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EXISTENCE RESULTS FOR A CLASS OF NONLINEAR DEGENERATE (p,q)-BIHARMONIC OPERATORS IN WEIGHTED SOBOLEV SPACES

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Abstract. In this paper we are interested in the existence of solutions for Navier problem associated with the degenerate nonlinear elliptic equation

$$\Delta\big[\omega(x)\,|\Delta u|^{p-2}\Delta u + v(x)|\Delta u|^{q-2}\Delta u\big] - \sum_{j=1}^{n} D_j\left[\omega(x)\mathcal{A}_j(x,u,\nabla u)\right]$$
$$= f_0(x) - \sum_{j=1}^{n} D_j f_j(x), \quad \text{in} \ \ \Omega$$

in the setting of the weighted Sobolev spaces.

1. Introduction

In this paper we prove the existence of (weak) solutions in the weighted Sobolev space $X = W^{2,p}(\Omega,\omega)\cap W_0^{1,p}(\Omega,\omega)$ (see Definitions 2.3 and 2.4) for the Navier problem

$$(P)\begin{cases} Lu(x) = f_0(x) - \displaystyle\sum_{j=1}^{n} D_j f_j(x), & \text{in} \ \ \Omega, \\ u(x) = \Delta u(x) = 0, & \text{on} \ \ \partial\Omega, \end{cases}$$

where L is the partial differential operator

$$Lu(x) = \Delta\big[\omega(x)\,|\Delta u|^{p-2}\Delta u + v(x)\,|\Delta u|^{q-2}\Delta u\big] - \sum_{j=1}^{n} D_j\left[\omega(x)\mathcal{A}_j(x,u(x),\nabla u(x))\right]$$

where $D_j = \partial/\partial x_j$, Ω is a bounded open set in $\mathbb{R}^n$, ω and v are two weight functions, Δ is the usual Laplacian operator, $2\leq q < p < \infty$ and the functions $\mathcal{A}_j : \Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ $(j = 1,\ldots,n)$ satisfying the following conditions:

(H1) $x\mapsto\mathcal{A}_j(x,\eta,\xi)$ is measurable on Ω for all $(\eta,\xi)\in\mathbb{R}\times\mathbb{R}^n$, $(\eta,\xi)\mapsto\mathcal{A}_j(x,\eta,\xi)$ is continuous on $\mathbb{R}\times\mathbb{R}^n$ for almost all $x\in\Omega$;

(H2) there exists a constant $\theta_1 > 0$ such that $[\mathcal{A}(x,\eta,\xi) - \mathcal{A}(x,\eta',\xi')].\ (\xi-\xi')\geq\theta_1\,|\xi-\xi'|^p$, whenever $\xi,\xi'\in\mathbb{R}^n$, $\xi\neq\xi'$, where $\mathcal{A}(x,\eta,\xi) = (\mathcal{A}_1(x,\eta,\xi),\ldots,\mathcal{A}_n(x,\eta,\xi))$ (where the dot denotes here the Euclidean scalar product in $\mathbb{R}^n$);

(H3) $\mathcal{A}(x,\eta,\xi).\xi\geq\lambda_1|\xi|^p$, where λ_1 is a positive constant;

(H4) $|\mathcal{A}(x,\eta,\xi)|\leq K_1(x) + h_1(x)|\eta|^{p/p'} + h_2(x)|\xi|^{p/p'}$, where K_1, h_1 and h_2 are positive functions with h_1 and $h_2\in L^\infty(\Omega)$, and $K_1\in L^{p'}(\Omega,\omega)$ (with $1/p + 1/p' = 1$).

Let Ω be an open set in $\mathbb{R}^n$. By the symbol $\mathcal{W}(\Omega)$ we denote the set of all measurable a.e. in Ω positive and finite functions $\omega = \omega(x)$, $x\in\Omega$. Elements of $\mathcal{W}(\Omega)$ will be called *weight functions*. Every weight ω gives rise to a measure on the measurable subsets of $\mathbb{R}^n$ through integration. This measure will be denoted by μ_ω. Thus, $\mu_\omega(E) = \displaystyle\int_E \omega(x)\,dx$ for measurable sets $E\subset\mathbb{R}^n$.

In general, the Sobolev spaces $\mathrm{W}^{k,p}(\Omega)$ without weights occur as spaces of solutions for elliptic and parabolic partial differential equations. In the particular case for $p = q = 2$ and $\omega = v\equiv 1$, we have

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the equation

$$\Delta^2 u - \sum_{j=1}^{n} D_j \mathcal{A}_j(x, u, \nabla u) = f,$$

where $\Delta^2 u$ is the biharmonic operator. If $p = q$, $\omega = v \equiv 1$ and $\mathcal{A}(x, \eta, \xi) = |\xi|^{p-2}\,\xi$, we have the equation

$$\Delta(|\Delta|^{p-2}\,\Delta u) - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = f.$$

Biharmonic equations appear in the study of mathematical model in several real-life processes as, among others, radar imaging (see [1]) or incompressible flows (see [17]).

For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces (see [4], [5], [6], [3] and [9]). In various applications, we can meet the boundary value problems for elliptic equations whose ellipticity is disturbed in the sense that there appear some degeneration or singularity appears. There are several very concrete problems from practice which lead to such differential equations, e.g., from glaceology, non-Newtonian fluid mechanics, flows through porous media, differential geometry, celestial mechanics, climatology, petroleum extraction and reaction-diffusion problems (see some examples of applications of degenerate elliptic equations in [2] and [8]).

A class of weights, which is particularly well understood, is the class of A_p-weights (or Muckenhoupt class) that was introduced by B. Muckenhoupt (see [18]). These classes have found many useful applications in harmonic analysis (see [20]). Another reason for studying A_p-weights is the fact that powers of distance to submanifolds of $\mathbb{R}^n$ often belong to A_p (see [15]). There are, in fact, many interesting examples of weights (see [14] for p-admissible weights).

In the non-degenerate case (i.e., with $\omega(x) \equiv 1$), for all $f \in L^p(\Omega)$, the Poisson equation associated with the Dirichlet problem

$$\begin{cases} -\Delta u = f(x) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

is uniquely solvable in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ (see [13]), and the nonlinear Dirichlet problem

$$\begin{cases} -\Delta_p u = f(x) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

is uniquely solvable in $W_0^{1,p}(\Omega)$ (see [7]), where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator. In the degenerate case, the weighted p-Biharmonic operator has been studied by many authors (see [19] and the references therein), and the degenerated p-Laplacian was studied in [9].

The following theorem will be proved in Section 3.

Theorem 1.1. *Let $2 \le q < p < \infty$ and assume $(H1) - (H4)$. If*

(*H*5) *$\omega \in A_p$, $v \in \mathcal{W}(\Omega)$ and $\dfrac{v}{\omega} \in L^r(\Omega, \omega)$, where $r = p/(p-q)$;*

(*H*6) *$f_j/\omega \in L^{p'}(\Omega, \omega)$ $(j = 0, 1, \ldots, n)$.*

Then the problem (P) has a unique solution $u \in X = W^{2,p}(\Omega, \omega) \cap W_0^{1,p}(\Omega, \omega)$. Moreover, we have

$$\|u\|_X \le \frac{1}{\gamma^{p'/p}} \left(C_\Omega \|f_0/\omega\|_{L^{p'}(\Omega,\omega)} + \sum_{j=1}^{n} \|f_j/\omega\|_{L^{p'}(\Omega,\omega)} \right)^{p'/p},$$

where $\gamma = \min\{\lambda_1, 1\}$ and C_Ω is the constant in Theorem 2.2.

2. Definitions and Basic Results

Let ω be a locally integrable nonnegative function in $\mathbb{R}^n$ and assume that $0 < \omega < \infty$ almost everywhere. We say that ω belongs to the Muckenhoupt class A_p, $1 < p < \infty$, or that ω is an A_p-weight, if there is a constant $C = C_{p,\omega}$ such that

$$\left(\frac{1}{|B|} \int_B \omega \, dx \right) \left(\frac{1}{|B|} \int_B \omega^{1/(1-p)} \, dx \right)^{p-1} \le C,$$

for all balls $B\subset\mathbb{R}^n$, where $|.|$ denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$. If $1<q\leq p$, then $A_q\subset A_p$ (see [12], [14] or [20] for more information about A_p-weights). The weight ω satisfies the doubling condition if there exists a positive constant C such that

$$\mu(B(x;2r))\leq C\,\mu(B(x;r)),$$

for every ball $B=B(x;r)\subset\mathbb{R}^n$, where $\mu(B)=\int_B\omega(x)\,dx$. If $\omega\in A_p$, then μ is doubling (see [14], Corollary 15.7).

As an example of A_p-weight, the function $\omega(x)=|x|^\alpha$, $x\in\mathbb{R}^n$, is in A_p if and only if $-n<\alpha<n(p-1)$ (see [20], Corollary 4.4, Chapter IX, Corollary 4.4).

If $\omega\in A_p$, then

$$\left(\frac{|E|}{|B|}\right)^p\leq C\frac{\mu(E)}{\mu(B)},$$

whenever B is a ball in $\mathbb{R}^n$ and E is a measurable subset of B (for *a strong doubling property* see 15.5 in [14]). Therefore, if $\mu(E)=0$, then $|E|=0$. The measure μ and the Lebesgue measure $|\cdot|$ are mutually absolutely continuous, i.e., they have the same zero sets ($\mu(E)=0$ if and only if $|E|=0$); so, there is no need to specify the measure when using the ubiquitous expression almost everywhere and almost every, both abbreviated a.e..

Definition 2.1. Let ω be a weight, and let $\Omega\subset\mathbb{R}^n$ be open. For $0<p<\infty$ we define $L^p(\Omega,\omega)$ as the set of measurable functions f on Ω such that

$$\|f\|_{L^p(\Omega,\omega)}=\left(\int_\Omega|f|^p\omega\,dx\right)^{1/p}<\infty.$$

If $\omega\in A_p$, $1<p<\infty$, then $\omega^{-1/(p-1)}$ is locally integrable and we have $L^p(\Omega,\omega)\subset L^1_{\rm loc}(\Omega)$ for every open set Ω (see [21, Remark 1.2.4]). It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega,\omega)$.

