Transactions of A. Razmadze
Mathematical Institute
Vol. **174** (2020), issue 1, 61–69

THE DIRECTED GRAPHS OF SOME FUNCTIONS

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Dedicated to Professor Alexander Kharazishvili on the occasion of his 70th birthday

Abstract. A description of the digraphs associated with Hamel's coordinate functions and with some elementary functions is given. Some cardinal invariants of the corresponding mono-unary algebras are found. It is also proved that the digraph of the function tan is an universal graph for the class of digraphs of functions of a certain type.

1. Introduction

Given a function $f:\mathbb{R}\to\mathbb{R}$, it can be studied from different points of view including analysis, geometry, algebra, graph theory, combinatorics, etc. Many well-known functions, for example, functions constructed by Hamel [2] and having huge values in Linear algebra, Geometry, Functional analysis and Measure theory (see [1], [3], [5], [6], [9–11], [13]), or elementary functions are not sufficiently studied, especially, in algebra and graph theory. Every function f naturally generates the mono-unary algebra and the corresponding functional digraph [12]. In this article, we consider the mono-unary algebras and the corresponding digraphs for coordinate functions of Hamel's basis and for basic elementary functions. A description of the connected components of the digraphs of the above functions is given; the cardinalities of automorphism groups of such digraphs and the cardinality of the set of all mono-unary algebras, isomorphic to a given one, are established also. It is proved that for every coordinate function of Hamel's basis $f:\mathbb{R}\to\mathbb{R}$ there exists an effectively constructed (without the axiom of choice) simple function $g:\mathbb{R}\to\mathbb{R}$ such that the algebras $(\mathbb{R},f)$ and $(\mathbb{R},f)$ are isomorphic. It is also proved that for any basic elementary functions, except for a constant, there are 2^c many functions from $\mathbb{R}$ to $\mathbb{R}$ such that the mono-unary algebras (graphs) generated by any of them are isomorphic to the mono-unary algebras (graphs) generated by the function f. Obviously, most of these functions (in view of cardinality) are discontinuous.

2. Preliminary

We will use the standard algebraic, set-theoretic and graph theoretic notations. A partial mono-unary algebra is a pair (A,f), where A is a non-empty set and f is a map $f:B\to A$ for some subset $B\subset A$. If $B=A$, then the pair (A,f) is called a mono-unary algebra. For each partial mono-unary algebra, the corresponding digraph G_f is determined as follows:

$$G_f=(A,\{(x,f(x)):x\in \mathrm{Dom}(f)\}).$$

If (A,f) is a partial mono-unary algebra, we define a relation E on A in the following way: xEy, if and only if for some natural numbers n and m the equality $f^n(x)=f^m(y)$ holds, where

$$f^0(x)=x, f^{n+1}(x)=f(f^n(x)),\quad \text{for}\quad n\in\omega.$$

Then E is an equivalence relation on A, and we call E-equivalence classes of algebra A with the induced operation, connected components of the partial algebra (A,f). If $\{A_i:i\in I\}$ is the family of all E-equivalence classes, then we have $A=\underset{i\in I}{\cup}A_i$ and the family $\{(A_i,f_i):i\in I\}$, where $f_i=f_{|A_i}$ is called the injective family of all connected components of partial mono-unary algebra (A,f). The

2010 *Mathematics Subject Classification.* 05C63, 08A60, 26A09, 97I70.
Key words and phrases. Functional digraph; Hamel's coordinate function; Basic elementary functions; Cardinal invariant.

corresponding digraphs of the subalgebras $\{A_i, f_i\} : i \in I$ are connected components of the digraph G_f.

The ordinal ω, is the set of all naturals, i.e., of all nonnegative integers, at the same time, ω denotes the cardinality of the set of natural numbers. The cardinality of the continuum is denoted by $\mathbf{c}$. The symbols: $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ denote the sets of integers, of rational numbers and the set of reals, respectively. $|X|$ denotes the cardinality of a set X.

(a) If $n \geq 1$ is the smallest integer such that $f^n(x) = x$ for some element x, the following set $\{(x, f(x)), (f(x), f^2(x)), \ldots, (f^{n-1}(x), x)\}$ is called an n-cycle.

(b) If an injective family $\{a_n : n \in \omega\}$ of elements of A such that $f(a_{n+1}) = a_n(f(a_n) = a_{n+1}$ respectively), $n \in \omega$, then we say that the algebra (A, f) contains an ω-chain (ω^*-chain, respectively).

