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ASYMPTOTIC ANALYSIS OF AN OVER-REFLECTION EQUATION IN MAGNETIZED PLASMA

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Abstract. The equation describing the over-reflection of the slow magneto-sonic waves in plasma with background uniform shear flow is derived and analyzed in detail both analytically and numerically. Using the methods of asymptotic analysis, analytical expressions for reflection and transmission coefficients of the waves are obtained for relatively small shear rates.

1. Introduction

The aim of the present paper is to study mathematical aspects of the over-reflection phenomenon [5] in magnetized plasmas which is thought to be one of the possible sources of large scale perturbations in astrophysical and laboratory plasmas [3, 5]. Towards this end, we choose the simplest system, the two-dimensional magneto-hydrodynamical shear flow with domination of plasma energy (the so-called high beta-plasma). This will allow us to study the over-reflection phenomenon in a pure form; in more general cases, the phenomenon is accompanied by a mutual transformation of different wave modes, as well.

Consider the two-dimensional compressible unbounded shear flow along the x-axis with the constant shear parameter, i.e., with the velocity vector $\mathbf{U}_0(Ay, 0)$. We assume that the density ρ_0 and the pressure P_0 are uniform, and the magnetic field $\mathbf{B}_0$ is directed along the streamlines. Assuming $\rho_1,\ u_x,\ u_y,\ b_x, b_y$ are the perturbations of density, velocity and magnetic field, respectively, the linearized equations governing the evolution of the spatial Fourier harmonics of dimensionless perturbations in the uniform shear flow are [3]

$$\dot{d} = v_x + K(T)v_y, \tag{1}$$

$$\dot{v}_x = -Sv_y - \beta d, \tag{2}$$

$$\dot{v}_y = -\beta K(T)d + [1 + K(T)]\, b, \tag{3}$$

$$\dot{b} = -v_y, \tag{4}$$

where $S = A/V_A k_x$ is the dimensionless shear rate, k_x and k_y are parallel and perpendicular wave numbers, respectively, V_A is the Alfvén velocity, $T = V_A k_x t$ is the dimensionless time, $\beta = c_s^2/V_A^2$ is beta-plasma, c_s is the sound speed, $K(T) = k_y/k_x - ST$ is the dimensionless perpendicular wave number, and $d(\mathbf{k}) = i\rho_1(\mathbf{k})/\rho_0,\ b(\mathbf{k}) = ib_y(\mathbf{k})/B_0,\ \mathbf{v}(\mathbf{k}) = \mathbf{u}(\mathbf{k})/V_A$ are dimensionless perturbations of the density, perpendicular component of the magnetic field and the velocity, respectively. In the above equations, the over-dot denotes a derivative with respect to the dimensionless time T.

Equations (1)–(4), along with the over-reflection phenomenon, describe various dynamical effects of the linear perturbations, such as coupling and mutual transformation of different plasma waves [3]. To derive the equations that describe the over-reflection in a pure form, one has to consider evolution of low frequency perturbations in the weakly compressible medium that corresponds to the dynamics of the so-called slow magneto-sonic waves in low beta-plasmas (plasma beta $\beta \gg 1$). Physically this means that in this case we may neglect compressibility of the low frequency waves. From the mathematical point of view, in the case under consideration, equations (3) and (4) decouple and describe the evolution of low frequency perturbations in the shear flow. Then, introducing the new

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variable $\Psi = b[1 + K(T)^2]^{1/2}$, these equations can readily be reduced to the following second-order ordinary differential equation:

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}T^2} + \left[1 - \frac{S}{\left(1 + K(T)^2\right)^2}\right]\Psi = 0. \tag{5}$$

In the next section, we present detailed analysis of this equation. We show that in the vicinity of critical points (where the expression in the squire brackets is undefined or becomes zero) it can be reduced to the specific type of the Bessel equation, thus allowing us to derive analytical expressions for the reflection and transmission coefficients of the wave perturbations.

2. Asymptotic Analysis and Numerical Study

First of all, let us note that in the limit $|K(T)| \gg S$, the Liouville-Green asymptotical solution [2, 6](known also from physical literature [4] as the Wentzel-Kramers-Brillouin or quasi-classical approximation) is applicable, and we have

$$\Psi_\pm = \frac{C_\pm}{\sqrt{\omega(T)}} e^{\pm i \int \omega(T)\mathrm{d}T}, \tag{6}$$

where

$$\omega(T) = \sqrt{1 - \frac{S}{\left[1 + K(T)^2\right]^2}},$$

and $C_\pm$ are some constants determined by the initial conditions.

It is well known [4] that $\Psi_\pm$ correspond to the waves, propagating along and backward with respect to the x-axis, respectively.

If the dimensionless shear rate is high enough, then for the time period, when $|K(T)| \sim S$, the asymptotic solutions (6) are not valid, i.e., physically speaking, evolution of the waves is not adiabatic. Suppose that at the initial moment of time $T = 0$ we have $K(0) \gg S$. When T increases, $K(T)$ decreases and passes through the interval of non-adiabatic evolution, where the condition $|K(T)| \gg S$ is not valid. From the mathematical point of view, this means that the asymptotic amplitudes $C_\pm$ in this interval do not remain constant. On the other hand, when T tends to infinity, the condition $|K(T)| \gg S$ becomes valid again and equation (5) has asymptotic solutions (6) with different amplitudes $C_\pm(\infty)$. Assuming that initially $C_+(0) = 1$ and $C_-(0) = 0$, the reflection (R) and transmission (G) coefficients of the wave can be defined in the usual manner [3–5],

$$R = \left|\frac{C_-(\infty)}{C_+(0)}\right|^2 \text{ and } G = \left|\frac{C_-(\infty)}{C_+(0)}\right|^2. \tag{7}$$

The reflection and transmission coefficients are not independent and the conservation of the wave action implies [4] $1 + R = G$.