Definition 2.2. Let $\Omega\subset\mathbb{R}^n$ be a bounded open set, $1<p<\infty$, k be a nonnegative integer and $\omega\in A_p$. We shall denote by $W^{k,p}(\Omega,\omega)$ the weighted Sobolev spaces, the set of all functions $u\in L^p(\Omega,\omega)$ with weak derivatives $D^\alpha u\in L^p(\Omega,\omega)$, $1\leq|\alpha|\leq k$. The norm in the space $W^{k,p}(\Omega,\omega)$ is defined by

$$\|u\|_{W^{k,p}(\Omega,\omega)}=\left(\int_\Omega|u|^p\,\omega\,dx+\sum_{1\leq|\alpha|\leq k}\int_\Omega|D^\alpha u|^p\,\omega\,dx\right)^{1/p}. \tag{2.1}$$

If $\omega\in A_p$, then $W^{1,p}(\Omega,\omega)$ is the closure of $C^\infty(\Omega)$ with respect to the norm (2.1) (see [21, Theorem 2.1.4]). The spaces $W^{1,p}(\Omega,\omega)$ are Banach spaces.

The space $W^{1,p}_0(\Omega,\omega)$ is the closure of $C^\infty_0(\Omega)$ with respect to the norm (2.1). Equipped with this norm, $W^{1,p}_0(\Omega,\omega)$ is a reflexive Banach space (see [16] for more information about the spaces $W^{1,p}(\Omega,\omega)$). The dual of the space $W^{1,p}_0(\Omega,\omega)$ is the space

$$[W^{1,p}_0(\Omega,\omega)]^*=\{T=f_0-\operatorname{div}(F),\ F=(f_1,\ldots,f_n):\ \frac{f_j}{\omega}\in L^{p'}(\Omega,\omega),\ j=0,1,\ldots,n\}.$$

It is evident that a weight function ω which satisfies $0<c_1\leq\omega(x)\leq c_2$ for $x\in\Omega$ (where c_1 and c_2 are constants), gives nothing new (the space $\mathrm{W}^{1,p}_0(\Omega,\omega)$ is then identical with the classical Sobolev space $\mathrm{W}^{1,p}_0(\Omega)$). Consequently, we shall be interested above all in such weight functions ω which either vanish somewhere in $\bar\Omega$, or increase at infinity (or both).

In this paper we use the following results.

Theorem 2.1. *Let $\omega\in A_p$, $1<p<\infty$, and let Ω be a bounded open set in $\mathbb{R}^n$. If $u_m\to u$ in $L^p(\Omega,\omega)$ then there exist a subsequence $\{u_{m_k}\}$ and a function $\Phi\in L^p(\Omega,\omega)$ such that*

(i) *$u_{m_k}(x)\to u(x)$, $m_k\to\infty$ a.e. on Ω;*
(ii) *$|u_{m_k}(x)|\leq\Phi(x)$ a.e. on Ω.*

Proof. The proof of this theorem follows the lines of Theorem 2.8.1 in [11]. □

Theorem 2.2 (The weighted Sobolev inequality)**.** *Let Ω be an open bounded set in $\mathbb{R}^n$ and $\omega \in A_p$ $(1 < p < \infty)$. There exist the constants C_Ω and δ positive such that for all $u \in W_0^{1,p}(\Omega,\omega)$ and all k satisfying $1 \le k \le n/(n-1)+\delta$,*

$$\|u\|_{L^{kp}(\Omega,\omega)} \le C_\Omega \| \, |\nabla u| \, \|_{L^p(\Omega,\omega)}. \tag{2.2}$$

Proof. It suffices to prove the inequality for the functions $u \in C_0^\infty(\Omega)$ (see [10, Theorem 1.3]). To extend the estimates (2.2) to arbitrary $u \in W_0^{1,p}(\Omega,\omega)$, we let $\{u_m\}$ be a sequence of $C_0^\infty(\Omega)$ functions tending to u in $W_0^{1,p}(\Omega,\omega)$. Applying the estimates (2.2) to differences $u_{m_1} - u_{m_2}$, we see that $\{u_m\}$ will be a Cauchy sequence in $L^{kp}(\Omega,\omega)$. Consequently, the limit function u will lie in the desired spaces and satisfy (2.2). □

Lemma 2.3. *Let $1 < p < \infty$.*

(a) *There exists a constant $\alpha_p > 0$ such that*

$$\Big| \, |x|^{p-2}x - |y|^{p-2}y \Big| \le \alpha_p \, |x-y|(|x|+|y|)^{p-2},$$

for all $x, y \in \mathbb{R}^n$.

(b) *There exist two positive constants β_p, γ_p such that for every $x, y \in \mathbb{R}^n$,*

$$\beta_p \, (|x|+|y|)^{p-2}|x-y|^2 \le (|x|^{p-2}x - |y|^{p-2}y).(x-y) \le \gamma_p \, (|x|+|y|)^{p-2}|x-y|^2.$$

Proof. See [7], Proposition 17.2 and Proposition 17.3. □

Remark 2.4. If $2 \le q < p < \infty$ and $\dfrac{v}{\omega} \in L^r(\Omega,\omega)$ (where $r = p/(p-q)$), then there exists a constant $C_{p,q} = \|v/\omega\|_{L^r(\Omega,\omega)}^{1/q}$ such that

$$\|u\|_{L^q(\Omega,v)} \le C_{p,q}\|u\|_{L^p(\Omega,\omega)}.$$

In fact, by Hölder's inequality $(1/r + q/p = (p-q)/p + q/p = 1)$,

$$\begin{aligned} \|u\|_{L^q(\Omega,v)}^q &= \int_\Omega |u|^q \, v \, dx = \int_\Omega |u|^q \frac{v}{\omega} \, \omega \, dx \\ &\le \bigg(\int_\Omega |u|^p \omega \, dx \bigg)^{q/p} \bigg(\int_\Omega \left(v/\omega\right)^r \omega \, dx \bigg)^{1/r} \\ &= \|u\|_{L^p(\Omega,\omega)}^q \|v/\omega\|_{L^r(\Omega,\omega)}. \end{aligned}$$

Hence, $\|u\|_{L^q(\Omega,v)} \le C_{p,q}\|u\|_{L^p(\Omega,\omega)}$, with $C_{p,q} = \|v/\omega\|_{L^r(\Omega,\omega)}^{1/q}$.

Definition 2.3. We denote by $X = W^{2,p}(\Omega,\omega) \cap W_0^{1,p}(\Omega,\omega)$ with the norm

$$\|u\|_X = \bigg(\int_\Omega |\nabla u|^p \, \omega \, dx + \int_\Omega |\Delta u|^p \, \omega \, dx \bigg)^{1/p}.$$

Definition 2.4. We say that an element $u \in X = W^{2,p}(\Omega,\omega) \cap W_0^{1,p}(\Omega,,\omega)$ is a (weak) solution of problem (P) if

$$\begin{aligned} &\int_\Omega |\Delta u|^{p-2} \, \Delta u \, \Delta\varphi \, \omega \, dx + \int_\Omega |\Delta u|^{q-2} \, \Delta u \, \Delta\varphi \, v \, dx + \sum_{j=1}^n \int_\Omega \mathcal{A}_j(x,u,\nabla u) \, D_j\varphi \, \omega \, dx \\ &= \int_\Omega f_0 \, \varphi \, dx + \sum_{j=1}^n \int_\Omega f_j \, D_j\varphi \, dx, \end{aligned}$$

for all $\varphi \in X$.

3. Proof of Theorem 1.1

The basic idea is to reduce problem (P) to an operator equation $Au = T$ and apply the theorem below.

Theorem 3.1. *Let $A : X\to X^*$ be a monotone, coercive and hemicontinuous operator on the real, separable, reflexive Banach space X. Then the following assertions hold:*
(a) *for each $T\in X^*$, the equation $Au = T$ has a solution $u\in X$;*
(b) *if the operator A is strictly monotone, then the equation $Au = T$ is uniquely solvable in X.*

Proof. See Theorem 26. A in [23]. □

To prove Theorem 1.1, we define $B, B_1, B_2, B_3 : X\times X\to\mathbb{R}$ and $T : X\to\mathbb{R}$ by

$$B(u,\varphi) = B_1(u,\varphi) + B_2(u,\varphi) + B_3(u,\varphi),$$

$$B_1(u,\varphi) = \sum_{j=1}^{n}\int_\Omega \mathcal{A}_j(x,u,\nabla u)D_j\varphi\,\omega\,dx = \int_\Omega \mathcal{A}(x,u,\nabla u)\cdot\nabla\varphi\,\omega\,dx$$

$$B_2(u,\varphi) = \int_\Omega |\Delta u|^{p-2}\Delta u\,\Delta\varphi\,\omega\,dx$$

$$B_3(u,\varphi) = \int_\Omega |\Delta u|^{q-2}\Delta u\,\Delta\varphi\,v\,dx$$

$$T(\varphi) = \int_\Omega f_0\,\varphi\,dx + \sum_{j=1}^{n}\int_\Omega f_j\,D_j\varphi\,dx.$$

Then $u\in X$ is a (weak) solution to problem (P) if

$$B(u,\varphi) = B_1(u,\varphi) + B_2(u,\varphi) + B_3(u,\varphi) = T(\varphi),$$

for all $\varphi\in X$.

Step 1. For $j = 1,\ldots,n$, we define the operator $F_j : X\to L^{p'}(\Omega,\omega)$ as

$$(F_ju)(x) = \mathcal{A}_j(x,u(x),\nabla u(x)).$$

We now show that the operator F_j is bounded and continuous.