(c) If there is an injective family $\{a_q : q \in \mathbb{Z}\}$ of elements of A such that $f(a_q) = a_{q+1}, q \in \mathbb{Z}$, then we say that the algebra (A, f) contains an $\omega^* + \omega$ -chain.

(d) Every graph, which is considered in this article is a digraph of a partial function, respectively, a root tree is called a tree, in which each vertex is oriented in the direction to the root.

(e) The points, whose in-degree and out-degree is 0, are called isolated points.

3. The Digraphs of Hamel Coordinate Functions

Definition 3.1. Let b be an element of some Hamel basis of the vector space $\mathbb{R}(\mathbb{Q})$. The coordinate function f of Hamel basis is defined as follows: for any $x \in \mathbb{R}$, $f(x)$ is the b-th coordinate of the vector x [1]; The set of all coordinate functions of some Hamel basis of the vector spaces $\mathbb{R}(\mathbb{Q})$ is denoted by H.

The following lemma is trivial to prove.

Lemma 3.2. *If $f \in H$, then the following hold:*

(a) $f(0) = 0$,

(b) *for each $r \in \mathbb{Q}$ the set $f^{-1}(\{r\})$ has the cardinality* $\mathbf{c}$,

(c) *for each $x \in \mathbb{R} \backslash \mathbb{Q}$ we have $f^{-1}(\{x\}) = \varnothing$,*

(d) $|H| = 2^c$.

Definition 3.3.

3.1.1. A root tree of cardinality $\mathbf{c}$ whose root is an incident to any vertex, except the root, is called a tree of type H_0.

3.1.2. If T is a tree of type H_0 and r_0 is its root, then the digraph $T \cup (r_0, r_0)$ is called a graph of type H_1 (see Figure 1).

3.1.3. Let $(T_i)_{i \in \omega}$ be a family of disjoint digraphs of type H_0 and r_i be a root of the tree T_i for each $i \in \omega$, then the following graph $(\cup T_i) \cup \{(r_i, r_0) : i \in \omega\}$ is called a graph of type H_2 (see Figure 2).

3.1.4. Let $(T_k)_{k \in \mathbb{Z}}$ be a family of disjoint graphs of type H_0 and r_k be a root of the tree T_k for each $k \in \mathbb{Z}$, then the following digraph $(\cup_{k \in \mathbb{Z}} T_k) \cup \{(r_k, r_{k+1}) : k \in \mathbb{Z}\}$ is called a graph of type H_3 (see Figure 3).

Remark 3.4. It is easy to verify that the graphs of the same types are pairwise isomorphic, and those of different types are pairwise non-isomorphic.

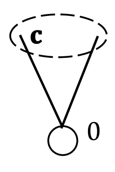

Figure 1

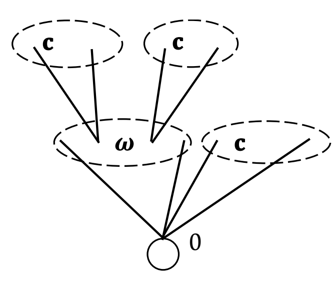

Figure 2

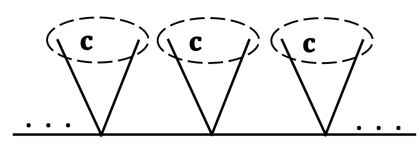

Figure 3

Theorem 3.5. *If $f \in H$, then the following hold:*

(a) *if $f(1) = 0$, then G_f is the graph of type H_2;*

(b) *if $f(1) = 1$, then the graph G_f consists of infinitely countably many connected components of type H_1;*

(c) *if $f(1) \in \mathbb{Q} \backslash \{0; 1\}$, then the graph G_f consists of countably infinitely many connected components, among them, only one component is of type H_1, all other components are of types H_3.*

Proof. (a) It is clear that $f(1) = 0$ iff for any $r \in \mathbb{Q}$ we have $f(r) = 0$. Therefore, from Lemma 3.2 it follows that if $f(1) = 0$, then the digraph G_f of the function f will be of type H_2.

(b) If $f(1) = 1$, then we have $f(r) = r$, $r \in \mathbb{Q}$. Let $\{r_i : i \in \mathbf{Z}\}$ be an injective family of all rational numbers. It is obvious that for each $i \in \mathbb{Z}$ the cardinality of the set $f^{-1}(r_i)$ is $\mathbf{c}$ and the digraph $(f^{-1}(r_i), \{(x, r_i) : x \in f^{-1}(r_i)\})$ is a connected component of type H_1 of the digraph G_f.

