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A NOTE ON THE MAXIMAL OPERATORS OF THE NÖRLUND LOGARITMIC MEANS OF VILENKIN-FOURIER SERIES

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Abstract. The main aim of this paper is to investigate the (H_p, L_p)- type inequalities for the maximal operators of Nörlund logarithmic means for $0 < p < 1$.

1. Introduction

It is well-known that (see e.g., [1], [8] and [16]) Vilenkin systems do not form bases in the Lebesgue space $L_1(G_m)$. Moreover, there exists a function in the Hardy space H_1 such that the partial sums of f are not bounded in L_1-norm.

In [19] (see also [21]), it was proved that the following is true:

Theorem T1. Let $0 < p < 1$. Then the maximal operator

$$\widetilde{S}_p^* f := \sup_{n \in \mathbb{N}} \frac{|S_n f|}{(n+1)^{1/p-1}}$$

is bounded from the Hardy space $H_p(G_m)$ to the space $L_p(G_m)$. Here, S_n denotes the n-th partial sum with respect to the Vilenkin system. Moreover, it was proved that the rate of the factor $(n+1)^{1/p-1}$ is in a sense sharp.

In the case $p = 1$, it was proved that the maximal operator $\widetilde{S}^*$ defined by

$$\widetilde{S}^* := \sup_{n \in \mathbb{N}} \frac{|S_n|}{\log(n+1)}$$

is bounded from the Hardy space $H_1(G_m)$ to the space $L_1(G_m)$. Moreover, the rate of the factor $\log(n+1)$ is in a sense sharp. Similar problems for the Nörlund logarithmic means in the case, where $p = 1$, was considered in [15].

Móricz and Siddiqi [9] investigated the approximation properties of some special Nörlund means of Walsh-Fourier series of $L_p(G_m)$ functions in L_p-norm. Fridli, Manchanda and Siddiqi [5] improved and extended the results of Móricz and Siddiqi [9] to the Martingale Hardy spaces. However, the case for $\{q_k = 1/k : k \in \mathbb{N}_+\}$ was excluded, since the methods are not applicable to the Nörlund logarithmic means. In [6], Gt and Goginava proved some convergence and divergence properties of Walsh-Fourier series of the Nörlund logarithmic means of functions in the Lebesgue space $L_1(G_m)$. In particular, they proved that there exists a function in the space $L_1(G_m)$ such that

$$\sup_{n \in \mathbb{N}} \|L_n f\|_1 = \infty.$$

In [2] (see also [15,17]), it was proved that there exists a martingale $f \in H_p(G_m)$, $(0 < p < 1)$ such that

$$\sup_{n \in \mathbb{N}} \|L_n f\|_p = \infty.$$

Analogous problems for the Nörlund means with respect to Walsh, Kaczmarz and unbounded Vilenkin systems were considered in Blahota, and Tephnadze, [3,4], Goginava and Nagy [7], Nagy and Tephnadze [10–12], Persson, Tephnadze and Wall [13,14], Tephnadze [18,20,21], Tutberidze [22].

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In this paper, we discuss the boundedness of the weighted maximal operators from the Hardy space $H_p(G_m)$ to the Lebesgue space $L_p(G_m)$ for $0<p<1$.

2. Definitions and Notation

Let $\mathbb{N}_+$ denote the set of the positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$.

Let $m := (m_0, m_1, \dots)$ denote a sequence of the positive integers, not less than 2.

Denote by

$$Z_{m_k} := \{0, 1, \dots, m_k - 1\}$$

the additive group of integers modulo m_k.

Define the group G_m as the complete direct product of the group Z_{m_j} with the product of the discrete topologies of Z_{m_j}.

The direct product μ of the measures

$$\mu_k(\{j\}) := 1/m_k \qquad (j \in Z_{m_k})$$

is the Haar measure on G_m with $\mu(G_m) = 1$.

