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GENERALIZED SCHWARTZ TYPE SPACES AND LCT BASED PSEUDO DIFFERENTIAL OPERATOR

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Abstract. In connection with the LCT, in this paper, we define the Schwartz type spaces $\mathcal{S}_{\Delta,\alpha,A}$, $\mathcal{S}^{\Delta,\beta,B}$, $\mathcal{S}^{\Delta,\beta,B}_{\Delta,\alpha,A}$ and study the mapping properties of LCT between these spaces. Moreover, we define a generalized Δ-pseudo differential operator and investigate its mapping properties in the framework of the above Schwartz type spaces.

1. Introduction

The Fourier transform

$$\hat{f}(\xi) := \mathcal{F}[f;\xi] = \int_{\mathbb{R}} f(x)e^{-ix\xi}\,dx$$

and the related convolution

$$(f * g)(\xi) = \int_{\mathbb{R}} f(\xi - x)g(x)\,dx$$

have become an essential tool for solving many practical problems over the last few decades. Because of their usefulness, these notions have been generalized and extended by several people to give rise more general transforms and convolutions such as fractional Fourier transform [8], [12], [25], [33]. One such generalizaion is the so-called linear canonical transform (LCT) introduced in 1971 [26] which is connected with the 2×2 matrix M given by

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad \text{with } ad - bc = 1.$$

The LCT is defined by

$$\mathcal{L}_M[f;\xi] = \int_{\mathbb{R}} f(x)K_M(x,\xi)\,dx,$$

where the kernel K_M is defined by

$$K_M(x,\xi) = \begin{cases} \frac{1}{\sqrt{2\pi bi}} \exp\left[\frac{i}{2}\left(\frac{a}{b}x^2 - \frac{2}{b}x\xi + \frac{d}{b}\xi^2\right)\right], & \text{if } b \neq 0 \\ \frac{1}{\sqrt{a}} e^{i\left(\frac{c}{2a}\right)\xi^2} \delta\left(x - \frac{\xi}{a}\right), & \text{if } b = 0. \end{cases}$$

The convolution related to LCT is given by

$$(f \star_M g)(x) = \int_{\mathbb{R}} f(\xi)g(x-\xi)\exp\left[i\frac{a}{b}\xi(x-\xi)\right]d\xi$$

and the inverse LCT is defined by

$$\mathcal{L}_{M^{-1}}[f;x] = \int_{\mathbb{R}} f(\xi)K_{M^{-1}}(\xi,x)\,d\xi,$$

where M^{-1} is the inverse of the matrix M.

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At present, "Fourier Analysis" is usually termed as "Time Frequency Analysis". In this context, the Fourier transform rotates the signals from the time axis to the frequency axis by 90 degrees. It has been observed that certain optical systems rotated the signals by an arbitrary angle which requires the notion of fractional Fourier transforms, i.e., a one-parameter family of transforms. The linear canonical transforms (LCT) form a class of three-parameter family of transforms involving many known transforms. For notational convenience, if we write the matrix M as $(a, b; c, d)$, then the matrices $(0, 1; -1, 0)$ and $(\cos\alpha, \sin\alpha; -\sin\alpha, \cos\alpha)$ correspond, respectively, to the Fourier and fractional Fourier transforms. More special matrices lead to some other known integral transforms, e.g., Fresnel transform, chirp functions etc. Various applications of LCT have been realized in the field of electromagnetic, acoustic and other wave propagation problems. As mentioned in [10], LCT is known under other terminology as well, such as a quadratic phase integral [2], generalized Huygens integral [28], generalized Fresnel transform [9], [13], etc.

Recently, in [23], the authors have studied certain mappings properties of LCT and the associated pseudo-differential operators in a variant of Schwartz space denoted by $\mathcal{S}_M \equiv \mathcal{S}_M(\mathbb{R})$.