(c) Let $f(1) = r \in \mathbb{Q} \backslash \{0; 1\}$. If $r_0 \in \mathbb{Q} \backslash \{0\}$, then for each $k \in \mathbb{Z}$ we have $f(r^k \cdot r_0) = r^{k+1} \cdot r_0$. Therefore $\{(r^k \cdot r_0, r^{k+1} \cdot r_0) : k \in \mathbb{Z}\}$ is an $\omega^* + \omega$ chain in the digraph G_f. If $r_1 \neq r_2$, then $f(r_1) \neq f(r_2)$. Thus, for every nonzero rational number q, the set $f^{-1}(\{q\})$ contains a unique rational number, and the graph G_f contains a unique loop $(0, 0)$. Consequently, each nonzero rational number forms a component of type H_3. It is easy to prove that there are infinitely many different components of type H_3. In addition to the connected components of type H_3, the graph G_f will contain a single-connected component of type H_1 corresponding to the number 0. □

Corollary 3.6. *The maximal family of pairwise non-isomorphic mono-unary algebras $(\mathbb{R}, f)$, where f is an element of H, consists of 3 elements.*

Corollary 3.7. *For $f \in H$, the automorphisms group of mono-unary algebra $(\mathbb{R}, f)$ has the cardinality $2^{\mathbf{c}}$.*

Theorem 3.8. *For each $f \in H$, there exists an effectively (without the axiom of choice) constructed function $g : \mathbb{R} \to \mathbb{R}$ such that the digraphs G_g and G_f are isomorphic.*

Proof. (a) If $f(1) = 0$, consider the function defined as follows:

$$g_a(x) = \begin{cases} 0, & \text{if} \quad x \in \mathbb{Z} \quad \text{or} \qquad -1 < x < 1, \\ n, & \text{if} \quad n < x < n + 1, \qquad n \in \omega \backslash \{0\}, \\ -n, & \text{if} \quad -n - 1 < x < -n, \quad n \in \omega \backslash \{0\}. \end{cases}$$

It is easy to prove that in this case the digraphs G_f and G_{g_a} are isomorphic.

(b) If $f(1) = 1$, consider the function defined as follows:

$$g_b(x) = \begin{cases} n, & \text{if} \quad n \le x < n + 1, \qquad n \in \omega \backslash \{0\}, \\ 0, & \text{if} \quad -1 < x < 1 \\ -n, & \text{if} \quad -n - 1 < x \le -n, \quad n \in \omega \backslash \{0\}. \end{cases}$$

It is easy to prove that in this case the digraphs G_f and G_{g_b} are isomorphic.

(c) Let $f(1) = r$, for some $r \in Q \backslash \{0; 1\}$.

Then for any prime number p, define the function $g_p(x)$ by the following equality:

$$g_p(x) = \begin{cases} p^{k+1}, & \text{if} \quad x \in (p^k - 1; p^k], \qquad \text{for} \quad k \in \omega \backslash \{0\} \\ -p^k, & \text{if} \quad x \in [-p^{k+1}; -p^{k+1} + 1), \quad \text{for} \quad k \in \omega \backslash \{0\}). \end{cases}$$

Evidently, for any prime number p, the digraph G_{g_p} is a digraph of type H_3.

If we now combine the functions g_p for all prime numbers and consider a function g_c defined as follows:

$$g_c(x) = \begin{cases} g_p(x), & \text{if} \quad x \in \mathrm{Dom}(g_p), \qquad \text{for some prime number } p, \\ 0, & \qquad \text{for all other values of } x \in \mathbb{R}, \end{cases}$$

then it is easy to verify that in this case the digraphs G_f and G_{g_c} are isomorphic. □

Remark 3.9. Despite the fact that the functions g_a, g_b and g_c are constructed effectively, the proof of the existence of the corresponding isomorphisms requires a countable form of the Axiom of Choice.

4. The Digraph of the Algebra $(\mathbb{R}, \sin)$

Definition 4.1.

4.1.1. A countably infinite root tree in which each vertex, except the root, is connected to the root, is called a tree of type S_0 (Figure 4).

