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ON SOME PROPERTIES OF PRIMITIVE POLYHEDRONS

SHALVA BERIASHVILI

Abstract. It is shown that in the three-dimensional Euclidean space a convex pentagonal prism is not a primitive polyhedron. Some properties of primitive polyhedrons are investigated and the associated dual graphs are considered.

Decomposing a geometric object into simpler parts is one of the most fundamental topics in geometry (especially in combinatorial, discrete and computational geometry).

In the Euclidean plane $\mathbf{R}^2$ one can consider triangulations of a simple polygon, without adding new vertices, and it is well known that every simple polygon can be triangulated in such a manner (see especially [2]). It is also known that any simple n-gon can be decomposed into $n-2$ triangles by using exactly $n-3$ its interior diagonals. In fact, the natural numbers $n-3$ and $n-2$ turn out to be invariants for triangulations of a simple n-gon without adding new vertices. Also, for any natural number $n \geq 3$, there exists a simple n-gon in $\mathbf{R}^2$ which admits only one triangulation without adding new vertices.

For the Euclidean space $\mathbf{R}^m$ whose dimension m is strictly greater than 2, the situation is radically different. Recall that in [14] one can find an example of a simple three-dimensional polyhedron P in $\mathbf{R}^3$ such that the number of all vertices of P is equal to 6 and P does not admit a triangulation without adding new vertices. At the same time, P admits triangulations via adding the necessary number of new vertices.

It should be mentioned that if Q is a convex polyhedron in the space $\mathbf{R}^3$, with a given number n of its vertices, then, in general, there are no invariants similar to $n-2$ and $n-3$ as in the case of the Euclidean plane $\mathbf{R}^2$. Indeed, it may happen that there are two triangulations of Q, without adding new vertices, such that the total number of tetrahedra in the first triangulation differs from the total number of tetrahedra in the second triangulation. Thus, one may conclude that in the Euclidean space $\mathbf{R}^m$, where $m > 2$, any convex polyhedron Q admits a triangulation without adding new vertices, but the total number of simplexes of the triangulation is not uniquely determined by Q. So, one can only speak of certain lower and upper estimates for this number, e.g., in terms of $v(Q)$, where $v(Q)$ denotes the total number of vertices of Q.

Many works and monographs were devoted to those questions and topics which are connected (more or less) with triangulations and decompositions of simple and convex polyhedrons in the Euclidean space (see, e.g., [1–5], [7–9], [10,12,13]).

In this article we would like to consider a certain class of convex polyhedrons (primarily, in the space $\mathbf{R}^3$), which will be called primitive polyhedrons (cf. [8]).

Throughout the article, we use the following standard notation:

$\mathbf{N}$ is the set of all natural numbers;

$\mathbf{R}$ is the set of all real numbers;

$\mathbf{R}^m$ is the m-dimensional Euclidean space, where $m \geq 1$.

For our further purpose, we shall need some notions and lemmas.

If P is an m-dimensional convex polyhedron in the Euclidean space $\mathbf{R}^m$, then $s(P)$ denotes the smallest number of m-dimensional simplexes into which this P can be decomposed.

Let Q be a convex m-dimensional polyhedron in the Euclidean space $\mathbf{R}^m$.

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A convex m-dimensional polyhedron $Q^{'}$ is called a primitive extension of Q if there exist an $(m-1)$-dimensional face D of Q and an m-dimensional simplex T in $\mathbf{R}^m$ such that D is also a face of T and the following two conditions are fulfilled:

(*) $T \cap Q = D$.

(**) The set of all vertices of $Q^{'}$ coincides with the union of the sets of all vertices of Q and T.

In particular, $Q^{'}$ can be obtained by adding to Q some m-dimensional simplex T, whose base is one of the facets of Q (and no vertex of Q is lost after adding T to Q).

In our further considerations we shall say that the above-mentioned simplex T is extreme for the polyhedron Q'. In the case $m = 2$, the term "ear" is commonly used for such T in Q'.

Let now $\{Q_1, Q_2, \dots, Q_k\}$ be a finite sequence of convex m-dimensional polyhedrons in the space $\mathbf{R}^m$.

We shall say that this sequence is primitive if Q_1 is an m-dimensional simplex and, for each natural index $i \in [1, k-1]$, the polyhedron Q_{i+1} is a primitive extension of Q_i.

A convex m-dimensional polyhedron $Q \subset \mathbf{R}^m$ is called primitive if $Q = Q_k$ for some primitive sequence $\{Q_1, Q_2, \dots, Q_k\}$ of convex m-dimensional polyhedrons in $\mathbf{R}^m$.

