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EXTENDING THE APPLICABILITY OF AN ULM-NEWTON-LIKE METHOD UNDER GENERALIZED CONDITIONS IN A BANACH SPACE

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Abstract. The aim of this paper is to extend the applicability of an Ulm-Newton-like method for approximating a solution of a nonlinear equation in a Banach space setting. The sufficient local convergence conditions are weaker than those in the earlier works leading to a larger radius of convergence and more precise error estimations on the distances involved. Numerical examples are also provided.

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution x_* of the equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet–differentiable operator defined on a convex subset Ω of a Banach space $\mathcal{B}_1$ with values in a Banach space $\mathcal{B}_2$.

A large number of problems in applied mathematics and also in engineering are solved by finding the solutions of certain equations. For example, dynamic systems are mathematically modeled by the difference or differential equations, and their solutions represent usually the states of the systems. For the sake of simplicity, assume that a time–invariant system is driven by the equation $\dot{x} = R(x)$, for some suitable operator R, where x is the state. Then the equilibrium states are determined by solving equation (1.1). Similar equations are used in the case of discrete systems. The unknowns of engineering equations may be functions (difference, differential, and integral equations), vectors (systems of linear or nonlinear algebraic equations) and real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative, that is, when starting from one or several initial approximations, a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework.

Moser in [13] proposed the following Ulm's-like method for generating a sequence $\{x_n\}$ approximating x_*:

$$x_{n+1} = x_n - B_n F(x_n), \qquad B_{n+1} = 2B_n - B_n F'(x_n) B_n. \tag{1.2}$$

Method (1.2) is useful when the derivative $F'(x_n)$ is not continuously invertible (as in the case of small divisors [1–8, 10, 11, 13–15]). Moser studied the semi-local convergence of method (1.2) and showed that the order of convergence is $1 + \sqrt{2}$ if $F'(x_*) \in L(\mathcal{B}_2, \mathcal{B}_1)$. However, the order of convergence is faster than the Secant method (i.e., $\frac{1+\sqrt{5}}{2}$). The quadratic convergence can be obtained if one uses Ulm's method [14, 15] defined for each $n = 0, 1, 2, \dots$ by

$$\begin{aligned} x_{n+1} &= x_n - B_n F(x_n), \\ B_{n+1} &= 2B_n - B_n F'(x_{n+1}) B_n. \end{aligned} \tag{1.3}$$

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The semi-local convergence of method (1.3) has also been studied in [1–9]. As far as we know, the local convergence analysis of methods (1.2) and (1.3) has not been given. In the present paper, we study the local convergence of Ulm's-like method defined for each $n = 0, 1, 2, 3, \dots$ by

$$x_{n+1} = x_n - B_n F(x_n),\ B_{n+1} = 2B_n - B_n A_{n+1} B_n, \tag{1.4}$$

where A_n is an approximation of $F'(x_n)$. Notice that method (1.4) is inverse free, the computation of $F'(x_n)$ is not required and the method produces successive approximations $\{B_n\} \approx F'(x_*)^{-1}$.

In Section 2, we present the local convergence analysis of method (1.4) and in Section 3, we present the numerical examples.

2. Local Convergence Analysis

The local convergence analysis of method (1.4) is given in this section. Denote by $U(v, \xi)$ and $\overline{U}(x, \xi)$ the open and closed balls in $\mathcal{B}_1$, respectively, with center $v \in \mathcal{B}_1$ and of radius $\xi > 0$.

Let $w_0 : [0, +\infty) \longrightarrow [0, +\infty)$ and $w : [0, +\infty) \longrightarrow [0, +\infty)$ be continuous and nondecreasing functions satisfying $w_0(0) = w(0) = 0$. Let also $q \in [0, 1)$ be a parameter. Define functions φ and ψ on the interval $[0, +\infty)$ by

$$\varphi(t) = \left[q\left(\int_0^1 w(\theta t) d\theta + 1\right) + w_0(t)\right] t$$

and

$$\psi(t) = \varphi(t) - 1.$$

We have that $\psi(0) = -1$ and for sufficiently large $t_0 \geq t$, $\psi(t_0) > 0$. By the intermediate value theorem equation $\psi(t) = 0$ has solutions in the interval $(0, t_0)$. Denote by ρ the smallest such a solution. Then for each $t \in [0, \rho)$, we have

$$0 \leq \psi(t) < 1. \tag{2.1}$$

We need to show an auxiliary perturbation result for method (1.4).