4.1.2. If $\mathbf{T}$ is a tree of type S_0 and r_0 is its root, then the digraph $\mathbf{T} \cup \{(r_0, r_0)\}$ is called a digraph of type S_1 (Figure 5).

4.1.3. Let $(\mathbf{T}_i)_{i\in\omega}$ be a family of disjoint digraphs of types S_0 and r_i be a root of the tree $\mathbf{T}_i$, for each $i \in \omega$, the following digraph $\cup \{\mathbf{T}_i : i \in \omega\} \cup \{(r_i, r_{i+1}) : i \in \omega\}$ is called a graph of type S_2 (Figure 6).

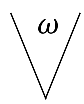

Figure 4

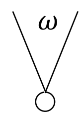

Figure 5

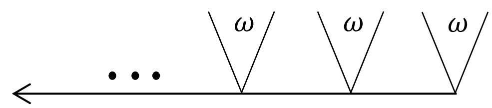

Figure 6

Theorem 4.2. *The set of all connected components of mono-unary algebra* $(\mathbb{R}, \sin)$ *has the cardinality of the continuum, among them one component is a graph of type* S_1 *and all the other components are the graphs of type* S_2.

Proof. It is obvious that the in-degree of each vertex of the graph $G_{\sin}$ is 0 or ω. It is also obvious that the component containing the number 0, is a component of type S_1.

If $x \in \mathbb{R}\backslash\{\pi k : k \in \mathbb{Z}\}$, then it is clear that

$$|x| > \big|\sin|x|\big| > \big|\sin\big|\sin|x|\big|\big| > \cdots$$

holds. Therefore, there aren't any cycles in the component, which doesn't contain 0. It is clear that such components contain an ω^*-chain. From the trivial inequality

$$|x - y| > |\sin x - \sin y|, \quad (x \neq y)$$

it follows that the sequence $x, \arcsin x, \arcsin^2 x, \dots$ is always finite. Therefore, there aren't any ω-chains in the digraph $G_{\sin}$.

Obviously, each connected component of the digraph $G_{\sin}$ contains a countably infinite set of vertices, hence the set of all connected components of $G_{\sin}$ has the cardinality of the continuum. □

5. The digraph of the algebra $(\mathbb{R}, \cos)$

Definition 5.1.

5.1.1. If $\mathbf{T}_0$ and $\mathbf{T}_1$ are two disjoint trees of types S_0, and r_0 and r_1 are their roots, then the digraph $\mathbf{T}_0 \cup \mathbf{T}_1 \cup \{(r_0, r_1)\}$ is a tree of type C_0 (Figure 7).

5.1.2. If $\mathbf{T}$ is a tree of type C_0 and r_0 is its root, then $\mathbf{T} \cup \{(r_0, r_0)\}$ is a digraph of type C_1 (Figure 8).

5.1.3. Let $(\mathbf{T}_i)_{i\in\omega}$ be a family of disjoint root trees of type C_0 and r_i be a root of the tree $\mathbf{T}_i$, $i \in \omega$, the following digraph $\cup \{\mathbf{T}_i : i \in \omega\} \cup \{(r_i, r_{i+1}) : i \in \omega\}$ is called a graph of type C_2 (Figure 9).

5.1.4. Let $\mathbf{T}$ be a tree of type S_0 whose root is r, $(\mathbf{T}_i)_{i\in\omega}$ be a family of disjoint root trees of type C_0 and let r_i be the root of a tree $\mathbf{T}_i$, $i \in \omega$. Then the graph $\mathbf{T} \cup \mathbf{T}_i \cup \{(r_i, r_{i+1}) : i \in \omega\} \cup \{(r, r_0)\}$ is called the graph of type C_3 (Figure 10).

Theorem 5.2. *The set of all connected components of mono-unary algebra* $(\mathbb{R}, \cos)$ *has the cardinality of the continuum, among them there is one component of type* C_1*, one component of type* C_2 *and all others are of type* C_3.