First, let us consider the limit $S \ll 1$. In this case, the exact asymptotic solution of equation (5) can be derived. Introducing new variable $\tau = K(T)$, equation (5) can be rewritten as

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}\tau^2} + \left[\frac{1}{S^2} - \frac{1}{\left(1 + \tau^2\right)^2}\right]\Psi = 0. \tag{8}$$

From the mathematical point of view, the non-adiabatic evolution (i.e., failure of the solutions (6)) is related to the critical (singular and turning) points of equation (8) [2, 6]. In the case of equation (8), there exist two second-order regular singular points $\tau_{12} = i$ and their complex conjugate, and also four turning points $\tau_{3-6} = \pm i(1 + S)^{1/2}$. In the above-considered limit $S \ll 1$, the turning points tend to coincide with the regular singular points.

Assume that initially there exists only a wave with a positive phase velocity, i.e., $C_-(0) = 0$. To derive the reflection coefficient, one has to consider equation (8) in the complex τ-plane [6] along the

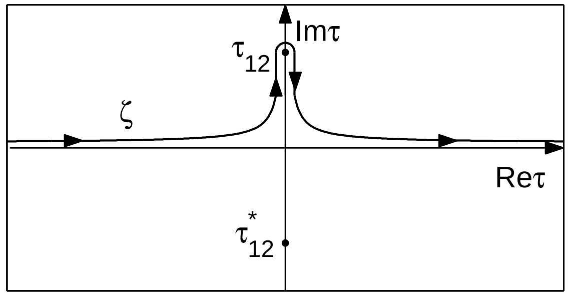

FIGURE 1. The path ζ of integration in the complex τ plane around critical point τ_{12}.

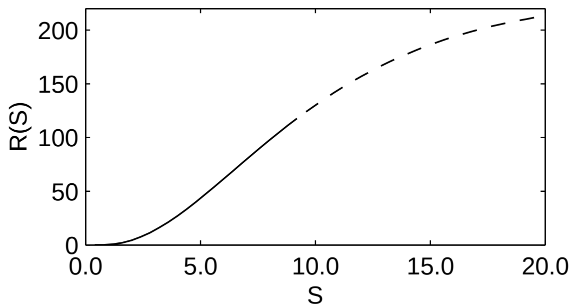

FIGURE 2. The reflection coefficient R as a function of the normalized shear parameter S.

path ζ presented in Figure 1. The evolution is adiabatic everywhere except a small vicinity of the singular point τ_{12}, where equation (8) reduces to the following equation:

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}\tau^2} + \left[\frac{1}{S^2} + \frac{1}{(\tau - i)^2}\right]\Psi = 0.$$

This equation represents specific case of the Bessel equation and, as is known [1], its solution can be expressed in terms of the zeroth order Hankel functions

$$\Psi_{1,2}(\tau) = \frac{1}{(\tau - i)^{1/2}} H^{(0)}_{1,2}\left(\frac{\tau - i}{S}\right).$$

Comparing asymptotic expansions of the Hankel functions [1] with equations (6), it can be readily seen that far away from the singular point, for $\tau = 0$, $\Psi_{1,2}$ coincide with $\Psi_{\pm}$. Then, the well-known analytical continuation formulas for the Hankel functions [1], together with the definition of the reflection coefficient (7), give

$$R = e^{-4/S}.$$

This equation represents the exact asymptotic solution for the reflection coefficient in the limit $S \to 0$. As it can be seen from this expression, the reflection coefficient in the limit under consideration is exponentially small with respect to the parameter $1/S$, in accordance with the solutions of similar equations in quantum mechanics [4].

The influence of the velocity shear becomes much more significant if the dimensionless shear parameter S is of the order of unity or higher. The asymptotic mathematical method of the phase integrals [2] is not applicable in this case and, hence, no analytical expression for the reflection coefficient can be obtained and the problem can be solved only numerically.

The dependence of the reflection coefficient R on the normalized shear parameter S obtained by numerical solution of equation (8) is presented in Figure 2. The initial conditions are chosen as the Liouville-Green asymptotical solutions (6). According to the numerical study, the amplitude of the reflected wave exceeds that of the incident wave (i.e., $R > 1$) if $S > 1.4$. This condition indicates that the phenomenon of the over-reflection can take place in plasma with high beta-parameter ($V_A \to 0$) even for small values of the shear parameter A. The amplification of the energy density of perturbations due to the over-reflection is always finite, but it may become arbitrarily large under due increase of the shear parameter.

3. Conclusion

In the presented paper, we have studied the phenomenon of over-reflection in the two-dimensional magneto-hydrodynamical shear flow in high beta-plasma. The equation describing the over-reflection of the slow magneto-sonic waves has been derived and analyzed both analytically and numerically. Using methods of asymptotic analysis analytical expressions for reflection coefficient of the waves are derived for small dimensionless shear parameter. It was shown that in high beta plasmas the over-reflection can take place even for relatively small shear rates.

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