Some nontrivial properties of primitive polyhedrons, connected with their decompositions into simplexes, are discussed in [8]. In particular, it is shown in Chapter 6 of [8] that for $m > 3$ no m-dimensional cube (parallelepiped) is a primitive polyhedron. On the other hand, it is easy to show that if $m \leq 3$, then all m-dimensional parallelepipeds are primitive polyhedrons.

We also need several simple notions from the theory of finite graphs (see, for example, the well-known monographs [6] or [11]).

Let P be a simple (in particular, convex) m-dimensional polyhedron in the Euclidean space $\mathbf{R}^m$ and let $\{T_i : 1 \leq i \leq n\}$ be a triangulation of P into m-dimensional simplexes.

The dual graph $\Gamma = (V, E)$ of this triangulation is defined as follows. The set V of vertices of Γ is obtained by choosing in every simplex T_i some interior point of T_i (more concretely, one may choose the barycenter t_i of T_i). Two verices t_i and t_j from V are connected by an edge from E if and only if the simplexes T_i and T_j are neighbors, i.e., if and only if there exists a common $(m-1)$-dimensional face of T_i and T_j.

Example 1. Let P be a simple n-gon in the plane $\mathbf{R}^2$ and let $\{T_i : 1 \leq i \leq n-2\}$ be any triangulation of P into triangles, without adding new vertices. It is not hard to see that the dual graph of this triangulation is a tree and every vertex of the dual graph is incident to at most three edges. Conversely, if one has a tree, all vertices of which are incident to at most three edges, then there exist a convex polygon in $\mathbf{R}^2$ and its triangulation, without adding new vertices, such that the dual graph of the triangulation is isomorphic to the given tree. The analogous statement fails to be valid for all convex polyhedrons in the space $\mathbf{R}^3$, but in some another form holds true for primitive polyhedrons (see Theorem 1 below).

Lemma 1. *For any convex three-dimensional polyhedron P in the space $\mathbf{R}^3$ with the number of vertices $v(P)$, the inequality*

$$v(P) - 3 \leq s(P)$$

holds true.

Sketch of the proof. Suppose to the contrary that the above-mentioned inequality fails to be satisfied for some convex three-dimensional polyhedra $P \subset \mathbf{R}^3$. Obviously, in such a case we may choose a convex three-dimensional polyhedron P for which

$$s(P) + 3 < v(P)$$

and the value $s(P)$ is minimal. Consider a dissection

$$\{T_i : 1 \leq i \leq s(P)\}$$

of P into $s(P)$ many tetrahedra.

Only the following two cases are possible.

Case 1. For some natural index $j \in \{1, 2, \dots, s(P)\}$, the tetrahedron T_j has three facets each of which lies in the corresponding facet of P.

Case 2. Every tetrahedron from the family $\{T_i : 1 \leq i \leq s(P)\}$ has at most two facets lying in the corresponding facets of P.

A combinatorial argument based on the classical Euler formula for convex three-dimensional polyhedrons, in both these cases leads to a contradiction with the minimality of $s(P)$ (for more details, see Chapter 7 in [8]). The obtained contradiction shows that neither Case 1 nor Case 2 is possible, so the inequality

$$v(P) - 3 \leq s(P)$$

must be valid for all three-dimensional convex polyhedrons $P \subset \mathbf{R}^3$.

Lemma 2. *Let P be a three-dimensional convex polyhedron in the space $\mathbf{R}^3$ and let $v = v(P)$ denote the number of all vertices of P.*

Then the polyhedron P is primitive if and only if the equality $s(P) = v - 3$ holds true.

The proof of Lemma 2 can be found in [8].

Lemma 3. *Let P be a three-dimensional simple polyhedron in the space $\mathbf{R}^3$ and let $\{T_i : 1 \leq i \leq n\}$ be a triangulation of P into tetrahedra, without adding new vertices.*

Then the dual graph of this triangulation is connected (but, in general, it is not a tree).

We omit an easy proof of this lemma.

Lemma 4. *Let P be a convex m-dimensional primitive polyhedron in the space $\mathbf{R}^m$.*

Then there exists a triangulation of P, without adding new vertices, such that the corresponding dual graph is a tree.

Proof. We use the method of induction according to the complexity of geometric structure of P or, equivalently, we use induction on the total number $v(P)$ of vertices of P.