Proof. Let d be a fixed point of the function cos. It is obvious that the in-degree of each vertex of the digraph of cos is 0 or ω. It is also obvious that the component containing the fixed point d, is a component of type C_1, in this component only two vertices d and $-d$ will have in-degrees

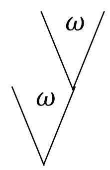

FIGURE 7

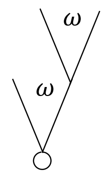

FIGURE 8

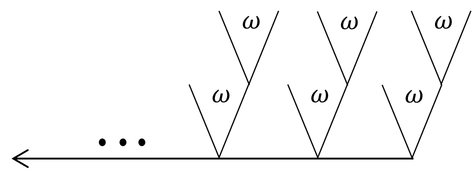

FIGURE 9

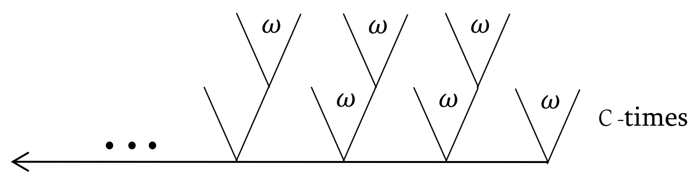

FIGURE 10

equal to ω. From the fact that the function cos has only one fixed point, it follows that the graph has only one component, which has a loop. For any $x \in \mathbb{R}\backslash\{d\}$, the injective sequence of iterations $x, \cos(x), \cos(\cos(x)), \ldots$ converges to d, so the digraph $G_{\cos}$ does not contain cycles, except for a single loop (d, d), and contains an ω^* chain. It follows from the parity of function cos that the other connected components are of type C_2 or of type C_3, clearly that type C_2 will have a single component containing the number 0. Evidently, that the set of all such components has the cardinality $\mathbf{c}$. □

Remark 5.3. It should be remarked that the digraphs $G_{\sin}$ and $G_{\cos}$ are not isomorphic, they don't even have components, isomorphic to each other.

6. THE DIGRAPH OF THE PARTIAL MONO-UNARY ALGEBRA $(\mathbb{R}, \tan)$

Definition 6.1.

6.1.1. Let's define the component of type Tan_0 as a root tree, whose in-degree of every vertex is countably infinite (Figure 11).

6.1.2. Let $(T_i)_i \in \omega$ be a family of disjoint components of type Tan_0 and r_i be the root of the tree T_i for each $i \in \omega\backslash\{0\}$. For every $n \in \omega\backslash\{0\}$, the following digraph

$$\Big(\bigcup_{i=1}^{n} T_i\Big) \cup \Big(\bigcup_{i=1}^{n-1} \{(r_i, r_{i+1})\}\Big) \cup \{(r_n, r_1)\}$$

is called a component of type Tan_n (Figure 12) and $(\cup_{i\in\omega} T_i) \cup \{(r_i, r_{i+1}) : i \in \omega\}$ is called a component of type Tan_∞ (Figure 13).

Theorem 6.2. *The set of connected components of partial mono-unary algebra* $(\mathbb{R}, \tan)$ *consists of countably infinitely many components of type* Tan_n*, for each* $n \in \omega$*, and continuum-many components of type* Tan_∞.

Proof. It is obvious that:

- The in-degree of each vertex of the graph of the function tan is ω.
- For each $r_k = \pi/2 + \pi k$, $k \in \mathbb{Z}$, there is a component T_k in the graph of the partial mono-unary algebra $(\mathbb{R}, \tan)$, which is a component of type Tan_0 and whose root is r_k. There aren't any other components of type Tan_0.
- Due to the reason that for each natural n, the equality-

$$\tan^n(x) = x$$

has countably infinitely many solutions, therefore there are countably infinitely many components of type Tan_n, for each natural n, in the graph of partial mono-unary algebra $(\mathbb{R}, \tan)$.

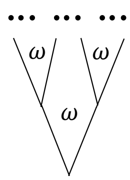

FIGURE 11

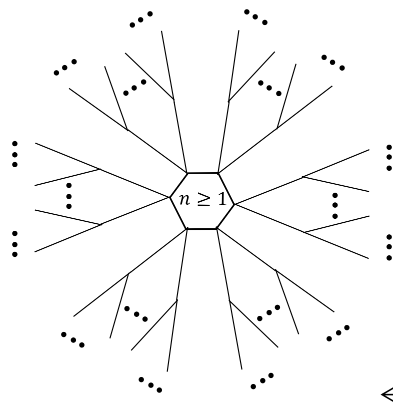

FIGURE 12

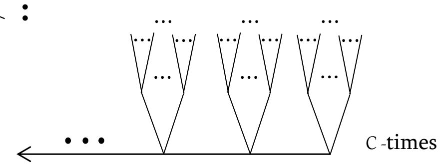

FIGURE 13

It's obvious that the union of sets of vertices of all components of type Tan_n, $n \in \omega$, is countably infinite. We can say that each of the remaining points is in the set of domains of the function tan as in the set of ranges of the function tan, the cardinality of the set of all such points is c. In addition, we know that the cardinality of the set of vertexes of each component of the graph of $(\mathbb{R}, \tan)$ is countably infinite. Therefore, the components which are not of type Tan_n, $n \in \omega$, are the components of type Tan_∞ and the cardinality of the set of all such components is $\mathbf{c}$. $\square$