Lemma 2.1. *Let $F : \Omega \subseteq \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be a continuously Fréchet-differentiable operator. Suppose that there exist $x_* \in \Omega$, $\{M_n\} \in L(\mathcal{B}_2, \mathcal{B}_1)$, $\{q_n\}$, $q \in \mathbb{R}_0^+$, continuous and nondecreasing functions $w_0 : [0, +\infty) \longrightarrow [0, +\infty)$ and $w : [0, +\infty) \longrightarrow [0, +\infty)$ such that for each $x \in \Omega$, $n = 0, 1, 2, \dots$ and $\theta \in [0, 1]$*

$$F(x_*) = 0,\ F'(x_*)^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1),$$

$$\|F'(x_*)^{-1}(F'(x_* + \theta(x - x_*)) - F'(x_*))\| \leq w(\theta\|x - x_*\|), \tag{2.2}$$

$$\|F'(x_*)^{-1}(F'(x) - F'(x_*))\| \leq w_0(\theta\|x - x_*\|), \tag{2.3}$$

$$\|F'(x_*)^{-1}(A_n - F'(x_n))\| \leq q_n \|F'(x_*)^{-1} F(x_n)\| \tag{2.4}$$

$$\text{for each}\, x, x_n \in \Omega_0 := \Omega \cap B(x_*, \rho),$$

$$\sup_{n \geq 0} q_n \leq q, \tag{2.5}$$

$$x_n \in B(x_*, r_0)$$

and

$$B(x_*, r_0) \subseteq \Omega,$$

where

$$r_0 \in (0, \rho). \tag{2.6}$$

Then the following items hold

$$\|F'(x_*)^{-1}F(x_n)\| \leq \Big(\int_0^1 w(\theta\|x_n - x_*\|)d\theta + 1\Big)\|x_n - x_*\|, \tag{2.7}$$

$$\|F'(x_*)^{-1}[A_n - F'(x_n)]\| \leq q\Big(\int_0^1 w(\theta\|x_n - x_*\|)d\theta + 1\Big)\|x_n - x_*\|, \tag{2.8}$$

$$A_n^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1) \tag{2.9}$$

and

$$\|A_n^{-1}F'(x_*)\| \leq \frac{1}{1 - \varphi(\|x_n - x_*\|)} \textit{hold}. \tag{2.10}$$

Proof. We shall show first that estimation (2.8) holds. Using (2.1), we have the identity

$$\begin{aligned} F(x_n) =& F(x_n) - F(x_*) = F(x_n) - F(x_*) - F'(x_*)(x_n - x_*) + F'(x_*)(x_n - x_*) \\ =& \int_0^1 [F'(x_* + \theta(x_n - x_*)) - F'(x_*)](x_n - x_*)d\theta. \end{aligned} \tag{2.11}$$

Then by (2.3) and (2.11), we have

$$\begin{aligned} \|F'(x_*)^{-1}F(x_n)\| \leq & \int_0^1 \|F'(x_*)^{-1}[F'(x_* + \theta(x_n - x_*)) - F'(x_*)]\|d\theta\|x_n - x_*\| \\ & + \|x_n - x_*\| \\ \leq & \Big(\int_0^1 w(\theta\|x_n - x_*\|)d\theta + 1\Big)\|x_n - x_*\|, \end{aligned}$$

which shows estimation (2.7). Moreover, by (2.4), (2.5) and (2.7), we obtain

$$\|F'(x_*)^{-1}[A_n - F'(x_n)]\| \leq q_n\|F'(x_*)^{-1}F(x_n)\| \leq q\Big(\int_0^1 w(\theta\|x_n - x_*\|)d\theta + 1\Big)\|x_n - x_*\|,$$

which shows estimation (2.8). Furthermore, using (2.2), (2.3), (2.7), (2.8) and the definition of r_0, we get

$$\begin{aligned} \|F'(x_*)^{-1}[A_n - F'(x_*)]\| \leq & \|F'(x_*)^{-1}[A_n - F'(x_n)]\| \\ & + \|F'(x_*)^{-1}[F'(x_n) - F'(x_*)]\| \\ \leq & \varphi(\|x_n - x_*\|) \\ \leq & \varphi(r_0) < 1. \end{aligned} \tag{2.12}$$