In this paper, we first introduce further variants of the space $\mathcal{S}_M$, denoted by $\mathcal{S}_{\Delta,\alpha,A}$, $\mathcal{S}^{\Delta,\beta,B}$ and $\mathcal{S}^{\Delta,\beta,B}_{\Delta,\alpha,A}$, where Δ is a differential operator defined and studied in Section 2, and α, β, A and B are certain constants. These spaces extend the spaces $\mathcal{S}_\alpha$, $\mathcal{S}^\beta$ and $\mathcal{S}^\beta_\alpha$ (see [5]). We study the mapping properties of LCT in the spaces $\mathcal{S}_{\Delta,\alpha,A}$, $\mathcal{S}^{\Delta,\beta,B}$ and $\mathcal{S}^{\Delta,\beta,B}_{\Delta,\alpha,A}$. This is done in Section 3. Finally, in Section 4, we define a generalized Δ-pseudo differential operator and study its mapping properties in the framework of the spaces $\mathcal{S}_{\Delta,\alpha,A}$, $\mathcal{S}^{\Delta,\beta,B}$ and $\mathcal{S}^{\Delta,\beta,B}_{\Delta,\alpha,A}$.

2. LCT Based Convolution and Differential Operators

We begin this section with mentioning that a Young type inequality can be proved for the convolution $\star_M$, and this can be done on lines, similar to those obvious modifications performed in [23]. We only state the result.

Theorem 2.1. *Let $1 \le p < \infty$, $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$. Then $(f \star_M g) \in L^p(\mathbb{R})$ with*

$$\|f \star_M g\|_{L^p(\mathbb{R})} \le \|f\|_{L^1(\mathbb{R})} \|g\|_{L^p(\mathbb{R})}.$$

Next, we prove the following

Theorem 2.2. *Let f be continuous and g be continuous with a compact support. Then $f \star_M g$ is continuous.*

Proof. Let $h \in \mathbb{R}$. Then

$$\begin{aligned}
&|(f \star_M g)(x+h) - (f \star_M g)(x)| \\
&\quad = \left| \int_{\mathbb{R}} f(y) g(x+h-y) \exp[i(a/b)y(x+h-y)]\, dy \right. \\
&\qquad \left. - \int_{\mathbb{R}} f(y) g(x-y) \exp[i(a/b)y(x-y)]\, dy \right| \\
&\quad = \left| \int_{\mathbb{R}} f(y)(g(x+h-y)\exp[i(a/b)yh] - g(x-y)) \exp[i(a/b)y(x-y)]\, dy \right| \\
&\quad \le \int_{\mathbb{R}} \left| f(y)\Big(g(x+h-y)\exp[i(a/b)yh] - g(x-y)\Big) \right| dy \\
&\quad = \int_{\mathbb{R}} \left| f(y)\Big(g(x+h-y)\exp[i(a/b)yh] - g(x-y)\exp[i(a/b)yh] \right. \\
&\qquad \left. + g(x-y)\exp[i(a/b)yh] - g(x-y)\Big) \right| dy
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_{\mathbb{R}} |f(y)||g(x+h-y)-g(x-y)|\,dy \\
&\qquad + \int_{\mathbb{R}} |f(y)||g(x-y)||\exp[i(a/b)yh]-1|\,dy \\
&=: I_1 + I_2.
\end{aligned}
$$

Let $K := \text{supp }(g)$ be compact. Then for any fixed x,

$$
x - K = \{x - y : y \in K\}
$$

is compact and therefore, f is uniformly continuous on $x-K$. Thus, for each $\varepsilon > 0$, there exists $\eta > 0$ such that if $|h| < \eta$, then $I_1 \to 0$ as $h \to 0$. Further, on $x-K$, f, g are bounded, therefore

$$
I_2 \leq \int_{\mathbb{R}} |f(y)||g(x-y)|2|\sin(a/2b)yh|\,dy
$$

which tends to 0 as $h \to 0$. Hence $|(f \star_M g)(x+h) - (f \star_M g)(x)| \to 0$ as $h \to 0$ and the assertion follows. □

A stronger version of Theorem 2.2 is the following

Theorem 2.3. *If $f \in C^\infty(\mathbb{R})$ and g is continuous with a compact support, then $f \star_M g$ is C^∞.*