(i) Using (H4), we obtain

$$\begin{aligned}\|F_ju\|^{p'}_{L^{p'}(\Omega,\omega)} &= \int_\Omega |F_ju(x)|^{p'}\omega\,dx\\ &= \int_\Omega |\mathcal{A}_j(x,u,\nabla u)|^{p'}\omega\,dx\\ &\le \int_\Omega \left(K_1 + h_1|u|^{p/p'} + h_2|\nabla u|^{p/p'}\right)^{p'}\omega\,dx\\ &\le C_p\int_\Omega \left[(K_1^{p'} + h_1^{p'}|u|^p + h_2^{p'}|\nabla u|^p)\omega\right]dx\\ &= C_p\left[\int_\Omega K_1^{p'}\,\omega\,dx + \int_\Omega h_1^{p'}|u|^p\,\omega\,dx + \int_\Omega h_2^{p'}|\nabla u|^p\omega\,dx\right],\end{aligned} \tag{3.1}$$

where the constant C_p depends only on p.

We have, by Theorem 2.2 (with $k = 1$),

$$\int_\Omega h_1^{p'}|u|^p\,\omega\,dx \le \|h_1\|^{p'}_{L^\infty(\Omega)}\int_\Omega |u|^p\,\omega\,dx$$

$$\leq C_\Omega^p \, \|h_1\|^{p'}_{L^\infty(\Omega)} \int_\Omega |\nabla u|^p \, \omega \, dx$$

$$\leq C_\Omega^p \, \|h_1\|^{p'}_{L^\infty(\Omega)} \, \|u\|^p_X,$$

and $\int_\Omega h_2^{p'} |\nabla u|^p \omega \, dx \leq \|h_2\|^{p'}_{L^\infty(\Omega)} \int_\Omega |\nabla u|^p \, \omega \, dx \leq \|h_2\|^{p'}_{L^\infty(\Omega)} \|u\|^p_X$. Therefore, in (3.1) we obtain

$$\|F_j u\|_{L^{p'}(\Omega,\omega)} \quad \leq \quad C_p^{1/p'} \Big(\|K\|_{L^{p'}(\Omega,\omega)} + (C_\Omega^{p/p'} \|h_1\|_{L^\infty(\Omega)} + \|h_2\|_{L^\infty(\Omega)}) \, \|u\|_X^{p/p'} \Big).$$

(ii) Let $u_m \to u$ in X as $m \to \infty$. We need to show that $F_j u_m \to F_j u$ in $L^{p'}(\Omega, \omega)$. We will apply the Lebesgue Dominated Convergence Theorem. If $u_m \to u$ in X, then $|\nabla u_m| \to |\nabla u|$ in $L^p(\Omega, \omega)$. Using Theorem 2.1, there exist a subsequence $\{u_{m_k}\}$ and a function Φ_1 such that

$$D_j u_{m_k}(x) \to D_j u(x), \text{ a.e. in } \Omega,$$
$$|\nabla u_{m_k}(x)| \leq \Phi_1(x), \text{ a.e. in } \Omega.$$

By Theorem 2.2, we obtain

$$\|u_{m_k}\|_{L^p(\Omega,\omega)} \leq C_\Omega \, \| \, |\nabla u_{m_k}| \, \|_{L^p(\Omega,\omega)} \leq C_\Omega \, \|\Phi_1\|_{L^p(\Omega,\omega)}.$$

Next, applying (H4), we obtain

$$\begin{aligned}
\|F_j u_{m_k} - F_j u\|^{p'}_{L^{p'}(\Omega,\omega)} &= \int_\Omega |F_j u_{m_k}(x) - F_j u(x)|^{p'} \omega \, dx \\
&= \int_\Omega |\mathcal{A}_j(x, u_{m_k}, \nabla u_{m_k}) - \mathcal{A}_j(x, u, \nabla u)|^{p'} \, \omega \, dx \\
&\leq C_p \int_\Omega \Big(|\mathcal{A}_j(x, u_{m_k}, \nabla u_{m_k})|^{p'} + |\mathcal{A}_j(x, u, \nabla u)|^{p'} \Big) \omega \, dx \\
&\leq C_p \Bigg[\int_\Omega \Big(K_1 + h_1 |u_{m_k}|^{p/p'} + h_2 |\nabla u_{m_k}|^{p/p'} \Big)^{p'} \omega \, dx \\
&+ \int_\Omega \Big(K_1 + h_1 |u|^{p/p'} + h_2 |\nabla u|^{p/p'} \Big)^{p'} \omega \, dx \Bigg] \\
&\leq C_p \Bigg[\int_\Omega K_1^{p'} \, \omega \, dx + \|h_1\|^{p'}_{L^\infty(\Omega)} \int_\Omega |u_{m_k}|^p \, \omega \, dx + \|h_2\|^{p'}_{L^\infty(\Omega)} \int_\Omega |\nabla u_{m_k}|^p \, \omega \, dx \\
&+ \int_\Omega K_1^{p'} \, \omega \, dx + \|h_1\|^{p'}_{L^\infty(\Omega)} \int_\Omega |u|^p \, \omega \, dx + \|h_2\|^{p'}_{L^\infty(\Omega)} \int_\Omega |\nabla u|^p \, \omega \, dx \Bigg] \\
&\leq 2 \, C_p \Bigg[\int_\Omega K_1^{p'} \, \omega \, dx + C_\Omega^p \|h_1\|^{p'}_{L^\infty(\Omega)} \int_\Omega \Phi_1^p \, \omega \, dx + \|h_2\|^{p'}_{L^\infty(\Omega)} \int_\Omega \Phi_1^p \, \omega \, dx \Bigg] \\
&= 2 \, C_p \Bigg[\|K_1\|^{p'}_{L^{p'}(\Omega,\omega)} + \big(C_\Omega^p \|h_1\|^{p'}_{L^\infty(\Omega)} + \|h_2\|^{p'}_{L^\infty(\Omega)} \big) \|\Phi_1\|^p_{L^p(\Omega,\omega)} \Bigg].
\end{aligned}$$

By condition (H1), we have

$$F_j u_{m_k}(x) = \mathcal{A}_j(x, u_{m_k}(x), \nabla u_{m_k}(x)) \to \mathcal{A}_j(x, u(x), \nabla u(x)) = F_j u(x),$$

as $m_k \to +\infty$. Therefore, by the Lebesgue Dominated Convergence Theorem, we obtain

$$\|F_j u_{m_k} - F_j u\|_{L^{p'}(\Omega,\omega)} \to 0,$$

that is,

$$F_j u_{m_k} \to F_j u \quad \text{in} \quad L^{p'}(\Omega, \omega).$$

We conclude from the Convergence Principle in Banach spaces (see [22, Proposition 10.13]) that

$$F_j u_m \to F_j u \quad \text{in} \quad L^{p'}(\Omega,\omega). \tag{3.2}$$

Step 2. We define the operator $G_1 : X \to L^{p'}(\Omega,\omega)$ by

$$(G_1 u)(x) = |\Delta u(x)|^{p-2}\, \Delta u(x).$$

This operator is continuous and bounded. In fact,
(i) we have

$$\begin{aligned}\|G_1 u\|^{p'}_{L^{p'}(\Omega,\omega)} &= \int_\Omega \big|\, |\Delta u|^{p-2}\, \Delta u\big|^{p'}\, \omega\, dx \\ &= \int_\Omega |\Delta u|^{(p-1)\, p'}\, \omega\, dx \\ &= \int_\Omega |\Delta u|^{p}\, \omega\, dx \\ &\le \|u\|^p_X.\end{aligned}$$

Hence, $\|G_1 u\|_{L^{p'}(\Omega,\omega)} \le \|u\|^{p-1}_X$.

(ii) If $u_m \to u$ in X, then $\Delta u_m \to \Delta u$ in $L^p(\Omega,\omega)$. By Theorem 2.1, there exist a subsequence $\{u_{m_k}\}$ and a function $\Phi_2 \in L^p(\Omega,\omega)$ such that

$$\Delta u_{m_k}(x) \to \Delta u(x), \ \text{ a.e. in } \Omega, \tag{3.3}$$

$$|\Delta u_{m_k}(x)| \le \Phi_2(x), \ \text{ a.e. in } \Omega. \tag{3.4}$$

Hence, using Lemma 2.3 (a), $\theta = \dfrac{p}{p'} = p-1$ and $\theta' = \dfrac{(p-2)}{(p-1)}$, we obtain (since $2 \le q < p < \infty$),

$$\begin{aligned}&\|G_1 u_{m_k} - G_1 u\|^{p'}_{L^{p'}(\Omega,\omega)} = \int_\Omega |\, G_1 u_{m_k} - G_1 u|^{p'}\, \omega\, dx \\ &= \int_\Omega \left|\, |\Delta u_{m_k}|^{p-2}\, \Delta u_{m_k} - |\, \Delta u|^{p-2}\, \Delta u \right|^{p'} \omega\, dx \\ &\le \int_\Omega \Big[\alpha_p\, |\Delta u_{m_k} - \Delta u|\, (\, |\Delta u_{m_k}| + |\Delta u|)^{(p-2)}\Big]^{p'} \omega\, dx \\ &\le \alpha_p^{p'} \int_\Omega |\, \Delta u_{m_k} - \Delta u|^{p'}\, (2\, \Phi_2)^{(p-2)\, p'}\, \omega\, dx \\ &\le 2^{(p-2)p'} \alpha_p^{p'} \left(\int_\Omega |\Delta u_{m_k} - \Delta u|^{p'\theta} \omega\, dx\right)^{1/\theta} \left(\int_\Omega \Phi_2^{(p-2)p'\theta'}\, \omega\, dx\right)^{1/\theta'} \\ &\le \alpha_p^{p'}\, 2^{(p-2)p'} \left(\int_\Omega |\Delta u_{m_k} - \Delta u|^{p}\, \omega\, dx\right)^{p'/p} \left(\int_\Omega \Phi_2^{p}\, \omega\, dx\right)^{(p-2)/(p-1)} \\ &\le \alpha_p^{p'}\, 2^{(p-2)\, p'}\, \|u_{m_k} - u\|^{p'}_X\, \|\Phi_2\|^{(p-2)p'}_{L^p(\Omega,\omega)},\end{aligned}$$

since $(p-2)p'\theta' = (p-2)\dfrac{p}{(p-1)}\dfrac{(p-1)}{(p-2)} = p$ if $p \ne 2$. Then

$$\|G_1 u_{m_k} - G_1 u\|_{L^{p'}(\Omega,\omega)} \le 2^{(p-2)} \alpha_p\, \|\Phi_2\|^{p-2}_{L^p(\Omega,\omega)} \|u_{m_k} - u\|_X.$$