Remark 6.3. In the same way we can find out that $(\mathbb{R}, \mathrm{cotan})$ has the same structure of the digraph as $(\mathbb{R}, \tan)$. Therefore, we have $(\mathbb{R}, \tan) \cong (\mathbb{R}, \mathrm{cotan})$.

7. The Digraphs of Some Basic Elementary Functions

It's easy to show, what kinds of graphs have the following functions: see Figure 14

$$F_0 = \{\mathbf{arcsin, arccos, arctan, arccotan}, \boldsymbol{a^x}, \mathbf{log}_{\boldsymbol{a}} \boldsymbol{x}, \boldsymbol{x^n}, \boldsymbol{x^{\frac{1}{n}}}\ (\boldsymbol{n} = \mathbf{1, 2, 3, \dots})\}$$

8. Universality of the Digraph of the Function tan

Theorem 8.1. *For each mono-unary algebra* $(\mathbb{R}, f)$*,* $f \in F$*, there is a monomorphism from* $(\mathbb{R}, f)$ *into the* $(\mathbb{R}, \tan)$*.*

Proof. The corresponding monomorphisms can be easily constructed. We construct a monomorphism of the algebra $(\mathbb{R}, \cos)$ into the partial algebra $(\mathbb{R}, \tan)$. The remaining monomorphisms are constructed more simply.

First, we build a monomorphism from the component C_1^0 of type C_1 with root d, which is the fixed point of cos, into any component T_1 of type $\mathrm{Tan}_{\mathbf{1}}$, whose root is r.

Define the sets: $A_1' = \{x : \cos(x) = d\}$; $B_1' = \{x : \tan(x) = r\}$.

Let f_1' be a bijection between these two sets.

Now we define the following sets: $A_1'' = \{x : \cos(x) = -d\}$ and $B_1'' = \{x : \tan(x) = f_1'(-d)\}$. Let f_1'' be a bijection between these two sets.

So, we can define monomorphism f' from the component of type C_1^0, into the component of type T_1 as follows:

$$f'(x) = \begin{cases} r, & \text{if} \quad x = d; \\ f_1'(x), & \text{if} \quad x \in A_1'; \\ f_1''(x), & \text{if} \quad x \in A_1''. \end{cases}$$

Second, we build a monomorphism from the component C_2^0 of type $\mathbf{C}_2$ into any component T_∞ of type Tan_∞.

Let b be a point from the component T_∞.