Proof. We have

$$
\begin{aligned}
&\frac{1}{h}[(f \star_M g)(x+h) - (f \star_M g)(x)] \\
&\quad = \frac{1}{h}\int_{\mathbb{R}} g(y)\Big(f(x+h-y)\exp[i(a/b)yh] - f(x-y)\Big)\exp[i(a/b)y(x-y)]\,dy \\
&\quad = \frac{1}{h}\int_{\mathbb{R}} g(y)\Big(f(x+h-y)\exp[i(a/b)yh] - \exp[i(a/b)yh]f(x-y) \\
&\qquad\quad + \exp[i(a/b)yh]f(x-y)) - f(x-y)\Big)\exp[i(a/b)y(x-y)]\,dy \\
&\quad = \frac{1}{h}\int_{\mathbb{R}} g(y)\Big(f(x+h-y) - f(x-y)\Big)\exp[i(a/b)y(x+h-y)]\,dy \\
&\qquad\quad + \frac{1}{h}\int_{\mathbb{R}} g(y)(\exp[i(a/b)yh]-1)f(x-y)\exp[i(a/b)y(x-y)]\,dy. \\
&\quad \to (Df \star_M g)(x) + (f \star_M (ia/b)(\cdot)g)(x)
\end{aligned}
$$

as $h \to 0$. Therefore, it follows that $f \star_M g$ is differentiable if f is differentiable. It can be proved by induction that

$$
D_x^n(f \star_M g)(x) = \sum_{r=0}^{n} A_{n,r}(D^{n-r}f \star_M (ia/b(\cdot))^r g)(x),
$$

where $A_{n,r}$ are appropriate constants. Hence, $f \star_M g \in C^\infty$. □

Remark 2.4. Since $f \star_M g$ is commutative, therefore, if $g \in C^\infty$ and f is continuous with a compact support, then

$$
D_x^n(f \star_M g)(x) = \sum_{r=0}^{n} A_{n,r}((ia/b(\cdot))^{n-r}f \star_M D^r g)(x)
$$

and, consequently, $f \star_M g \in C^\infty$.

Denote $D_x := \dfrac{d}{dx}$. Let us define the following generalized differential operators based on the LCT:

$$\Delta_{x,a} = D_x - i\frac{a}{b}x$$
$$\Delta^*_{x,a} = -\left(D_x + i\frac{a}{b}x\right).$$

Remark 2.5. The following can be observed immediately:

(i) $\triangle_{x,a} K_M(x,\xi) = \left(\frac{-i\xi}{b}\right) K_M(x,\xi)$.

(ii) $\triangle_{\xi,d} K_M(x,\xi) = \left(\frac{-ix}{b}\right) K_M(x,\xi)$.

(iii) $\triangle^*_{x,a} K_{M^{-1}}(\xi,x) = \left(\frac{-i\xi}{b}\right) K_{M^{-1}}(\xi,x)$.

(iv) $\triangle^*_{\xi,d} K_{M^{-1}}(\xi,x) = \left(\frac{-ix}{b}\right) K_{M^{-1}}(\xi,x)$.

Let us recall that the Schwartz space $\mathcal{S}(\mathbb{R})$ consists of all functions $\phi \in C^\infty$ such that

$$\sup_{x\in\mathbb{R}} |x^k \phi^{(q)}(x)| \le m_{kq}, \quad k,q = 0,1,2,\dots .$$

Some of the properties of the operator $\triangle_{x,a}$ are given below which can be proved in a way, similar to [23].

Proposition 2.6.

(i) *For $\phi \in \mathcal{S}(\mathbb{R})$, the follwoing*

$$(\triangle_{\xi,d})^n \mathcal{L}_M[\phi;\xi] = \mathcal{L}_M\left[\left(\frac{-ix}{b}\right)^n \phi;\xi\right] \textit{holds.}$$

(ii) *For $\phi,\psi \in \mathcal{S}(\mathbb{R})$, the following Leibnitz type rule*

$$\triangle_{x,a}(\phi(x)\psi(x)) = \sum_{r=0}^{n} A_{n,r} D_x^r \phi(x) \cdot \triangle^{n-r}_{x,a}\psi(x) \textit{holds.}$$

Remark 2.7. The results similar to those of Proposition 2.6 can also be proved for $\triangle^*_{x,a}, \triangle_{\xi,d}$ and $\triangle^*_{\xi,d}$.