If $\sup_{n\in\mathbb{N}} m_n < \infty$, then we call G_m a bounded Vilenkin group. If the generating sequence m is not bounded, then G_m is said to be an unbounded one. **In this paper we discuss the bounded Vilenkin groups only.**

The elements of G_m are represented by the sequences

$$x := (x_0, x_1, \dots, x_j, \dots) \qquad (x_k \in Z_{m_k}).$$

It is easy to give a base for the neighborhood of G_m,

$$I_0(x) := G_m,$$

$$I_n(x) := \{y \in G_m \mid y_0 = x_0, \dots, y_{n-1} = x_{n-1}\} \ (x \in G_m,\ n \in \mathbb{N})$$

Denote $I_n := I_n(0)$, for $n \in \mathbb{N}$ and $\overline{I_n} := G_m \backslash I_n$.

If we define the so-called generalized number system based on m in the following way :

$$M_0 := 1, \qquad M_{k+1} := m_k M_k \qquad (k \in \mathbb{N})$$

then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j}$ $(j \in \mathbb{N})$ and only a finite number of n_j's differs from zero. Let $|n| := \max\{j \in \mathbb{N}; \quad n_j \neq 0\}$.

The norm (or quasi-norm) of the space $L_p(G_m)$ is defined by

$$\|f\|_p^p := \int_{G_m} |f|^p \, d\mu \qquad (0<p<\infty).$$

The space $weak - L_p(G_m)$ consists of all measurable functions f for which

$$\|f\|_{weak-L_p(G_m)}^p := \sup_{\lambda>0} \lambda^p \mu(x : |f(x)| > \lambda) < +\infty.$$

Next, we introduce on G_m an orthonormal system which is called the Vilenkin system. First we define the complex-valued function $r_k(x) : G_m \to C$, the generalized Rademacher functions as

$$r_k(x) := \exp(2\pi i x_k / m_k) \qquad \left(i^2 = -1, \quad x \in G_m, \quad k \in \mathbb{N}\right).$$

Now, define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on G_m as:

$$\psi_n := \prod_{k=0}^{\infty} r_k^{n_k}, \qquad (n \in \mathbb{N}).$$

Specifically, we call this system the Walsh-Paley one if m=2.

The Vilenkin system is orthonormal and complete in $L_2(G_m)$ [1,23].

Now we introduce analogues of the usual definitions in the Fourier analysis.

If $f\in L_1(G_m)$, we can establish the Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels with respect to the Vilenkin system ψ in the usual manner:

$$\widehat{f}(k):=\int_{G_m} f\overline{\psi}_k d\mu, \qquad (k\in\mathbb{N}),$$

$$S_n f:=\sum_{k=0}^{n-1}\widehat{f}(k)\psi_k, \qquad (n\in\mathbb{N}_+, \quad S_0 f:=0),$$

$$D_n:=\sum_{k=0}^{n-1}\psi_k, \qquad (n\in\mathbb{N}_+).$$

Recall that (for details see e.g. [1])

$$D_{M_n}(x)=\begin{cases} M_n & x\in I_n \\ 0 & x\notin I_n. \end{cases} \tag{1}$$

The σ-algebra generated by the intervals $\{I_n(x): x\in G_m\}$ will be denoted by F_n $(n\in\mathbb{N})$. Denote by $f=(f_n:n\in\mathbb{N})$ a martingale with respect to F_n $(n\in\mathbb{N})$ (for details see e.g. [24,25]). The maximal function of a martingale f is defined by

$$f^*=\sup_{n\in\mathbb{N}}|f_n|.$$

In the case, where $f\in L_1$, the maximal function is also given by

$$f^*(x)=\sup_{n\in\mathbb{N}}\frac{1}{|I_n(x)|}\left|\int_{I_n(x)} f(u)\mu(u)\right|.$$

For $0<p<\infty$, the Hardy martingale spaces $H_p(G_m)$ consist of all martingales for which

$$\|f\|_{H_p}:=\|f^*\|_p<\infty.$$

If $f\in L_1$, then it is easy to show that the sequence $(S_{M_n}f:n\in\mathbb{N})$ is a martingale. If $f=(f_n:n\in\mathbb{N})$ is a martingale, then the Vilenkin-Fourier coefficients should be defined in a slightly different manner:

$$\widehat{f}(i):=\lim_{k\to\infty}\int_{G_m} f_k\overline{\psi}_i d\mu.$$

The Vilenkin-Fourier coefficients of $f\in L_1(G_m)$ are the same as those of the martingale $(S_{M_n}f:n\in\mathbb{N})$ obtained from f.

