American Mathematical Society* **80** (1974), no. 3, 490–494.
4. F. Šipp, Certain rearrangements of series in the Walsh system. (Russian) *Mat. Zametki* **18** (1975), no. 2, 193–201.

(Received 11.12.2019)

Department of Mathematics, Uppsala University, Box 480, 751 06 UPPSALA, Sweden
E-mail address: rostom.getsadze@math.uu.se; rostom.getsadze@telia.com