It follows from (2.12) and the Banach lemma on invertible operators [1, 5, 6, 11] that (2.9) and (2.10) hold. □

Remark 2.2. In earlier studies the Lipschitz condition [1–15]

$$\|F'(x_*)^{-1}[F'(x) - F'(y)]\| \leq w_1(\|x - y\|) \text{ for each } x, y, \in \Omega \tag{2.13}$$

is used which is stronger than our conditions (2.2) and (2.3). Notice also that since $\Omega_0 \subseteq \Omega$,

$$w(t) \leq w_1(t) \tag{2.14}$$

and

$$w_0(t) \leq w_1(t), \tag{2.15}$$

where the function w_1 is the same as the function w, but defined on Ω instead of Ω_0. The ratio $\frac{w_0}{w_1}$ may be arbitrarily large [1, 5, 6]. Moreover, if (2.13) is used instead of (2.2) and (2.3) in the proof of

Lemma 2.1, then the conclusions hold provided that r_0 is replaced by r_1 which is the smallest positive solution of the equation

$$\psi_1(t) = 0, \tag{2.16}$$

where $\psi_1(t) = \varphi_1(t) - 1$ and $\varphi_1(t) = [q(\int_0^1 w_1(\theta t)d\theta + 1) + w_1(t)]t$. It follows from (2.7), (2.14), (2.15), (2.16) that

$$r_1 \leq r_0. \tag{2.17}$$

Furthermore, the strict inequality holds in (2.17), if (2.14) or (2.15) hold as strict inequalities. Finally, estimations (2.8) and (2.9) are tighter than the corresponding ones (using (2.13)) given by

$$\|F'(x_*)^{-1}[A_n - F'(x_n)]\| \leq q\Big(\int_0^1 w_1(\theta\|x_n - x_*\|)d\theta + 1\Big)\|x_n - x_*\|.$$

Let λ be a parameter satisfying $\lambda \in [0, 1)$. Let also $w_2 : [0, \rho) \longrightarrow [0, +\infty)$ be a continuous and nondecreasing function. Moreover, define the functions $\alpha : [0, \rho) \longrightarrow [0, +\infty), \beta : [0, \rho) \longrightarrow [0, +\infty), f : [0, \rho) \longrightarrow [0, +\infty)$ and $g : \beta : [0, \rho) \longrightarrow [0, +\infty)$, by $\alpha(t) = \frac{1}{1-\varphi(t)}$, $\beta(t) = 2q(1 + \int_0^1 w(\theta t)d\theta)t + 2w_0(t)$, $f(t) = \alpha(t)\beta(t) - \lambda$ $g(t) = \lambda^2 + (1 + \lambda^2)\alpha(t)\int_0^1 w_2((1-\theta)t)d\theta - 1$, sequences $\alpha_k, \beta_k, \gamma_k$ by $\alpha_k = \frac{1}{1-\varphi(\|x_k - x_*\|)}$, $\beta_k := q(1 + \int_0^1 w(\theta\|x_{k+1} - x_*\|)d\theta)\|x_{k+1} - x_*\| + q(1 + \int_0^1 w(\theta\|x_k - x_*\|)d\theta)\|x_k - x_*\| + w_0(\|x_{k+1} - x_*\|) + w_0(\|x_k - x_*\|)$, $d_0 = \gamma_0$, $\gamma_k = \|I - B_kA_k\|^2 + 2\|I - B_kA_k\|\|A_{k+1} - A_k\| + \|B_k\|^2\|A_{k+1} - A_k\|^2$, parameters α, β by $\alpha = \alpha(r_0)$, $\beta = \beta(r_0)$ and quadratic equation $(1 + \alpha\beta)t^2 + 2\alpha\beta(1 + \alpha\beta)t + (\alpha\beta)^2 - \lambda^2 = 0$. Then we have $f(0) = -\lambda < 0$ and $f(t) \longrightarrow +\infty$ as $t \longrightarrow \rho^-$. Denote by ρ_0 the smallest solution of equation $f(t) = 0$ in $(0, \rho)$. Then we find that for each $t \in (0, \rho_0)$,

$$0 < \alpha(t)\beta(t) < \lambda.$$

In view of the above inequality, the preceding quadratic equation has both a unique positive solution denoted by ρ_+ and a negative solution. Define parameter γ by

$$0 \leq \gamma < \gamma_0 = \min\{\rho_+, \rho_0, r_0\}. \tag{2.18}$$

Then we have

$$(1 + \alpha\beta)\gamma^2 + 2\alpha\beta(1 + \alpha\beta)\gamma + (\alpha\beta)^2 < \lambda^2.$$

Notice that we also have $\alpha_k \leq \alpha$ and $\beta_k \leq \beta$.