Therefore, by the Lebesgue Dominated Convergence Theorem, we obtain (as $m_k \to \infty$)

$$\|G_1 u_{m_k} - G_1 u\|_{L^{p'}(\Omega,\omega)} \to 0,$$

that is, $G_1u_{m_k}\to G_1u$ in $L^{p'}(\Omega,\omega)$. By the Convergence Principle in Banach spaces, we have

$$G_1u_m\to G_1u \text{ in } L^{p'}(\Omega,\omega). \tag{3.5}$$

Step 3. We define the operator $G_2: X\to L^s(\Omega,\omega)$, where $s=p/(q-1)$, by

$$(G_2u)(x)=|\Delta u(x)|^{q-2}\Delta u(x).$$

We also have that the operator G_2 is continuous and bounded. In fact,
(i) we have

$$\begin{aligned}\|G_2u\|^s_{L^s(\Omega,\omega)} &= \int_\Omega \left||\Delta u|^{q-2}\Delta u\right|^s \omega\,dx\\ &= \int_\Omega |\Delta u|^{(q-1)\,s}\,\omega\,dx\\ &= \int_\Omega |\Delta u|^p\,\omega\,dx\\ &\le \|u\|^p_X,\end{aligned}$$

and $\|G_2u\|_{L^s(\Omega,\omega)}\le\|u\|_X^{q-1}$.

(ii) If $u_m\to u$ in X, then $\Delta u_m\to\Delta u$ in $L^p(\Omega,\omega)$. If $2<q<p<\infty$, by (3.3), (3.4) and Lemma 2.3(a), we have

$$\begin{aligned}\|G_2u_{m_k}-G_2u\|^s_{L^s(\Omega,\omega)} &= \int_\Omega \left||\Delta u_{m_k}|^{q-2}\,\Delta u_{m_k}-|\Delta u|^{q-2}\Delta u\right|^s\omega\,dx\\ &\le \int_\Omega \left[\alpha_q|\Delta u_{m_k}-\Delta u|\left(|\Delta u_{m_k}|+|\Delta u|\right)^{q-2}\right]^s\omega\,dx\\ &= \alpha_q^s\int_\Omega |\Delta u_{m_k}-\Delta u|^s\left(|\Delta u_{m_k}|+|\Delta u|\right)^{(q-2)s}\omega\,dx\\ &\le 2^{(q-2)s}\alpha_q^s\int_\Omega |\Delta u_{m_k}-\Delta u|^s\,\Phi_2^{(q-2)s}\,\omega\,dx. \end{aligned}\tag{3.6}$$

For $\delta=q-1$ and $\delta'=(q-1)/(q-2)$, in (3.6) we have

$$\begin{aligned}&\|G_2u_{m_k}-G_2u\|^s_{L^s(\Omega,\omega)}\\ &\le 2^{(q-2)s}\alpha_q^s\int_\Omega |\Delta u_{m_k}-\Delta u|^s\,\Phi_2^{(q-2)s}\,\omega\,dx\\ &\le 2^{(q-2)s}\alpha_q^s\left(\int_\Omega |\Delta u_{m_k}-\Delta u|^{s\,\delta}\omega\,dx\right)^{1/\delta}\left(\int_\Omega \Phi_2^{(q-2)s\,\delta'}\,\omega\,dx\right)^{1/\delta'}\\ &= 2^{(q-2)s}\alpha_q^s\left(\int_\Omega |\Delta u_{m_k}-\Delta u|^p\,\omega\,dx\right)^{1/(q-1)}\left(\int_\Omega \Phi_2^p\,\omega\,dx\right)^{1/\delta'}\\ &\le 2^{(q-2)s}\,\alpha_q^s\,\|u_{m_k}-u\|_X^{p/(q-1)}\,\|\Phi_2\|^{p/\delta'}_{L^p(\Omega,\omega)}.\end{aligned}$$

Hence, $\|G_2u_{m_k}-G_2u\|_{L^s(\Omega,\omega}\le 2^{(q-2)}\,\alpha_q\,\|u_{m_k}-u\|_X\,\|\Phi_2\|^{(q-2)}_{L^p(\Omega,\omega)}$.
In the case $2=q<p<\infty$, we have $(G_2u)(x)=\Delta u(x)$ and $s=p$. Hence,

$$\begin{aligned}&\|G_2u\|_{L^p(\Omega,\omega)}\le\|u\|_X,\\ &\|G_2u_{m_k}-G_2u\|_{L^p(\Omega,\omega)}\le\|u_{m_k}-u\|_X.\end{aligned}$$

Therefore, for $2 \le q < p < \infty$, by the Lebesgue Dominated Convergence Theorem, we obtain (as $m_k \to \infty$)

$$\|G_2 u_{m_k} - G_2 u\|_{L^s(\Omega,\omega)} \to 0,$$

that is, $G_2 u_{m_k} \to G_2 u$ in $L^s(\Omega,\omega)$. By the Convergence Principle in Banach spaces, we have

$$G_2 u_m \to G_2 u \quad \text{in} \quad L^s(\Omega,\omega). \tag{3.7}$$

Step 4. Since $\dfrac{f_j}{\omega} \in L^{p'}(\Omega,\omega)$ $(j = 0,1,\ldots,n)$, therefore $T \in [W_0^{1,p}(\Omega,\omega)]^* \subset X^*$. Moreover, by Theorem 2.2, we have

$$\begin{aligned}
|T(\varphi)| &\le \int_\Omega |f_0||\varphi|\,dx + \sum_{j=1}^n \int_\Omega |f_j||D_j\varphi|\,dx \\
&= \int_\Omega \frac{|f_0|}{\omega}|\varphi|\omega\,dx + \sum_{j=1}^n \int_\Omega \frac{|f_j|}{\omega}|D_j\varphi|\,\omega\,dx \\
&\le \|f_0/\omega\|_{L^{p'}(\Omega,\omega)}\|\varphi\|_{L^p(\Omega,\omega)} + \sum_{j=1}^n \|f_j/\omega\|_{L^{p'}(\Omega,\omega)}\|D_j\varphi\|_{L^p(\Omega,\omega)} \\
&\le C_\Omega\,\|f_0/\omega\|_{L^{p'}(\Omega,\omega)}\|\,|\nabla\varphi|\,\|_{L^p(\Omega,\omega)} + \left(\sum_{j=1}^n \|f_j/\omega\|_{L^{p'}(\Omega,\omega)}\right)\|\,|\nabla\varphi|\,\|_{L^p(\Omega,\omega)} \\
&\le \left(C_\Omega\,\|f_0/\omega\|_{L^{p'}(\Omega,\omega)} + \sum_{j=1}^n \|f_j/\omega\|_{L^{p'}(\Omega,\omega)}\right)\|\varphi\|_X.
\end{aligned}$$

Moreover, we also have

$$\begin{aligned}
|B(u,\varphi)| &\le |B_1(u,\varphi)| + |B_2(u,\varphi)| + |B_3(u,\varphi)| \\
&\le \sum_{j=1}^n \int_\Omega |\mathcal{A}_j(x,u,\nabla u)||D_j\varphi|\,\omega\,dx + \int_\Omega |\,\Delta u|^{p-2}\,|\,\Delta u|\,|\,\Delta\varphi|\,\omega\,dx \\
&\quad + \int_\Omega |\Delta u|^{q-2}|\Delta u||\Delta\varphi|\,v\,dx.
\end{aligned} \tag{3.8}$$

In (3.8), by (H4), we have

$$\begin{aligned}
&\int_\Omega |\mathcal{A}(x,u,\nabla u)|\,|\nabla\varphi|\,\omega\,dx \le \int_\Omega \left(K_1 + h_1|u|^{p/p'} + h_2|\nabla u|^{p/p'}\right)|\nabla\varphi|\,\omega\,dx \\
&\le \|K_1\|_{L^{p'}(\Omega,\omega)}\|\,|\nabla\varphi|\,\|_{L^p(\Omega,\omega)} + \|h_1\|_{L^\infty(\Omega)}\|u\|^{p/p'}_{L^p(\Omega,\omega)}\|\,|\nabla\varphi|\,\|_{L^p(\Omega,\omega)} \\
&\ + \|h_2\|_{L^\infty(\Omega)}\|\,|\nabla u|\,\|^{p/p'}_{L^p(\Omega,\omega)}\|\,|\nabla\varphi|\,\|_{L^p(\Omega,\omega)} \\
&\le \left(\|K_1\|_{L^{p'}(\Omega,\omega)} + (C_\Omega^{p/p'}\,\|h_1\|_{L^\infty(\Omega)} + \|h_2\|_{L^\infty(\Omega)})\|u\|_X^{p/p'}\right)\|\varphi\|_X,
\end{aligned}$$

and

$$\begin{aligned}
\int_\Omega |\,\Delta u|^{p-2}\,|\,\Delta u|\,|\,\Delta\,\varphi|\,\omega\,dx &= \int_\Omega |\,\Delta u|^{p-1}\,|\,\Delta\,\varphi|\,\omega\,dx \\
&\le \left(\int_\Omega |\,\Delta u|^p\,\omega\,dx\right)^{1/p'}\left(\int_\Omega |\,\Delta\varphi|^p\,\omega\,dx\right)^{1/p} \\
&\le \|u\|_X^{p/p'}\,\|\varphi\|_X,
\end{aligned}$$