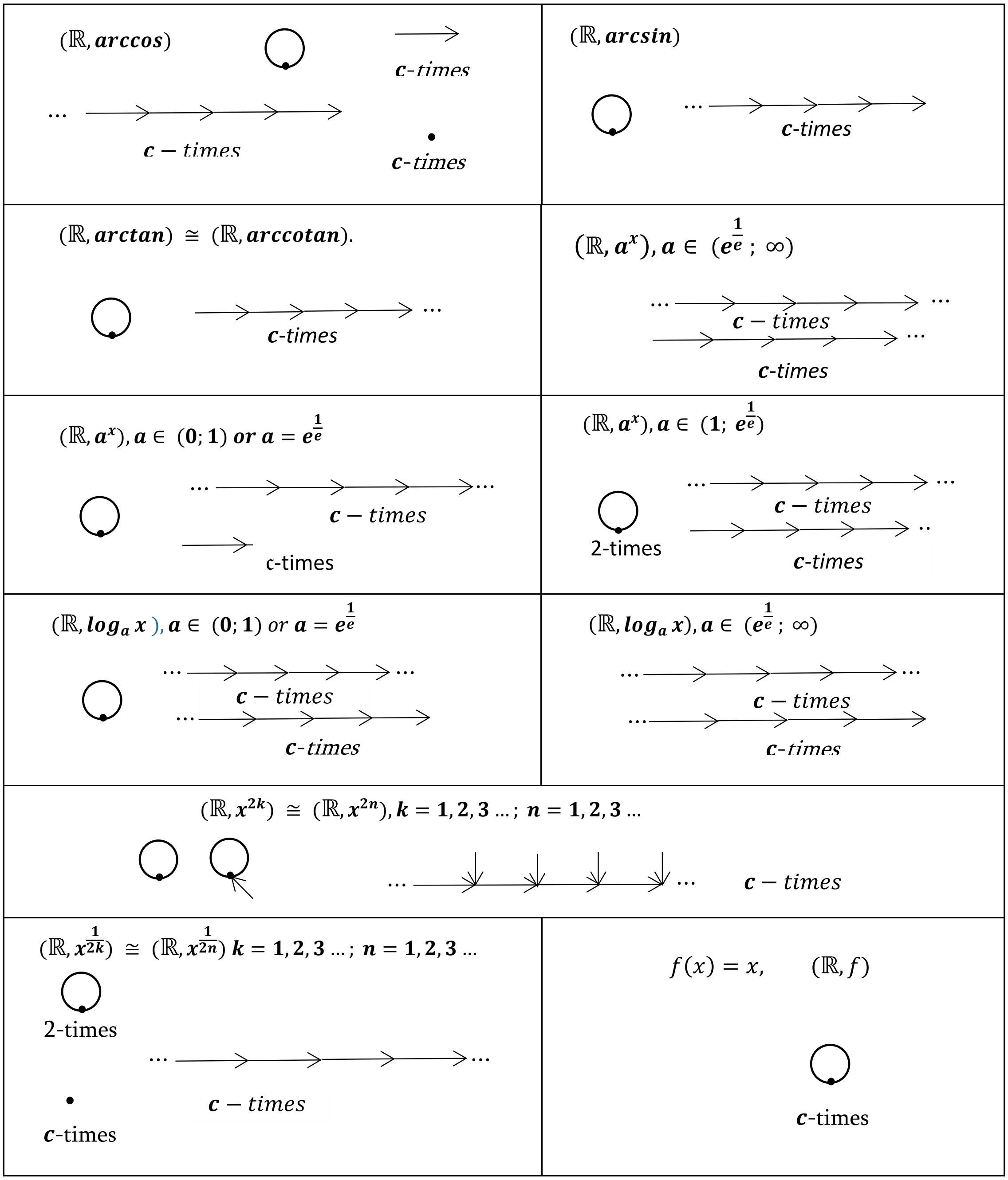

FIGURE 14

Define the sets

$$A_i = \{x : \cos(x) = \cos^{i+1}(\pi/2) \& x \neq \cos^i(\pi/2)\},\ i \in \omega,$$
$$A_i' = \{x : \cos(x) = -\cos^{i+1}(\pi/2)\},\ i \in \omega \backslash \{0\},$$
$$B_i = \{x : \tan(x) = \tan^{i+1}(b) \& x \neq \tan^i(b)\},\ i \in \omega \backslash \{0\}.$$

For each $i \in \omega\backslash\{0\}$, let b_i be a point for which:

$$\tan(b_i) = \tan^{i+2}(b) \& b_i \neq \tan^{i+1}(b) \quad \text{and let} \quad B'_i = \{x : \tan(x) = b_i\},\ i \in \omega\backslash\{0\}.$$

Let us denote a bijection between the sets A_i and B_i by f_i and the bijection between the sets A'_i and B'_i by f'_i. Let's define a monomorphism from the component C_2^0 into $\mathbf{T}_\infty$ as follows:

$$f_0(x) = \begin{cases} \tan^i(b), & \text{if} \quad x = cos^i(\pi/2)\ i \in \omega; \\ f_i(x), & \text{if} \quad x \in A_i\ i \in \omega; \\ f'_i(x), & \text{if} \quad x \in A'_i\ i \in \omega\backslash\{0\}. \end{cases}$$

Since $\mathbf{C}_2$ can be presented by the union of component of type $\mathbf{C}_3$ and component of type $\mathbf{S}_0$, there is a monomorphism from the component of type $\mathbf{C}_3$ into the component of type Tan_∞.

Therefore, there is a monomorphism from mono-unary algebra $(\mathbb{R}, \cos)$ into the $(\mathbb{R}, \tan)$. For the other mono-unary algebras $(\mathbb{R}, f)$, $f \in F$, the proof is similar. □

Remark 8.2. Since the cardinality of the set of all connected components of $(\mathbb{R}, \cos)$ mono-unary algebra is $\mathbf{c}$, we have used the continuum form of the axiom of choice.