If P is an m-dimensional simplex, it is clear that its dual graph is a singleton, hence is a trivial tree with only one vertex and without edges.

Suppose now that P is a convex m-dimensional primitive polyhedron in $\mathbf{R}^m$ with $v(P) \geq m + 2$. From the definition of m-dimensional convex primitive polyhedrons it follows that P has at least one extreme simplex. So, we can pick an extreme simplex T of P and consider the reduced polyhedron P which is obtained from P by removing this simplex T. Obviously, we have the inequality $v(P') < v(P)$. Applying the inductive assumption to P', we can construct one of the triangulations of P', without adding new vertices, such that its dual graph is a tree Γ. It is not hard to see, keeping in mind the fact that T has a common facet with P', this tree can be expanded to a tree which will be the dual graph of the initial polyhedron PS. Indeed, it suffices to add to Γ one additional vertex and one additional edge corresponding to the extreme simplex T and incident to this vertex. □

Example 2. In the space $\mathbf{R}^3$, consider an arbitrary trigonal bi-pyramid P which has 5 vertices, i.e., $v(P) = 5$, and which is a primitive polyhedron. It is easy to see that there are two types of the dual graphs that are associated with two triangulations of P, without adding new vertices:

(a) one edge, when P is decomposed into two tetrahedra;

(b) 3-cycle, when P is decomposed into three tetrahedra.

This simple example shows that even for a primitive polyhedron Q in the space $\mathbf{R}^3$, a triangulation of Q, without adding new vertices, should be carefully chosen if one wants to obtain a tree as the dual graph of the triangulation.

The following theorem is valid.

Theorem 1. *Let $\Gamma = (V, E)$ be a tree such that the degrees of all vertices of this tree are less than or equal to $m + 1$.*

Then there exist both a convex m-dimensional primitive polyhedron P in the Euclidean space $\mathbf{R}^m$ and a triangulation of P without adding new vertices such that the dual graph of the triangulation is isomorphic to Γ.

Proof. We use the method of induction on $\mathrm{card}(V)$.

If $\mathrm{card}(V) = 1$, then Γ is trivially isomorphic to the dual graph of an m-dimensional simplex, so there is nothing to prove.

Suppose now that $\mathrm{card}(V) \geq 2$ and that the assertion of this theorem has already been established for all those trees whose cardinalities are strictly less than $\mathrm{card}(V)$. As is well known from the graph theory, there exists at least one vertex $v \in V$ incident to exactly one edge e from E. In our argument below, this vertex will be called a leaf of the tree. So, we can pick a leaf v in (V, E).

Consider the reduced graph $\Gamma' = (V', E')$, where

$$V' = V \setminus \{v\}, \qquad E' = E \setminus \{e\}.$$

Obviously, the graph Γ' is also a tree. Applying the inductive assumption to (V', E'), we can find an m-dimensional convex polyhedron P and its triangulation $\{T_i : 1 \leq i \leq n\}$, without adding new vertices, such that the dual graph of $\{T'_i : 1 \leq i \leq n\}$ is isomorphic to Γ. Moreover, we may assume that all facets of P' are the $(m-1)$-dimensional simplexes. Now, it is easy to see how one can construct a primitive extension P of P' and some triangulation

$$\{T'_i : 1 \leq i \leq n\} \cup \{T\}$$

of P such that the dual graph of this extended triangulation would be isomorphic to Γ. Only one delicate moment should be emphasized here: in order to guarantee the convexity of the required polyhedron P, the new vertex of P, being one of the vertices of T, can be taken in the vicinity of the barycenter of a certain facet of P'. □

Lemma 5. *Let P be a convex polygon with $v = v(P)$ vertices in the Euclidean plane $\mathbf{R}^2$ and let P be decomposed into some finitely many triangles $\{T_i : 1 \leq i \leq n\}$, i.e.,*

$$P = \bigcup \{T_i : 1 \leq i \leq n\}$$

and these triangles pairwise have no common interior points.

Then the inequality $n \geq v - 2$ holds true.

Proof. Consider the sum of all interior angles of the triangles $\{T_i : 1 \leq i \leq n\}$. Clearly, it is equal to $\pi \cdot n$. As is well known, the sum of all interior angles of P is equal to $\pi \cdot (v-2)$. So, we can write

$$\pi \cdot (v-2) \leq \pi \cdot n,$$

whence it follows that $v - 2 \leq n$. □

Example 3. For any convex polygon $P \subset \mathbf{R}^2$ denote again by $s(P)$ the minimal cardinality of a dissection of P into triangles. Then we have the equality

$$s(P) = v - 2,$$

where $v = v(P)$ denotes again the number of all vertices of P. This fact is an immediate consequence of Lemma 5. It should be remarked that the analogous statement fails to be true for simple polygons in the plane. For instance, there is a simple polygon in $\mathbf{R}^2$ with 6 vertices which can be decomposed into two triangles.