and since $s = p/(q-1)$, $r = p/(p-q)$ and $\dfrac{1}{s}+\dfrac{1}{r}+\dfrac{1}{p}=1$, by the generalized Hölder inequality, we obtain

$$\begin{aligned}
&\int_\Omega |\Delta u|^{q-2}\,|\Delta u|\,|\Delta\varphi|\,v\,dx = \int_\Omega |\Delta u|^{q-1}\,|\Delta\varphi|\,\frac{v}{\omega}\,\omega\,dx\\
&\leq \bigg(\int_\Omega |\Delta u|^{(q-1)s}\,\omega\,dx\bigg)^{1/s}\bigg(\int_\Omega |\Delta\varphi|^p\,\omega\,dx\bigg)^{1/p}\bigg(\int_\Omega \big(\frac{v}{\omega}\big)^r\,\omega\,dx\bigg)^{1/r}\\
&= \bigg(\int_\Omega |\Delta u|^p\,\omega\,dx\bigg)^{(q-1)/p}\bigg(\int_\Omega |\Delta\varphi|^p\,\omega\,dx\bigg)^{1/p}\bigg(\int_\Omega \big(\frac{v}{\omega}\big)^r\,\omega\,dx\bigg)^{1/r}\\
&\leq \|u\|_X^{(q-1)}\,\|\varphi\|_X\,\|v/\omega\|_{L^r(\Omega,\omega)}.
\end{aligned}$$

Hence, in (3.8), for all $u,\varphi\in X$, we obtain

$$\begin{aligned}
&|B(u,\varphi)|\\
&\leq\Big[\|K_1\|_{L^{p'}(\Omega,\omega)} + C_\Omega^{p/p'}\,\|h_1\|_{L^\infty(\Omega)}\|u\|_X^{p/p'} + \|h_2\|_{L^\infty(\Omega,\omega)}\|u\|_X^{p/p'} + \|u\|_X^{p/p'}\\
&+\|v/\omega\|_{L^r(\Omega,\omega)}\,\|u\|_X^{q-1}\Big]\|\varphi\|_X.
\end{aligned}$$

Since $B(u,.)$ is linear, for each $u\in X$, there exists a linear and continuous functional on X denoted by Au such that $\langle Au,\varphi\rangle = B(u,\varphi)$, for all $u,\ \varphi\in X$ (here $\langle f,x\rangle$ denotes the value of the linear functional f at the point x). Moreover,

$$\begin{aligned}
\|Au\|_*\leq&\|K_1\|_{L^{p'}(\Omega,\omega)} + C_\Omega^{p/p'}\,\|h_1\|_{L^\infty(\Omega)}\|u\|_X^{p/p'} + \|h_2\|_{L^\infty(\Omega,\omega)}\|u\|_X^{p/p'} + \|u\|_X^{p/p'}\\
&+\|v/\omega\|_{L^r(\Omega,\omega)}\|u\|_X^{q-1},
\end{aligned}$$

where $\|Au\|_* = \sup\{|\langle Au,\varphi\rangle| = |B(u,\varphi)| : \varphi\in X, \|\varphi\|_X = 1\}$ is the norm of the operator Au.

Hence, we obtain the operator

$$\begin{aligned}
A :&X\to X^*\\
&u\mapsto Au.
\end{aligned}$$

Consequently, problem (P) is equivalent to the operator equation

$$Au = T,\ u\in X.$$

Step 5. Using condition (H2) and Lemma 2.3 (b), we have

$$\begin{aligned}
&\langle Au_1 - Au_2, u_1-u_2\rangle = B(u_1,u_1-u_2) - B(u_2,u_1-u_2)\\
&=\int_\Omega \mathcal{A}(x,u_1,\nabla u_1)\cdot\nabla(u_1-u_2)\,\omega\,dx + \int_\Omega |\,\Delta u_1|^{p-2}\,\Delta u_1\,\Delta(u_1-u_2)\,\omega\,dx\\
&+\int_\Omega |\,\Delta u_1|^{q-2}\,\Delta u_1\,\Delta(u_1-u_2)\,v\,dx\\
&-\int_\Omega \mathcal{A}(x,u_2,\nabla u_2)\cdot\nabla(u_1-u_2)\,\omega\,dx - \int_\Omega |\,\Delta u_2|^{p-2}\,\Delta u_2\,\Delta(u_1-u_2)\,\omega\,dx\\
&-\int_\Omega |\,\Delta u_2|^{q-2}\,\Delta u_2\,\Delta(u_1-u_2)\,v\,dx\\
&=\int_\Omega \Big(\mathcal{A}(x,u_1,\nabla u_1) - \mathcal{A}(x,u_2,\nabla u_2)\Big)\cdot\nabla(u_1-u_2)\,\omega\,dx\\
&+\int_\Omega (|\,\Delta u_1|^{p-2}\,\Delta u_1 - |\,\Delta u_2|^{p-2}\,\Delta u_2)\,\Delta(u_1-u_2)\,\omega\,dx
\end{aligned}$$

$$
\begin{aligned}
&+\int_\Omega (|\Delta u_1|^{q-2}\Delta u_1 - |\Delta u_2|^{q-2}\Delta u_2)\,\Delta(u_1-u_2)\,v\,dx\\
\geq&\theta_1\int_\Omega |\nabla(u_1-u_2)|^p\,\omega\,dx + \beta_p\int_\Omega (|\Delta u_1|+|\Delta u_2|)^{p-2}\,|\Delta u_1-\Delta u_2|^2\,\omega\,dx\\
&+\beta_q\int_\Omega (|\Delta u_1|+|\Delta u_2|)^{q-2}\,|\Delta u_1-\Delta u_2|^2\,v\,dx\\
\geq&\theta_1\int_\Omega |\nabla(u_1-u_2)|^p\,\omega\,dx + \beta_p\int_\Omega (|\Delta u_1-\Delta u_2|)^{p-2}\,|\Delta u_1-\Delta u_2|^2\,\omega\,dx\\
&+\beta_q\int_\Omega (|\Delta u_1-\Delta u_2|)^{q-2}\,|\Delta u_1-\Delta u_2|^2\,v\,dx\\
=&\theta_1\int_\Omega |\nabla(u_1-u_2)|^p\,\omega\,dx + \beta_p\int_\Omega |\Delta u_1-\Delta u_2|^p\,\omega\,dx\\
&+\beta_q\int_\Omega |\Delta u_1-\Delta u_2|^q\,v\,dx\\
\geq&\theta_1\int_\Omega |\nabla(u_1-u_2)|^p\,\omega\,dx + \beta_p\int_\Omega |\Delta u_1-\Delta u_2|^p\,\omega\,dx\\
\geq&\theta\,\|u_1-u_2\|_X^p,
\end{aligned}
$$

where $\theta=\min\{\theta_1,\beta_p\}$. Therefore, the operator A is strongly monotone, and this implies that A is strictly monotone. Moreover, from (H3), we obtain

$$
\begin{aligned}
&\langle Au,u\rangle = B(u,u) = B_1(u,u)+B_2(u,u)+B_3(u,u)\\
&=\int_\Omega \mathcal{A}(x,u,\nabla u)\cdot\nabla u\,\omega\,dx + \int_\Omega |\Delta u|^{p-2}\,\Delta u\,\Delta u\,\omega\,dx \; + \int_\Omega |\Delta u|^{q-2}\,\Delta u\,\Delta u\,v\,dx\\
&\geq \int_\Omega \lambda_1|\nabla u|^p\,\omega\,dx + \int_\Omega |\Delta u|^p\,\omega\,dx + \int_\Omega |\Delta u|^q\,v\,dx\\
&\geq \int_\Omega \lambda_1|\nabla u|^p\,\omega\,dx + \int_\Omega |\Delta u|^p\,\omega\,dx\\
&\geq\gamma\,\|u\|_X^p,
\end{aligned}
$$

where $\gamma=\min\{\lambda_1,1\}$. Hence, since $2\leq q<p<\infty$, we have

$$
\frac{\langle Au,u\rangle}{\|u\|_X}\to+\infty, \text{ as } \|u\|_X\to+\infty,
$$

that is, A is coercive.

Step 6. We need to show that the operator A is continuous.

Let $u_m\to u$ in X as $m\to\infty$. We have

$$
\begin{aligned}
|B_1(u_m,\varphi)-B_1(u,\varphi)|\leq&\sum_{j=1}^n\int_\Omega |\mathcal{A}_j(x,u_m,\nabla u_m)-\mathcal{A}_j(x,u,\nabla u)||D_j\varphi|\,\omega\,dx\\
=&\sum_{j=1}^n\int_\Omega |F_ju_m-F_ju||D_j\varphi|\,\omega\,dx\\
\leq&\sum_{j=1}^n \|F_ju_m-F_ju\|_{L^{p'}(\Omega,\omega)}\|D_j\varphi\|_{L^p(\Omega,\omega)}
\end{aligned}
$$

$$\leq \Big(\sum_{j=1}^{n} \|F_j u_m - F_j u\|_{L^{p'}(\Omega,\omega)} \Big) \|\varphi\|_X,$$

and

$$\begin{aligned} &|B_2(u_m,\varphi) - B_2(u,\varphi)| \\ &= \Big| \int_\Omega |\,\Delta u_m|^{p-2} \Delta u_m \,\Delta\varphi\,\omega\,dx - \int_\Omega |\,\Delta u|^{p-2}\Delta u\,\Delta\varphi\,\omega\,dx \Big| \\ &\leq \int_\Omega \Big| |\,\Delta u_m|^{p-2}\,\Delta u_m - |\,\Delta u|^{p-2}\Delta u \Big|\,|\,\Delta\varphi|\,\omega\,dx \\ &= \int_\Omega |G_1 u_m - G_1 u|\,|\Delta\varphi|\,\omega\,dx \\ &\leq \|G_1 u_m - G_1 u\|_{L^{p'}(\Omega,\omega)}\,\|\varphi\|_X, \end{aligned}$$