3. Schwartz Type Spaces Based on LCT

The space S_Δ was defined in [17] (see also [23]) as the space of all $\phi \in C^\infty$ for which

$$\sup_{t\in\mathbb{R}} \left|x^k \triangle^q_{x,a}\phi(x)\right| < \infty, \qquad k,q \in \mathbb{N}_0 \equiv \mathbb{N}\cup\{0\}.$$

When $\triangle_{x,a}$ is the differential operator $\frac{d}{dx}$, the space S_Δ coincides with the standard Schwartz space $\mathcal{S}$. Let us note from the construction of the Schwartz space $\mathcal{S} := \mathcal{S}$ that the sequence m_{kq} depends on both k and q. The Gelfand and Shilov type spaces are the variants of the space $\mathcal{S}$, in which the sequence m_{kq} depends only on k, or only on q, or on both. Such spaces are denoted, respectively, by S_α, S^β and S^β_α. These spaces have further been generalized to give rise to the spaces $S_{\alpha,A}$, $S^{\beta,B}$ and $S^{\beta,B}_{\alpha,A}$. For a systematic study and related results about these spaces, one may refer to [5].

Below, we define and study further generalizations of the spaces $S_{\alpha,A}$, $S^{\beta,B}$ and $S^{\beta,B}_{\alpha,A}$ in which the derivative $\dfrac{d}{dx}$ is replaced by more general operators Δ and Δ^*.

In the literature (see, e.g., [5]), various spaces of type $\mathcal{S}$ such as $\mathcal{S}_\alpha, \mathcal{S}^\beta, \mathcal{S}^\beta_\alpha$ have been defined and studied. In this section, we define and study similar variants of the space $\mathcal{S}_\Delta$.

Definition 3.1. Let $\delta > 0$. We define the space $S_{\Delta,\alpha,A}$ that consists of all $\phi \in C^\infty$ such that

$$|x^k \triangle^q_{x,a}\phi(x)| \le C_{q,\delta}(A+\delta)^k k^{k\alpha},$$

where $k,q \in \mathbb{N}_0$ and $C_{q,\delta}$ depends on ϕ.

Definition 3.2. Let $\rho > 0$. We define the space $\mathcal{S}^{\Delta,\beta,B}$ that consists of all $\phi \in C^\infty$ such that

$$|x^k \triangle_{x,a}^q \phi(x)| \le C_{k,\rho}(B+\rho)^q q^{q\beta},$$

where $k, q \in \mathbb{N}_0$ and $C_{k,\rho}$ depends upon ϕ.

Definition 3.3. Let $\delta, \rho > 0$. We define the space $\mathcal{S}_{\Delta,\alpha,A}^{\Delta,\beta,B}$ that consists of all $\phi \in C^\infty$ such that

$$|x^k \triangle_{x,a}^q \phi(x)| \le C_k (A+\delta)^k (B+\rho)^q k^{k,\alpha} q^{q\beta},$$

where $k, q \in \mathbb{N}_0$ and C_k depends on ϕ.

Remark 3.4. We also define the spaces $\mathcal{S}_{\Delta^*,\alpha,A}$, $\mathcal{S}^{\Delta^*,\beta,B}$ and $\mathcal{S}_{\Delta^*,\alpha,A}^{\Delta^*,\beta,B}$, where Δ in Definitions 3.1, 3.2 and 3.3, is replaced by Δ^*.

Theorem 3.5. *Let $\phi \in \mathcal{S}_{\Delta^*,\alpha,A}$. Then $\mathcal{L}_M[\phi;\cdot] \in \mathcal{S}^{\Delta,\alpha,B}$.*

Proof. We have

$$\begin{aligned}
\xi^k \triangle_{\xi,d}^q \mathcal{L}_M[\phi;\xi] &= \xi^k \triangle_{\xi,d}^q \int_{\mathbb{R}} K_M(x,\xi)\phi(x)\,dx \\
&= \xi^k \int_{\mathbb{R}} \triangle_{\xi,d}^q K_M(x,\xi)\phi(x)\,dx \\
&= \xi^k \int_{\mathbb{R}} \left(\frac{-ix}{b}\right)^q K_M(x,\xi)\phi(x)\,dx \\
&= \left(\frac{-i}{b}\right)^{q-k} \int_{\mathbb{R}} \left(\frac{-i\xi}{b}\right)^k K_M(x,\xi)x^q\phi(x)\,dx \\
&= \left(\frac{-i}{b}\right)^{q-k} \int_{\mathbb{R}} (\triangle_{x,a})^k K_M(x,\xi)x^q\phi(x)\,dx \\
&= \left(\frac{-i}{b}\right)^{q-k} \int_{\mathbb{R}} K_M(x,\xi)(\triangle_{x,a}^*)^k (x^q\phi(x))\,dx \\
&= \left(\frac{-i}{b}\right)^{q-k} \int_{\mathbb{R}} K_M(x,\xi)\left(\sum_{r=0}^k A_{k,r} D_x^r x^q (\triangle_{x,a}^*)^{(k-r)}\phi(x)\right)dx \\
&= \left(\frac{-i}{b}\right)^{q-k} \left(\sum_{r=0}^k A_{k,r} \int_{\mathbb{R}} K_M(x,\xi) D_x^r x^q (\triangle_{x,a}^*)^{(k-r)}\phi(x)\,dx\right)
\end{aligned}$$