Lemma 6. *Let P be an arbitrary convex pentagonal prism in the space $\mathbf{R}^3$ and let T be a tetrahedron lying in P.*

Then the inequality $\lambda(T) < \frac{1}{3}\lambda(P)$ holds true, where $\lambda(P)$ denotes the volume of P and $\lambda(T)$ denotes the volume of T.

It is easy to see that any convex quadrilateral prism in the space $\mathbf{R}^3$ is a primitive polyhedron. Moreover, any convex polyhedron in $\mathbf{R}^3$, combinatorially isomorphic to such a prism, is primitive. On the other hand, the next statement is valid.

Theorem 2. *If P is a convex pentagonal prism in the space $\mathbf{R}^3$, then P is not a primitive polyhedron.*

Proof. We use the method of volumes presented and developed in [8]. Suppose to the contrary that the given convex pentagonal prism P is a primitive polyhedron.

By virtue of Lemma 2, we must have the equality

$$s(P) = v(P) - 3,$$

where $v(P)$ denotes the number of vertices of P and $s(P)$ denotes the smallest number of tetrahedra into which the pentagonal prism P can be decomposed. In our case,

$$v(P) = 10, \ s(P) = 10 - 3 = 7.$$

Let us consider the two bases of the prism. By using Lemma 5, we can deduce that each of these bases needs at least 3 tetrahedra of a decomposition. Since no two facets of a tetrahedron are parallel to each other, any tetrahedron corresponding to one base differs from any tetrahedron corresponding to the other base. Consequently, every decomposition of P into tetrahedra contains at least 6 tetrahedra which correspond to the bases of P. The total volume of those tetrahedra does not exceed $(2/3)\lambda(P)$. Now, consider the tetrahedron distinct from all the above-mentioned 6 tetrahedra. Its volume, according to Lemma 6, is strictly less than $(1/3)\lambda(P)$. This circumstance implies that the total volume of seven tetrahedra is strictly less than $\lambda(P)$. Therefore, no seven tetrahedra can constitute a decomposition (dissection) of P. □

Let G be some subgroup of D_m, where D_m denotes the group of all isometric transformations of the space $\mathbf{R}^m$, and let X and Y be two polyhedrons of $\mathbf{R}^m$.

We recall (see, e.g., [1]) that these two polyhedrons are (finitely) G-equidecomposable if there exist two finite disjoint families

$$\{X_k : k \in K\}, \ \{Y_k : k \in K\}$$

of polyhedrons in $\mathbf{R}^m$ such that:

(1) $X = \cup\{X_k : k \in K\}$ and $Y = \cup\{Y_k : k \in K\}$;

(2) for each index $k \in K$, the polyhedron X_k is G-congruent to the polyhedron Y_k;

(3) the polyhedrons X_k (respectively, Y_k) have no pairwise common interior points.

Obviously, if polyhedrons X and Y in $\mathbf{R}^m$ are G-congruent, then they are also (finitely) G-equidecomposable, but the converse assertion is not true, in general.

The finite G-equidecomposability is an equivalence relation in the class of all polyhedrons in the space $\mathbf{R}^m$ (see [1]).

If $G = D_m$, then the G-equicomposability of two polyhedrons X and Y is called simply the equidecomposability of X and Y.

Theorem 3. *Let P and Q be two convex 3-dimensional polyhedrons in $\mathbf{R}^3$ with equal volumes. Then the following six cases can be realized (separately):*

(a) both P and Q are primitive polyhedrons and they are equidecomposable;

(b) both P and Q are primitive polyhedrons and they are not equidecomposable;

(c) P is a primitive polyhedron, Q is not a primitive polyhedron and they are equidecomposable;

(d) P is a primitive polyhedron, Q is not a primitive polyhedron and they are not equidecomposable;

(e) both P and Q are not primitive polyhedrons and they are equidecomposable;

(f) both P and Q are not primitive polyhedrons and they are not equidecomposable.

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Georgian Technical University, 77 Kostava Str., Tbilisi 0175, Georgia
E-mail address: shalva_89@yahoo.com