Next, we present the local convergence of method (1.4).

Theorem 2.3. *Under the hypotheses of Lemma* 2.1 *and with* r_0 *given in* (2.6) *for* $\lambda \in [0, 1)$, *we further suppose that there exists the function* $w_2 : [0, r_0) \longrightarrow [0, +\infty)$, *continuous and nondecreasing such that for each* $x \in B(x_*, r_0)$ $\theta \in [0, 1]$ *and*

$$\|A_n^{-1}\| \leq \frac{1}{1 - \varphi_1(\|x_n - x_*\|)} < \varphi_1(r_1)$$

we have

$$\|F'(x_*)^{-1}[F'(x_* + \theta(x - x_*)) - F'(x)]\| \leq w_2((1-\theta)\|x - x_*\|) \tag{2.19}$$

for each $x \in \Omega_0 = \Omega \cap B(x_*, r_0)$,

$$\|I - B_0A_0\| \leq d_0 < \lambda^2 \tag{2.20}$$

and

$$B(x_*, \gamma) \subseteq \Omega,$$

where γ *is given in* (2.18). *Then the sequence* $\{x_n\}$ *generated by method* (1.4) *for* $x_0 \in B(x_*, \gamma) - \{x_*\}$ *is well-defined, remains in* $B(x_*, \gamma)$ *and converges to* x_*.

Proof. By hypothesis (2.20), we have $\|I - B_0A_0\| \leq \gamma_0 < \lambda^2$, so

$$\|I - B_kA_k\| \leq \gamma_k < \lambda^2 \tag{2.21}$$

is true for $k = 0$. Suppose that (2.21) is true for all integers smaller or equal to k. Using Lemma 2.1, we have the estimations

$$\begin{aligned}\|B_k\| =&\|B_kA_kA_k^{-1}\| \leq \|B_kA_k\|\|A_k^{-1}\|\\ \leq&(1 + \|I - B_kA_k\|)\|A_k^{-1}\|\\ \leq&(1 + \gamma_k)\frac{1}{1 - \varphi(\|x_k - x_*\|)} \leq (1 + \gamma_k)\alpha_k.\end{aligned}$$

In view of method (1.4) for $n = k$, we can write in turn that

$$\begin{aligned}x_{k+1} - x_* =&x_k - x_* - B_k(F(x_k) - F(x_*))\\ =&[I - B_kF'(x_k)](x_k - x_*)\\ &+ \int_0^1 B_k(F'(x_k) - F'(x_* + \theta(x_k - x_*))(x_k - x_*)d\theta.\end{aligned} \tag{2.22}$$

Using (2.22), we get

$$\|x_{k+1} - x_*\| \leq \|I - B_kF'(x_k)\|\|x_k - x_*\| + \frac{L_2}{2}\|B_k\|\|x_k - x_*\|,$$

since $\|x_k - x_*\| \leq \rho$ and $\|x_* + \theta(x_k - x_*) - x_*\| \leq \theta\|x_k - x_*\| \leq \rho$. We By Lemma 2.1 and the induction hypotheses we also have