Let $\{q_k:k>0\}$ be a sequence of non-negative numbers. The n-th Nörlund means for the Fourier series of f is defined by

$$\frac{1}{Q_n}\sum_{k=1}^{n}q_{n-k}S_k f, \quad \text{where} \quad Q_n:=\sum_{k=1}^{n}q_k.$$

If $q_k=1/k$, then we get the Nörlund logarithmic means

$$L_n f:=\frac{1}{l_n}\sum_{k=0}^{n-1}\frac{S_k f}{n-k}, \quad \text{where} \quad l_n=\sum_{k=0}^{n-1}\frac{1}{n-k}=\sum_{j=1}^{n}\frac{1}{j}.$$

A bounded measurable function a is p-atom, if there exists a dyadic interval I such that

$$\int_I a d\mu=0, \quad \|a\|_\infty\leq\mu(I)^{-1/p}, \quad \operatorname{supp}(a)\subset I.$$

3. Formulation of Main Results

Theorem 1. a) *Let $0<p<1$. Then the maximal operator*

$$\tilde{L}_p^* f := \sup_{n\in\mathbb{N}} \frac{|L_n f|}{(n+1)^{1/p-1}}$$

is bounded from the Hardy space $H_p(G_m)$ to the space $L_p(G_m)$.

b) *Let $0<p<1$ and $\varphi:\mathbb{N}_+\to[1,\infty)$ be a non-decreasing function satisfying the condition*

$$\overline{\lim_{n\to\infty}} \frac{n^{1/p-1}}{\log n\varphi(n)} = +\infty.$$

Then there exists a martingale $f\in H_p(G_m)$ such that the maximal operator

$$\sup_{n\in\mathbb{N}} \frac{|L_n f|}{\varphi(n+1)}$$

is not bounded from the Hardy space $H_p(G_m)$ to the space $L_p(G_m)$.

4. Proof of the Theorem

Proof. Since

$$\frac{|L_n f|}{(n+1)^{1/p-1}} \le \frac{1}{(n+1)^{1/p-1}} \sup_{1\le k\le n} |S_k f| \le \sup_{1\le k\le n} \frac{|S_k f|}{(k+1)^{1/p-1}} \le \sup_{n\in\mathbb{N}} \frac{|S_n f|}{(n+1)^{1/p-1}},$$

if we use Theorem T1, we obtain

$$\sup_{n\in\mathbb{N}} \frac{|L_n f|}{(n+1)^{1/p-1}} \le \sup_{n\in\mathbb{N}} \frac{|S_n f|}{(n+1)^{1/p-1}}$$

and

$$\left\| \sup_{n\in\mathbb{N}} \frac{|L_n f|}{(n+1)^{1/p-1}} \right\|_p \le \left\| \sup_{n\in\mathbb{N}} \frac{|S_n f|}{(n+1)^{1/p-1}} \right\|_p \le c_p \|f\|_{H_p}.$$

Now, prove part b) of the Theorem. Let

$$f_{n_k} = D_{M_{2n_k+1}} - D_{M_{2n_k}}.$$

It is evident that

$$\widehat{f}_{n_k}(i) = \begin{cases} 1, & \text{if} \quad i = M_{2n_k},\dots,M_{2n_k+1}-1, \\ 0, & \text{otherwise.} \end{cases}$$

Then we can write that

$$S_i f_{n_k} = \begin{cases} D_i - D_{M_{2n_k}}, & \text{if} \quad i = M_{2n_k}+1,\dots,M_{2n_k+1}-1, \\ f_{n_k}, & \text{if} \quad i \ge M_{2n_k+1}, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

From (1), we get

$$\begin{aligned} \|f_{n_k}\|_{H_p} &= \left\| \sup_{n\in\mathbb{N}} S_{M_n} f_{n_k} \right\|_p = \left\| D_{M_{2n_k+1}} - D_{M_{2n_k}} \right\|_p \\ &\le \left\| D_{M_{2n_k+1}} \right\|_p + \left\| D_{M_{2n_k}} \right\|_p \le c M_{2n_k}^{1-1/p} < c < \infty. \end{aligned} \tag{3}$$