so that

$$\begin{aligned}
&|\xi^k \triangle_{\xi,d}^q \mathcal{L}_M[\phi;\xi]| \\
&= \left|\left(\frac{-i}{b}\right)^{q-k} \left(\sum_{r=0}^k A_{k,r} \int_{\mathbb{R}} K_M(x,\xi) \frac{q!}{(q-r)!}\psi(x)^{q-r} (\triangle_{x,a}^*)^{(k-r)}\phi(x)\,dx\right)\right|,
\end{aligned}$$

where

$$\psi(x) = \begin{cases} x & \text{if } q-r \ge 0, \\ 0 & \text{otherwise.} \end{cases}$$

Denote $|A_k| = \sup_r |A_{k,r}|$. Then

$$|\xi^k \triangle_{\xi,d}^q \mathcal{L}_M[\phi;\xi]|$$

$$\leq \left(\frac{1}{|b|}\right)^{q-k} \left(\sum_{r=0}^{k} |A_{k,r}| \int_{\mathbb{R}} |K_M(x,\xi)| \frac{q!}{(q-r)!} |\psi(x)|^{q-r} |(\triangle_{x,a}^{*})^{(k-r)} \phi(x)|\, dx\right)$$

$$\leq \left(\frac{1}{|b|}\right)^{q-k} |A_k| \left(\sum_{r=0}^{k} \int_{\mathbb{R}} |K_M(x,\xi)| \frac{q!}{(q-r)!} |\psi(x)|^{q-r} |(\triangle_{x,a}^{*})^{(k-r)} \phi(x)|\, dx\right)$$

$$= \left(\frac{1}{|b|}\right)^{q-k} |A_k| k! \left(\sum_{r=0}^{k} \int_{\mathbb{R}} |K_M(x,\xi)| \frac{q!}{k!(q-r)!} |\psi(x)|^{q-r} |(\triangle_{x,a}^{*})^{(k-r)} \phi(x)|\, dx\right)$$

$$\leq \left(\frac{1}{|b|}\right)^{q-k} |A_k| k! \left(\sum_{r=0}^{k} \int_{\mathbb{R}} |K_M(x,\xi)| \frac{q!}{r!(q-r)!} |\psi(x)|^{q-r} |(\triangle_{x,a}^{*})^{(k-r)} \phi(x)|\, dx\right)$$

$$= \left(\frac{1}{|b|}\right)^{q-k} |A_k| k! |K_M(x,\xi)| \left(\sum_{r=0}^{q} \frac{q!}{r!(q-r)!} \int_{\mathbb{R}} |\psi(x)|^{q-r} |(\triangle_{x,a}^{*})^{(k-r)} \phi(x)|\, dx\right)$$

$$= \left(\frac{1}{|b|}\right)^{q-k} |A_k| k! |K_M(x,\xi)| \left(\sum_{r=0}^{q} \frac{q!}{r!(q-r)!} \int_{\mathbb{R}} (1+|x|^2) |\psi(x)|^{q-r} \right.$$

$$\left. \times (\triangle_{x,a}^{*})^{(k-r)} \phi(x)| \frac{dx}{(1+|x|^2)}\right)$$

$$= \left(\frac{1}{|b|}\right)^{q-k} |A_k| k! |K_M(x,\xi)| \left(\sum_{r=0}^{q} \frac{q!}{r!(q-r)!} \left[\int_{\mathbb{R}} |\psi(x)|^{q-r} |(\triangle_{x,a}^{*})^{(k-r)} \phi(x)| \frac{dx}{(1+|x|^2)}\right.\right.$$