$$\begin{aligned}&\|F'(x_*)^{-1}(A_{k+1}) - A_k)\|\\ \leq&\|F'(x_*)^{-1}(A_{k+1} - F'(x_{k+1}))\|\\ &+ \|F'(x_*)^{-1}(F'(x_{k+1}) - F'(x_k))\| + \|F'(x_*)^{-1}(A_k - F'(x_*))\|\\ \leq&\|F'(x_*)^{-1}(A_{k+1} - F'(x_{k+1}))\| + \|F'(x_*)^{-1}(A_k - F'(x_k))\|\\ &+ \|F'(x_*)^{-1}(F'(x_{k+1}) - F'(x_*))\| + \|F'(x_*)^{-1}(F'(x_k) - F'(x_*))\|\\ \leq&\|F'(x_*)^{-1}(A_{k+1} - F'(x_{k+1}))\| + \|F'(x_*)^{-1}(A_k - F'(x_k))\|\\ &+ \|x_{k+1} - x_*\| + \|x_k - x_*\|\\ \leq&q\left(1 + \int_0^1 w(\theta\|x_{k+1} - x_*\|)d\theta\right)\|x_{k+1} - x_*\|\\ &+ q\left(1 + \int_0^1 w(\theta\|x_k - x_*\|)d\theta\right)\|x_k - x_*\|\\ &+ w_0(\|x_{k+1} - x_*\|) + w_0(\|x_k - x_*\|)\\ \leq&\beta_k \leq \beta.\end{aligned}$$

By the definition of method (1.4), we have the estimations

$$I - B_{k+1}A_{k+1} = I - (2B_k - B_kA_{k+1}B_k)A_{k+1} = (1 - B_kA_{k+1})^2. \tag{2.23}$$

Then by (2.23) and (2.22) for $n = k$, we get

$$\begin{aligned}\|I - B_{k+1}A_{k+1}\| \leq&(\|I - B_kA_k\| + \|B_k\|\|A_{k+1} - A_k\|)^2\\ \leq&\|I - B_kA_k\|^2 + 2\|I - B_kA_k\|\|B_k\|\|A_{k+1} - A_k\|\\ &+ \|B_k\|^2\|A_{k+1} - A_k\|^2\\ \leq&\gamma_k^2 + 2\gamma_k(1 + \gamma_k)\|A_k^{-1}\|\|A_{k+1} - A_k\|\\ &+ (1 + \gamma_k)^2\|A_k^{-1}\|^2\|A_{k+1} - A_k\|^2\\ \leq&\gamma_k^2 + 2\gamma_k(1 + \gamma_k)\alpha\beta + (1 + \gamma_k)^2\alpha^2\beta^2\\ =&(1 + \alpha\beta)^2\gamma_k^2 + 2\alpha\beta(1 + \alpha\beta)\gamma_k + \alpha_k^2\beta_k^2\end{aligned}$$

$$\leq (1+\alpha\beta)^2\gamma^2 + 2\alpha\beta(1+\alpha\beta)\gamma + \alpha^2\beta^2 < \lambda^2,$$

which shows (2.21) for $n = k+1$. Then, using the induction hypotheses, (2.19) and the definition of γ,

$$\begin{aligned}\|x_{k+1} - x_*\| \leq &(\lambda^2 + (1+\lambda^2)\alpha\|x_k - x_*\|)\\ &\times \int_0^1 w_2((1-\theta)\|x_k - x_*\| d\theta \|x_k - x_*\|\\ &< g(\gamma)\|x_k - x_*\| \leq g(\rho_+)\|x_k - x_*\| \leq c\|x_k - x_*\|,\end{aligned}$$

where $c = g(\gamma) \in [0,1)$, so $\lim_{k\longrightarrow\infty} x_k = x_*$ and $x_{k+1} \in B(x_*, \rho)$. □

Remark 2.4.

(a) As is noted in Remark 2.2, conditions (2.3) and (2.4) can be replaced by (2.19).

$$\|F'(x_*)^{-1}[F'(x_* + \theta(x - x_*)) - F'(x)]\| \leq w_3((1-\theta)\|x - x_*\|) \tag{2.24}$$

for each $x \in \Omega$ and $\theta \in [0,1]$, where the function w_3 is the same as w_1.

We have that $w_1(t) \leq w_3(t)$. Then in view of Remark 2.2 and (2.19), the radii of convergence as well as the error bounds are improved under the new approach, since old approaches use only (2.24) with the exception of our approach in [2, 4].

(b) The results obtained here can be used for operators F satisfying autonomous differential equations [1, 5, 6, 11] of the form

$$F'(x) = P(F(x)),$$

where $P : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous operator. Then, since $F'(x_*) = P(F(x_*)) = P(0)$, we can apply the results without actually knowing x_*. For example, let $F(x) = e^x - 1$. Then we can choose $P(x) = x+1$.