Let $0<p<1$ and $\{\lambda_k : k\in\mathbb{N}_+\}$ be an increasing sequence of the positive integers such that

$$\lim_{k\to\infty} \frac{\lambda_k^{1/p-1}}{\varphi(\lambda_k)} = \infty.$$

Let $\{n_k : k \in \mathbb{N}_+\} \subset \{\lambda_k : k \in \mathbb{N}_+\}$ such that

$$\lim_{k\to\infty} \frac{\left(M_{2_{n_k}}+2\right)^{1/p-1}}{\log\left(M_{2n_k}+2\right)\varphi\left(M_{2n_k+2}\right)} \geq c \lim_{k\to\infty} \frac{\lambda_k^{1/p-1}}{\varphi\left(\lambda_k\right)} = \infty.$$

According to (2), we can conclude that

$$\left|\frac{L_{M_{2n_k}+2}f_{n_k}}{\varphi\left(M_{2n_k+2}\right)}\right| = \frac{\left|D_{M_{2n_k}+1} - D_{M_{2n_k}}\right|}{l_{M_{2n_k}+1}\varphi\left(M_{2n_k+1}\right)}$$

$$= \frac{\left|\psi_{M_{2n_k}}\right|}{l_{M_{2n_k}+2}\varphi\left(M_{2n_k+1}\right)} = \frac{1}{l_{M_{2n_k}+1}\varphi\left(M_{2n_k+2}\right)}.$$

Hence,

$$\mu\left\{x \in G_m : \left|L_{M_{2n_k}+2}f_{n_k}\right| \geq \frac{1}{l_{M_{2n_k}+2}\varphi\left(M_{2n_k+2}\right)}\right\} = \mu\left(G_m\right) = 1. \tag{4}$$

By combining (3) and (4), we get

$$\frac{\frac{1}{l_{M_{2n_k}+2}\varphi\left(M_{2n_k+2}\right)}\left(\mu\left\{x \in G_m : \quad \left|L_{M_{2n_k}+2}f_{n_k}\right| \geq \frac{1}{l_{M_{2n_k}+2}\varphi\left(M_{2n_k+2}\right)}\right\}\right)^{1/p}}{\|f_{n_k}\|_p}$$

$$\geq \frac{M_{2_{n_k}}^{1/p-1}}{l_{M_{2n_k}+2}\varphi\left(M_{2n_k+2}\right)} \geq \frac{c\left(M_{2_{n_k}}+2\right)^{1/p-1}}{\log\left(M_{2n_k}+2\right)\varphi\left(M_{2n_k+2}\right)} \to \infty, \quad \text{as} \quad k \to \infty. \qquad \square$$

Open Problem. For any $0 < p < 1$, let us find a non-decreasing function $\Theta : \mathbb{N}_+ \to [1, \infty)$ such that the following maximal operator

$$\widetilde{L}_p^* f := \sup_{n\in\mathbb{N}} \frac{|L_n f|}{\Theta\left(n+1\right)}$$

is bounded from the Hardy space $H_p\left(G_m\right)$ to the Lebesgue space $L_p\left(G_m\right)$ and the rate of $\Theta : \mathbb{N}_+ \to [1, \infty)$ is sharp, that is, for any non-decreasing function $\varphi : \mathbb{N}_+ \to [1, \infty)$ satisfying the condition

$$\overline{\lim_{n\to\infty}} \frac{\Theta\left(n\right)}{\varphi\left(n\right)} = +\infty,$$

there exists a martingale $f \in H_p\left(G_m\right)$ such that the maximal operator

$$\sup_{n\in\mathbb{N}} \frac{|L_n f|}{\varphi\left(n+1\right)}$$

is not bounded from the Hardy space $H_p\left(G_m\right)$ to the space $L_p\left(G_m\right)$.

Remark 1. *According to Theorem* 1, *we can conclude that there exist absolute constants* C_1 *and* C_2 *such that*

$$\frac{C_1 n^{1/p-1}}{\log(n+1)} \leq \Theta\left(n\right) \leq C_2 n^{1/p-1}.$$

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