and since $\dfrac{1}{s} + \dfrac{1}{r} + \dfrac{1}{p} = 1$ (remember that $s = p/(q-1)$ (see Step 3) and $r = p/(p-q)$, by (H5)),

$$\begin{aligned} &|B_3(u_m,\varphi) - B_3(u,\varphi)| \\ &= \Big| \int_\Omega |\Delta u_m|^{q-2}\Delta u_m\,\Delta\varphi\,v\,dx - \int_\Omega |\Delta u|^{q-2}\Delta u\,\Delta\varphi\,v\,dx \Big| \\ &\leq \int_\Omega \Big| |\Delta u_m|^{q-2}\,\Delta u_m - |\Delta u|^{q-2}\Delta u \Big|\,|\Delta\varphi|\,v\,dx \\ &= \int_\Omega |G_2 u_m - G_2 u|\,|\Delta\varphi|\,\frac{v}{\omega}\,\omega\,dx \\ &\leq \Big(\int_\Omega |G_2 u_m - G_2 u|^s\,\omega\,dx \Big)^{1/s} \Big(\int_\Omega |\Delta\varphi|^p\,\omega\,dx \Big)^{1/p} \Big(\int_\Omega \big(\frac{v}{\omega}\big)^r\,\omega\,dx \Big)^{1/r} \\ &\leq \|G_2 u_m - G_2 u\|_{L^s(\Omega,\omega)}\,\|\varphi\|_X\,\|v/\omega\|_{L^r(\Omega,\omega)}, \end{aligned}$$

for all $\varphi \in X$. Hence,

$$\begin{aligned} &|B(u_m,\varphi) - B(u,\varphi)| \\ &\leq |B_1(u_m,\varphi) - B_1(u,\varphi)| + |B_2(u_m,\varphi) - B_2(u,\varphi)| + |B_3(u_m,\varphi) - B_3(u,\varphi)| \\ &\leq \Big[\sum_{j=1}^{n} \|F_j u_m - F_j u\|_{L^{p'}(\Omega,\omega)} + \|G_1 u_m - G_1 u\|_{L^{p'}(\Omega,\omega)} \\ &\quad + \|G_2 u_m - G_2 u\|_{L^s(\Omega,\omega)} \|v/\omega\|_{L^r(\Omega,\omega)} \Big] \|\varphi\|_X. \end{aligned}$$

Then we obtain

$$\begin{aligned} \|Au_m - Au\|_* \leq \sum_{j=1}^{n} \|F_j u_m - F_j u\|_{L^{p'}(\Omega,\omega)} &+ \|G_1 u_m - G_1 u\|_{L^{p'}(\Omega,\omega)} \\ &+ \|G_2 u_m - G_2 u\|_{L^s(\Omega,\omega)} \|v/\omega\|_{L^r(\Omega,\omega)}. \end{aligned}$$

Therefore, using (3.2), (3.5) and (3.7), we have $\|Au_m - Au\|_* \to 0$ as $m \to +\infty$, that is, A is continuous and this implies that A is hemicontinuous.

Therefore, by Theorem 3.1, the operator equation $Au = T$ has a unique solution $u \in X$ and it is the unique solution for problem (P).

Step 7. In particular, by setting $\varphi = u$ in Definition 2.4, we have

$$B(u,u) = B_1(u,u) + B_2(u,u) + B_3(u,u) = T(u). \tag{3.9}$$

Hence, using (H3) and $\gamma = \min\{\lambda_1, 1\}$, we obtain

$$\begin{aligned}
&B_1(u,u) + B_2(u,u) + B_3(u,u)\\
&= \int_\Omega \mathcal{A}(x,u,\nabla u)\cdot \nabla u\,\omega\,dx + \int_\Omega |\,\Delta u|^{p-2}\,\Delta u\,\Delta u\,\omega\,dx + \int_\Omega |\,\Delta u|^{q-2}\,\Delta u\,\Delta u\,v\,dx\\
&\geq \int_\Omega \lambda_1\,|\nabla u|^p + \int_\Omega |\,\Delta u|^p\,\omega\,dx + \int_\Omega |\,\Delta u|^q\,v\,dx\\
&\geq \lambda_1 \int_\Omega |\nabla u|^p + \int_\Omega |\,\Delta u|^p\,\omega\,dx\\
&\geq \gamma \|u\|_X^p
\end{aligned}$$

and

$$\begin{aligned}
T(u) &= \int_\Omega f_0\,u\,dx + \sum_{j=1}^n \int_\Omega f_j\,D_j u\,dx\\
&\leq \|f_0/\omega\|_{L^{p'}(\Omega,\omega)}\|u\|_{L^p(\Omega,\omega)} + \sum_{j=1}^n \|f_j/\omega\|_{L^{p'}(\Omega)}\|D_j u\|_{L^p(\Omega,\omega)}\\
&\leq C_\Omega\,\|f_0/\omega\|_{L^{p'}(\Omega,\omega)}\|\,|\nabla u|\,\|_{L^p(\Omega,\omega)} + \sum_{j=1}^n \|f_j/\omega\|_{L^{p'}(\Omega)}\|\,|\nabla u|\,\|_{L^p(\Omega,\omega)}\\
&\leq \left(C_\Omega\,\|f_0/\omega\|_{L^{p'}(\Omega,\omega)} + \sum_{j=1}^n \|f_j/\omega\|_{L^{p'}(\Omega)}\right)\|u\|_X.
\end{aligned}$$

Therefore, in (3.9),

$$\gamma\,\|u\|_X^p \leq \left(C_\Omega\,\|f_0/\omega\|_{L^{p'}(\Omega,\omega)} + \sum_{j=1}^n \|f_j/\omega\|_{L^{p'}(\Omega,\omega)}\right)\|u\|_X,$$

and we obtain

$$\|u\|_X \leq \frac{1}{\gamma^{p'/p}}\left(C_\Omega\,\|f_0/\omega\|_{L^{p'}(\Omega,\omega)} + \sum_{j=1}^n \|f_j/\omega\|_{L^{p'}(\Omega,\omega)}\right)^{p'/p}.$$

Example. Let $\Omega = \{(x,y)\in\mathbb{R}^2 : x^2+y^2 < 1\}$, the weight functions $\omega(x,y) = (x^2+y^2)^{-1/2}$ and $v(x,y) = (x^2+y^2)^{-1/3}$ ($\omega\in A_4$, $v\in A_3$, $p=4$ and $q=3$), and the function

$$\begin{aligned}
&\mathcal{A} : \Omega\times\times\mathbb{R}^2\to\mathbb{R}^2\\
&\mathcal{A}((x,y),\eta,\xi) = h_2(x,y)\,|\xi|\,\xi,
\end{aligned}$$

where $h(x,y) = 2\,\mathrm{e}^{(x^2+y^2)}$. Let us consider the partial differential operator

$$Lu(x,y) = \Delta\big[\omega(x,y)\,|\Delta u|^2\,\Delta u + v(x,y)\,|\Delta u|\Delta u\big] - \operatorname{div}\left((x^2+y^2)^{-1/2}\,\mathcal{A}((x,y),u,\nabla u)\right).$$

Therefore, by Theorem 1.1, the problem

$$(P)\begin{cases} Lu(x) = \dfrac{\cos(xy)}{(x^2+y^2)} - \dfrac{\partial}{\partial x}\left(\dfrac{\sin(xy)}{(x^2+y^2)}\right) - \dfrac{\partial}{\partial y}\left(\dfrac{\sin(xy)}{(x^2+y^2)}\right), & \text{in } \Omega\\ u(x) = 0, \quad \text{on } \partial\Omega \end{cases}$$

has a unique solution $u\in X = W^{2,4}(\Omega,\omega)\cap W_0^{1,4}(\Omega,\omega)$.

Corollary 3.2. *Under the assumptions of Theorem* 1.1 *with* $2 \le q < p < \infty$*, if* $u_1, u_2 \in X$ *are solutions of*

$$(P_1)\begin{cases} Lu_1 = f_0 - \sum_{j=1}^{n} D_j f_j & \text{in } \Omega,\\ u_1(x) = \Delta u_1(x) = 0 & \text{on } \partial\Omega,\end{cases}$$

and

$$(P_2)\begin{cases} Lu_2 = g_0 - \sum_{j=1}^{n} D_j g_j & \text{in } \Omega,\\ u_2(x) = \Delta u_2(x) = 0 & \text{on } \partial\Omega,\end{cases}$$

then

$$\|u_1 - u_2\|_X \le \frac{1}{\alpha^{1/(p-1)}}\left(C_\Omega \left\|\frac{f_0 - g_0}{\omega}\right\|_{L^{p'}(\Omega,\omega)} + \sum_{j=1}^{n}\left\|\frac{f_j - g_j}{\omega}\right\|_{L^{p'}(\Omega,\omega)}\right)^{1/(p-1)},$$

where α *is a positive constant and* C_Ω *is the constant in Theorem* 2.2*.*

Proof. If u_1 and u_2 are the solutions of $(P1)$ and $(P2)$, then for all $\varphi \in X$, we have

$$\begin{aligned} &\int_\Omega |\Delta u_1|^{p-2}\Delta u_1\,\Delta\varphi\,\omega\,dx + \int_\Omega |\Delta u_1|^{q-2}\Delta u_1\,\Delta\varphi\,v\,dx + \int_\Omega \mathcal{A}(x,u_1,\nabla u_1)\cdot\nabla\varphi\,\omega\,dx\\ &-\left(\int_\Omega |\Delta u_2|^{p-2}\Delta u_2\,\Delta\varphi\,\omega\,dx + \int_\Omega |\Delta u_2|^{q-2}\Delta u_2\,\Delta\varphi\,v\,dx + \int_\Omega \mathcal{A}(x,u_2,\nabla u_2)\cdot\nabla\varphi\,\omega\,dx\right)\\ &= \int_\Omega (f_0 - g_0)\,\varphi\,dx + \sum_{j=1}^{n}\int_\Omega (f_j - g_j) D_j\varphi\,dx. \end{aligned} \tag{3.10}$$