(c) The local results obtained here can be used for projection methods such as the Arnoldi's method, the generalized minimum residual method (GMRES), the generalized conjugate method (GCR) for combined Newton/finite projection methods, and in connection with the mesh independence principle can be used to develop the cheapest and most efficient mesh refinement strategies [1, 5, 6].

(d) Let L_0, L, L_1, L_2, L_3 be positive constants. Researchers choose $w_0(t) = L_0 t$, $w(t) = Lt$, $w_1(t) = L_1 t$, $w_2(t) = L_2 t$ and $w_3(t) = L_3 t$. Moreover, if we choose $\Omega_0 = \Omega$ and $L = L_1$, then our results reduce to the ones where the second order of convergence was shown with the Lipschitz conditions given in non-affine invariant form. In Example 3.1, we show that the radii are extended and the upper bounds on $\|x_n - x_*\|$ are tighter if we use w_0, w, w_2 instead of w_0 and w we have used in [4], or only w_3 as used in [2, 7–15].

3. Numerical Examples

Example 3.1. *Let $X = \mathbb{R}^3$, $D = \bar{U}(0,1)$, $x^* = (0,0,0)^T$. Define the function F on D for $w = (x,y,z)^T$ by*

$$F(w) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T.$$

Then the Fréchet-derivative is defined by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that using the Lipschitz conditions, we get $w_0(t) = L_0 t$, $w(t) = Lt$, $w_1(t) = L_1 t$, $w_2(t) = L_2 t$ and $w_3(t) = L_3 t$, where $L_0 = L = e - 1$, $L_1 = L_3 = e$ and $L_2 = e^{\frac{1}{L_0}}$. Moreover, choose $A_n = \frac{1}{2}F'(x_n)$ to obtain $q_n = q = \frac{1}{2}$. The parameters are

$$\rho = 0.5758, r_1 = 0.4739, \bar{\rho} = 0.5499, \bar{r}_1 = 0.4739,$$

where the bar answers corresponding to the case where only w_3 is used in the derivation of the radii.

Example 3.2. *Let $X = Y = \mathbb{R}^{m-1}$ for a natural integer $n \geq 2$. X and Y are equipped with the max-norm $\mathbf{x} = \max_{1\leq i\leq n-1} x_i$. The corresponding matrix norm is*

$$A = \max_{1\leq i\leq m-1} \sum_{j=1}^{j=m-1} |a_{ij}|$$

for $A = (a_{ij})_{1\leq i,j\leq m-1}$. On the interval $[0,1]$, we consider the following two point boundary value problem

$$\begin{cases} v'' + v^2 = 0 \\ v(0) = v(1) = 0 \end{cases} \tag{3.1}$$

[6,8,9,11]. *To discretize the above equation, we divide the interval $[0,1]$ into m equal parts with length of each part: $h = 1/m$ and coordinate of each point: $x_i = i\,h$ with $i = 0,1,2,\dots,m$. A second-order finite difference discretization of equation (3.1) results in the following set of nonlinear equations*

$$F(\mathbf{v}) := \begin{cases} v_{i-1} + h^2\, v_i^2 - 2v_i + v_{i+1} = 0 \\ \text{for } \ i = 1,2,\dots,(m-1) \ \text{ and from (3.1) } \ v_0 = v_m = 0, \end{cases}$$

where $\mathbf{v} = [v_1, v_2, \dots, v_{(m-1)}]^T$. For the above system-of-nonlinear-equations, we provide the Fréchet derivative

$$F'(\mathbf{v}) = \begin{bmatrix} \frac{2v_1}{m^2} - 2 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & \frac{2v_2}{m^2} - 2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \frac{2v_3}{m^2} - 2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & \frac{2v_{(m-1)}}{m^2} - 2 \end{bmatrix}.$$

We see that for $A_n = \frac{9}{10}F'(x_n)$, $w_0(t) = L_0 t$, $w(t) = Lt$, $w_1(t) = L_1 t$, $w_2(t) = L_2 t$, $w_3(t) = L_3 t$, where $L_0 = L = L_1 = L_2 = 3$, $L_3 = 4$, $q = \frac{1}{10}$ and $\|F'(x_*)^{-1}\| = \frac{1}{2}$. The parameters are

$$\rho = 0.5478, r_1 = 0.5478, \bar{\rho} = 0.4762, \bar{r}_1 = 0.4762,$$

where the bar answers corresponding to the case in which only w_3 is used in the derivation of the radii.

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