9. Some Cardinal Invariants

Definition 9.1.

9.1.1. Let (E, R) be a relational structure. By $\sigma(E, R)$ we denote the cardinality of the set of all relational structures (E, A) isomorphic to (E, R);

9.1.2. For a partial mono-unary algebra $(\mathbb{R}, f)$, let $\sigma(f)$ denote the cardinality of the set of all partial algebras $(\mathbb{R}, g)$, isomorphic to the $(\mathbb{R}, f)$.

9.1.3. Denote

$$F = \{\sin, \cos, \tan, \mathrm{cotan}, \arcsin, \arccos, \arctan, \mathrm{arccot}, \frac{1}{x}, a^x, \log_a x, x^n, x^{1/n},\ \ (n = 2, 3, \dots)\}.$$

Finding the cardinal invariants $\sigma(f)$ and $|\mathrm{Aut}(\mathbb{R}, f)|$ for any function f, are special cases of Ulam's product-isomorphism problems (see [14]). In the general case, the problem of finding the cardinal number $\sigma(E, R)$ depends on GCH (see [4]).

Lemma 9.2 ([7])**.** *Let (E, R) be an infinite relational structure, $|E| = \varepsilon, \Delta_E$ be a diagonal of E^2 and let R be the functional relation with respect to the first or second coordinate. Then:*

1) *if $|\Delta_E \cap R| = \varepsilon \& |\Delta_E \setminus R| = \delta$, then $\sigma(E, R) = \varepsilon^\delta$;*
2) *if $(\exists l)$ $(l \subset E^2 \& (l = \{x\} \times E \vee l = E \times \{x\}) \& |l \cap R| = \varepsilon \& |l \backslash R| = \delta)$, then $\sigma(E, R) = \varepsilon^{\delta+1}$;*
3) *if $\delta = \max\{|pr_1 R|, |pr_2 R|\} < \varepsilon$, then $\sigma(E, R) = \varepsilon^\delta$;*
4) *in all the remaining cases $\sigma(E, R) = 2^\varepsilon$.*

Theorem 9.3. *If $f \in F$, then*

$$\sigma(f) = |\mathrm{Aut}(\mathbb{R}, f)| = 2^c$$

holds.

Proof. If $f \in F$, then:

(i) f has at most countably many fixed points;

(ii) for any l, where $l = \{x\} \times \mathbb{R}$ or $l = \mathbb{R} \times \{x\}$, the function f has at most countable set of intersections with l;

(iii) the cardinalities of the sets $\mathrm{Dom}(f)$ and $\mathrm{Ran}(f)$ are equal to $\mathbf{c}$.

So, from Lemma 9.2 it follows that $\boldsymbol{\sigma}(f) = 2^c$ holds.

If $f \in F$, then the digraph G_f has continuum many pairwise isomorphic components, therefore $|\mathrm{Aut}(\mathbb{R}, f)| = 2^c$. □

Corollary 9.4. *If $f \in F$, then there are 2^c-many discontinuous functions that have isomorphic digraph with the digraph of f.*

Remark 9.5. For the values of cardinalities of the automorphism groups of mono-unary algebras in the general case, see [8].

Remark 9.6. If $f : A \to A$ is a bijection, then (A, f) and (A, f^{-1}) mono-unary algebras are isomorphic and the cardinality of the set of all isomorphisms between the algebras (A, f) and (A, f^{-1}) is $|\mathrm{Aut}(A, f)|$.

Proof. It is not difficult to produce isomorphism between those two mono-unary algebras by building an isomorphism between the digraphs of those two mono-unary algebras, because the components of digraphs of bijections can be only n-cycle for some an $n \in \omega \setminus \{0\}$ or an $\omega^* + \omega$ chain.

If $f : A \to A$ is bijection, h is any automorphism of algebra (A, f) and φ is any isomorphism between (A, f) and (A, f^{-1}) mono-unary algebras, then $\varphi \circ h$ will also be an isomorphism between (A, f) and (A, f^{-1}) mono-unary algebras. □

10. Open Problem

For which function $f \in F$ there exists a non-measurable function $g : \mathbb{R} \to \mathbb{R}$, whose digraph is isomorphic to the digraph of f?

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(Received 05.11.2019)

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