In particular, for $\varphi = u_1 - u_2$, we obtain in (3.10):

(i) By Lemma 2.3 (b) and since $2 \le q < p < \infty$, there exist two positive constants β_p and β_q such that

$$\begin{aligned} &\int_\Omega \left(|\Delta u_1|^{p-2}\Delta u_1 - |\Delta u_2|^{p-2}\Delta u_2\right)\Delta(u_1 - u_2)\,\omega\,dx\\ &\ge \beta_p \int_\Omega \left(|\Delta u_1| + |\Delta u_2|\right)^{p-2} |\Delta u_1 - \Delta u_2|^2\,\omega\,dx\\ &\ge \beta_p \int_\Omega |\Delta u_1 - \Delta u_2|^{p-2}|\Delta u_1 - \Delta u_2|^2\,\omega\,dx = \beta_p \int_\Omega |\Delta(u_1 - u_2)|^p\,\omega\,dx, \end{aligned}$$

and, analogously,

$$\int_\Omega \left(|\Delta u_1|^{q-2}\Delta u_1 - |\Delta u_2|^{q-2}\Delta u_2\right)\Delta(u_1 - u_2)\,v\,dx \ge \beta_q \int_\Omega |\Delta(u_1 - u_2)|^q\,v\,dx \ge 0.$$

(ii) By condition (H2)

$$\int_\Omega \left(\mathcal{A}(x,u_1,\nabla u_1) - \mathcal{A}(x,u_2,\nabla u_2)\right).\nabla(u_1 - u_2)\,\omega\,dx \ge \theta_1 \int_\Omega |\nabla u_1 - \nabla u_2|^p\,\omega\,dx.$$

(iii) By condition (H6) and Theorem 2.2, we also have

$$\begin{aligned} &\left|\int_\Omega (f_0 - g_0)\,(u_1 - u_2)\,dx + \sum_{j=1}^{n}\int_\Omega (f_j - g_j) D_j(u_1 - u_2)\,dx\right|\\ &\le C_\Omega \left\|\frac{f_0 - g_0}{\omega}\right\|_{L^{p'}(\Omega,\omega)} \|\nabla(u_1 - u_2)\|_{L^p(\Omega,\omega)} + \left(\sum_{j=1}^{n}\left\|\frac{f_j - g_j}{\omega}\right\|_{L^{p'}(\Omega,\omega)}\right)\|\nabla(u_1 - u_2)\|_{L^p(\Omega,\omega)} \end{aligned}$$

$$\leq\left(C_\Omega\left\|\frac{f_0-g_0}{\omega}\right\|_{L^{p'}(\Omega,\omega)}+\sum_{j=1}^{n}\left\|\frac{f_j-g_j}{\omega}\right\|_{L^{p'}(\Omega,\omega)}\right)\|u_1-u_2\|_X.$$

Hence, with $\alpha=\min\{\beta_p,\theta_1\}$, we obtain

$$\begin{aligned}\alpha\,\|u_1-u_2\|_X^p&\leq\beta_p\int_\Omega|\Delta(u_1-u_2)|^p\,\omega\,dx+\theta_1\int_\Omega|\nabla u_1-\nabla u_2)|^p\,\omega\,dx\\&\leq\left(C_\Omega\left\|\frac{f_0-g_0}{\omega}\right\|_{L^{p'}(\Omega,\omega)}+\sum_{j=1}^{n}\left\|\frac{f_j-g_j}{\omega}\right\|_{L^{p'}(\Omega,\omega)}\right)\|u_1-u_2\|_X.\end{aligned}$$

Therefore, since $2\leq q<p<\infty$,

$$\|u_1-u_2\|_X\leq\frac{1}{\alpha^{1/(p-1)}}\left(C_\Omega\left\|\frac{f_0-g_0}{\omega}\right\|_{L^{p'}(\Omega,\omega)}+\sum_{j=1}^{n}\left\|\frac{f_j-g_j}{\omega}\right\|_{L^{p'}(\Omega,\omega)}\right)^{1/(p-1)}.\qquad\square$$

Corollary 3.3. *Assume $2\leq q<p<\infty$. Let the assumptions of Theorem 1.1 be fulfilled, and let $\{f_{0m}\}$ and $\{f_{jm}\}$ $(j=1,\dots,n)$ be sequences of functions satisfying $\dfrac{f_{0m}}{\omega}\to\dfrac{f_0}{\omega}$ in $L^{p'}(\Omega,\omega)$ and $\dfrac{f_{jm}}{\omega}\to\dfrac{f_j}{\omega}$ in $L^{p'}(\Omega,\omega)$ as $m\to\infty$. If $u_m\in X$ is a solution of the problem*

$$(P_m)\begin{cases}Lu_m(x)=f_{0m}(x)-\displaystyle\sum_{j=1}^{n}D_jf_{jm} & \text{in}\ \ \Omega,\\ u_m(x)=\Delta u_m(x)=0 & \text{on}\ \ \partial\Omega,\end{cases}$$

then $u_m\to u$ in X and u is a solution of problem (P).

Proof. By Corollary 3.2, we have

$$\|u_m-u_k\|_X\leq\frac{1}{\alpha^{1/(p-1)}}\left(C_\Omega\left\|\frac{f_{0m}-f_{0k}}{\omega}\right\|_{L^{p'}(\Omega,\omega)}+\sum_{j=1}^{n}\left\|\frac{f_{jm}-f_{jk}}{\omega}\right\|_{L^{p'}(\Omega,\omega)}\right)^{1/(p-1)}.$$

Therefore $\{u_m\}$ is a Cauchy sequence in X. Hence, there is $u\in X$ such that $u_m\to u$ in X. We find that u is a solution of problem (P). In fact, since u_m is a solution of (P_m), for all $\varphi\in X$ we have

$$\begin{aligned}&\int_\Omega|\Delta u|^{p-2}\Delta u\,\Delta\varphi\,\omega\,dx+\int_\Omega|\Delta u|^{q-2}\Delta u\,\Delta\varphi\,v\,dx+\int_\Omega\mathcal{A}(x,u,\nabla u)\cdot\nabla\varphi\,\omega\,dx\\&=\int_\Omega\Big(|\Delta u|^{p-2}\Delta u-|\Delta u_m|^{p-2}\Delta u_m\Big)\,\Delta\varphi\,\omega\,dx+\int_\Omega\Big(|\Delta u|^{q-2}\Delta u-|\Delta u_m|^{q-2}\Delta u_m\Big)\Delta\varphi\,v\,dx\\&+\int_\Omega\Big(\mathcal{A}(x,u,\nabla u)-\mathcal{A}(x,u_m,\nabla u_m)\Big)\cdot\nabla\varphi\,\omega\,dx\\&+\int_\Omega|\Delta u_m|^{p-2}\Delta u_m\,\Delta\varphi\,\omega\,dx+\int_\Omega|\Delta u_m|^{q-2}\Delta u_m\,\Delta\varphi\,v\,dx+\int_\Omega\mathcal{A}(x,u_m,\nabla u_m)\cdot\nabla\varphi\,\omega\,dx\\&=I_1+I_2+I_3+\int_\Omega f_{0m}\varphi\,dx+\sum_{j=1}^{n}\int_\Omega f_{jm}D_j\varphi\,dx\\&=I_1+I_2+I_3+\int_\Omega f_0\,\varphi\,dx+\sum_{j=1}^{n}\int_\Omega f_jD_j\varphi\,dx\\&+\int_\Omega(f_{0m}-f_0)\,\varphi\,dx+\sum_{j=1}^{n}\int_\Omega(f_{jm}-f_j)D_j\varphi\,dx,\end{aligned}\tag{3.11}$$

where

$$I_1 = \int_\Omega \Big(|\Delta u|^{p-2}\Delta u - |\Delta u_m|^{p-2}\Delta u_m\Big)\,\Delta\varphi\,\omega\,dx,$$

$$I_2 = \int_\Omega \Big(|\Delta u|^{q-2}\Delta u - |\Delta u_m|^{q-2}\Delta u_m\Big)\Delta\varphi\,v\,dx,$$

$$I_3 = \int_\Omega \Big(\mathcal{A}(x,u,\nabla u) - \mathcal{A}(x,u_m,\nabla u_m)\Big)\cdot\nabla\varphi\,\omega\,dx.$$

We find that:
(1) by Lemma 2.3(a), there exists $\alpha_p > 0$ such that

$$\begin{aligned}|I_1| &\le \int_\Omega \big||\Delta u|^{p-2}\Delta u - |\Delta u_m|^{p-2}\Delta u_m\big|\,|\Delta\varphi|\,\omega\,dx\\ &\le \alpha_p \int_\Omega |\Delta u - \Delta u_m|\,(|\Delta u| + |\Delta u_m|)^{p-2}\,|\Delta\varphi|\,\omega\,dx.\end{aligned}$$

Let $\delta = p/(p-2)$. Since $\dfrac{1}{p} + \dfrac{1}{p} + \dfrac{1}{\delta} = 1$, by the Generalized Hölder inequality we obtain

$$\begin{aligned}|I_1| &\le \alpha_p\left(\int_\Omega |\Delta u - \Delta u_m|^p\,\omega\,dx\right)^{1/p}\left(\int_\Omega |\Delta\varphi|^p\,\omega\,dx\right)^{1/p}\left(\int_\Omega (|\Delta u| + |\Delta u_m|)^{(p-2)\delta}\,\omega\,dx\right)^{1/\delta}\\ &\le \alpha_p\|u-u_m\|_X\,\|\varphi\|_X\||\Delta u| + |\Delta u_m|\|^{(p-2)}_{L^p(\Omega,\omega)}.\end{aligned}$$

Now, since $u_m \to u$ in X, there exists a constant $M > 0$ such that $\|u_m\|_X \le M$. Hence,

$$\||\Delta u| + |\Delta u_m|\|_{L^p(\Omega,\omega)} \le \|u\|_X + \|u_m\|_X \le 2M. \tag{3.12}$$

Therefore,

$$\begin{aligned}|I_1| &\le \alpha_p\,(2M)^{p-2}\,\|u-u_m\|_X\,\|\varphi\|_X\\ &= C_1\,\|u-u_m\|_X\,\|\varphi\|_X,\end{aligned}$$

where $C_1 = \alpha_p(2M)^{p-2}$.
(2) By Lemma 2.3 (a) there exists a positive constant α_q such that

$$\begin{aligned}|I_2| &\le \int_\Omega \big||\Delta u|^{q-2}\Delta u - |\Delta u_m|^{q-2}\Delta u_m\big|\,|\Delta\varphi|\,v\,dx\\ &\le \alpha_q \int_\Omega |\Delta u - \Delta u_m|\,(|\Delta u| + |\Delta u_m|)^{q-2}\,|\Delta\varphi|\,v\,dx.\end{aligned}$$

Let $\varepsilon = q/(q-2)$ (if $2 < q < p < \infty$). Since $\dfrac{1}{q} + \dfrac{1}{q} + \dfrac{1}{\varepsilon} = 1$, by the Generalized Hölder inequality, we obtain

$$\begin{aligned}|I_2| &\le \alpha_q\left(\int_\Omega |\Delta u - \Delta u_m|^q\,v\,dx\right)^{1/q}\left(\int_\Omega |\Delta\varphi|^q\,v\,dx\right)^{1/q}\left(\int_\Omega (|\Delta u| + |\Delta u_m|)^{(q-2)\varepsilon}\,v\,dx\right)^{1/\varepsilon}\\ &= \alpha_q\,\|\Delta u - \Delta u_m\|_{L^q(\Omega,v)}\,\|\Delta\varphi\|_{L^q(\Omega,v)}\||\Delta u| + |\Delta u_m|\|^{q-2}_{L^q(\Omega,v)}.\end{aligned}$$

Now, by Remark 2.4 and (3.12), we have

$$\begin{aligned}|I_2| &\le \alpha_q\,C_{p,q}\|\Delta u - \Delta u_m\|_{L^p(\Omega,\omega)}\,C_{p,q}\|\Delta\varphi\|_{L^p(\Omega,\omega)}C_{p,q}^{q-2}\||\Delta u| + |\Delta u_m|\|^{q-2}_{L^p(\Omega,\omega)}\\ &\le \alpha_q\,C_{p,q}^q\|u-u_m\|_X\|\varphi\|_X\ (2M)^{q-2}\\ &= C_2\,\|u-u_m\|_X\ \|\varphi\|_X,\end{aligned}$$

where $C_2 = (2M)^{q-2}\alpha_q\, C^q_{p,q}$.

In case $q = 2$, we have $I_2 = \int\limits_\Omega (\,\Delta u - \Delta u_m)\,\Delta\varphi\, v\, dx$, and by Remark 2.4, we obtain

$$\begin{aligned}|I_2| \le& \|\Delta u - \Delta u_m\|_{L^2(\Omega,v)} \|\Delta\varphi\|_{L^2(\Omega,v)}\\ \le& C^2_{p,2} \|\Delta u - \Delta u_m\|_{L^p(\Omega,\omega)} \|\Delta\varphi\|_{L^p(\Omega,\omega)}\\ \le& C^2_{p,2}\|u - u_m\|_X \|\varphi\|_X.\end{aligned}$$

By Step 1 (Theorem 1.1), we also have

$$\begin{aligned}|I_3| \le& \sum_{j=1}^n \int\limits_\Omega |\mathcal{A}_j(x,u,\nabla u) - \mathcal{A}_j(x,u_m,\nabla u_m)|\,|D_j\varphi|\,\omega\, dx\\ =& \sum_{j=1}^n \int\limits_\Omega |F_j(u) - F_j(u_m)||D_j\varphi|\,\omega\, dx\\ \le& \left(\sum_{j=1}^n \|F_j(u) - F_j(u_m)\|_{L^{p'}(\Omega,\omega)}\right)\|\nabla\varphi\|_{L^p(\Omega,\omega)}\\ \le& \left(\sum_{j=1}^n \|F_j(u) - F_j(u_m)\|_{L^{p'}(\Omega,\omega)}\right)\|\varphi\|_X.\end{aligned}$$

Therefore, we have $I_1, I_2, I_3, \to 0$ as $m \to \infty$.

(3) We also have

$$\left|\int\limits_\Omega (f_{0m} - f_0)\,\varphi\, dx + \sum_{j=1}^n \int\limits_\Omega (f_{jm} - f_j) D_j\varphi\, dx\right|$$

$$\left(C_\Omega \left\|\frac{f_{0m} - f_0}{\omega}\right\|_{L^{p'}(\Omega,\omega)} + \sum_{j=1}^n \left\|\frac{f_{jm} - f_j}{\omega}\right\|_{L^{p'}(\Omega,\omega)}\right)\|\varphi\|_X \to 0,$$

as $m \to \infty$.

Therefore, in (3.11), as $m \to \infty$, we obtain

$$\begin{aligned}&\int\limits_\Omega |\Delta u|^{p-2}\Delta u\,\Delta\varphi\,\omega\, dx + \int\limits_\Omega |\Delta u|^{q-2}\Delta u\,\Delta\varphi\, v\, dx\\ &+ \int\limits_\Omega \mathcal{A}(x,u,\nabla u)\cdot\nabla\varphi\,\omega\, dx\\ &= \int\limits_\Omega f_0\,\varphi\, dx + \sum_{j=1}^n \int\limits_\Omega f_j j D_j\varphi\, dx,\end{aligned}$$

i.e., u is a solution of problem (P). □

References

1. L. E. Andersson, T. Elfving, G. H. Golub, Solution of biharmonic equations with application to radar imaging. *J. Comput. Appl. Math.* **94** (1998), no. 2, 153–180.
2. D. Bresch, J. Lemoine, F. Guíllen–Gonzalez, A note on a degenerate elliptic equation with applications for lakes and seas. *Electron. J. Differential Equations* **2004**, no. 42, 1–13.
3. A. C. Cavalheiro, Existence of solutions for Dirichlet problem of some degenerate quasilinear elliptic equations. *Complex Var. Elliptic Equ.* **53** (2008), no. 2, 185–194.
4. A. C. Cavalheiro, Existence and uniqueness of solutions for some degenerate nonlinear Dirichlet problems. *J. Appl. Anal.* **19** (2013), no. 1, 41–54.
5. A. C. Cavalheiro, Existence results for Dirichlet problems with degenerated p-Laplacian. *Opuscula Math.* **33** (2013), no. 3, 439–453.
6. A. C. Cavalheiro, *Topics on Degenerate Elliptic Equations.* Lambert Academic Publishing, Germany, 2018.

7. M. Chipot, *Elliptic Equations: An Introductory Course*. Birkhäuser Verlag, Basel, 2009.
8. M. Colombo, *Flows of Non-Smooth Vector Fields and Degenerate Elliptic Equations: With Applications to the Vlasov-Poisson and Semigeostrophic Systems*. 22. Edizioni della Normale, Pisa, 2017.
9. P. Drábek, A. Kufner, F. Nicolosi, *Quasilinear Elliptic Equations with Degenerations and Singularities*. Walter de Gruyter & Co., Berlin, 1997.
10. E. Fabes, C. Kenig, R. Serapioni, The local regularity of solutions of degenerate elliptic equations. *Comm. Partial Differential Equations* **7** (1982), no. 1, 77–116.
11. S. Fučik, O. John, A. Kufner, *Function Spaces*. Noordhoff International Publishing, Leyden; Academia, Prague, 1977.
12. J. Garcia-Cuerva, J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*. Weighted norm inequalities and related topics. North-Holland Mathematics Studies, 116. North-Holland Publishing Co., Amsterdam, 1985.
13. D. Gilbarg, N. S. Trudinger, *Elliptic Partial Equations of Second Order*. Second edition. Springer-Verlag, Berlin, 1983.
14. J. Heinonen, T. Kilpeläinen, O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations*. Oxford Math. Monographs. The Clarendon Press, Oxford University Press, New York, 1993.
15. A. Kufner, *Weighted Sobolev Spaces*. John Wiley & Sons, Inc., New York, 1985.
16. A. Kufner, B. Opic, How to define reasonably weighted Sobolev spaces. *Comment. Math. Univ. Carolin.* **25** (1984), no. 3, 537–554.
17. M. C. Lai, H. C. Liu, Fast direct solver for the biharmonic equation on a disk and its application to incompressible flows. *Appl. Math. Comput.* **164** (2005), no. 3, 679–695.
18. B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function. *Trans. Amer. Math. Soc.* **165** (1972), 207–226.
19. M. Talbi, N. Tsouli, On the spectrum of the weighted p-biharmonic operator with weight. *Mediterr. J. Math.* **4** (2007), no. 1, 73–86.
20. A. Torchinsky, *Real-Variable Methods in Harmonic Analysis*. Academic Press, Inc., Orlando, FL, 1986.
21. B. O. Turesson, *Nonlinear Potential Theory and Weighted Sobolev Spaces*. Lecture Notes in Mathematics, 1736. Springer-Verlag, Berlin, 2000.
22. E. Zeidler, *Nonlinear Functional Analysis and its Applications*. I. Springer-Verlag, New York, 1990.
23. E. Zeidler, *Nonlinear Functional Analysis and its Applications*. II/B. Springer-Verlag, New York, 1990.

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