$$\left.\left. + \int_{\mathbb{R}} |\psi(x)|^{q+2-r} |(\triangle_{x,a}^{*})^{(k-r)} \phi(x)| \frac{dx}{(1+|x|^2)}\right]\right)$$

$$\leq \left(\frac{1}{|b|}\right)^{q-k} |A_k| k! |K_M(x,\xi)| \left(\sum_{r=0}^{q} \frac{q!}{r!(q-r)!} \int_{\mathbb{R}} 2|\psi(x)|^{q+2-r} |(\triangle_{x,a}^{*})^{(k-r)} \phi(x)| \frac{dx}{(1+|x|^2)}\right)$$

$$\leq 2\left(\frac{1}{|b|}\right)^{q-k} |A_k| k! |K_M(x,\xi)| \left(\sum_{r=0}^{q} \frac{q!}{r!(q-r)!} C_{k-r,\delta} (A+\delta)^{q+2-r} (q+2-r)^{(q+2-r)\alpha}\right.$$

$$\left. \times \int_{\mathbb{R}} \frac{dx}{(1+|x|^2)}\right)$$

$$\leq 2\left(\frac{1}{|b|}\right)^{q-k} |A_k| k! |K_M(x,\xi)| \left(\sum_{r=0}^{q} \frac{q!}{r!(q-r)!} \int_{\mathbb{R}} C_{k-r,\delta} (A+\delta)^{q+2-r}\right.$$

$$\left. \times (q+2)^{(q+2)\alpha} \frac{dx}{(1+|x|^2)}\right)$$

$$\leq 2\left(\frac{1}{|b|}\right)^{q-k} |A_k| k! |K_M(x,\xi)| \left(\sum_{r=0}^{q} \frac{q!}{r!(q-r)!} C_{k,\delta} (A+\delta)^{q+2-r}\right.$$

$$\left. \times (q+2)^{(q+2)\alpha} \int_{\mathbb{R}} \frac{dx}{(1+|x|^2)}\right)$$

$$\leq 2\left(\frac{1}{|b|}\right)^{q-k} |A_k| k! |K_M(x,\xi)| C_{k,\delta} \left(\sum_{r=0}^{q+2} \frac{q!}{r!(q-r)!} (A+\delta)^{q+2-r}\right.$$

$$\left. \times (q+2)^{(q+2)\alpha} \int_{\mathbb{R}} \frac{dx}{(1+|x|^2)}\right)$$

$$
\begin{aligned}
&= 2\left(\frac{1}{|b|}\right)^{q-k} |A_k| k! |K_M(x,\xi)| C_{k,\delta}(1+A+\delta)^{q+2}(q+2)^{(q+2)\alpha} \int\limits_{\mathbb{R}} \frac{dx}{(1+|x|^2)} \\
&= 2\pi\left(\frac{1}{|b|}\right)^{-k-2} |A_k| k! |K_M(x,\xi)| C_{k,\delta}\left(\frac{1+A}{|b|}+\delta/|b|\right)^{q+2}(q+2)^{(q+2)\alpha} \\
&= 2\pi |b|^{k+2} |A_k| k! |K_M(x,\xi)| C_{k,\delta}\left(\frac{1+A}{|b|}+\delta/|b|\right)^{q+2}(q+2)^{(q+2)\alpha} \\
&= D_{k,\delta}\,(B+\rho)^{q+2}\,(q+2)^{(q+2)\alpha} \\
&= D_{k,\rho}\,(B+\rho)^{q+2}\,(q+2)^{(q+2)\alpha} \\
&= E_{k,\rho}\,(B+\rho)^{q}\,q^{q\alpha}.
\end{aligned} \tag{3.1}
$$

□

Theorem 3.6. *Let $\phi \in \mathcal{S}^{\Delta^*,\beta,B}$. Then $\mathcal{L}_M[\phi;\cdot] \in \mathcal{S}_{\Delta,\beta,A}$.*

Proof. Let $\phi \in \mathcal{S}_{\Delta^*,\beta,A}$ and $\rho > 0$ be arbitrary. Using (3.1), we get

$$
\begin{aligned}
&|\xi^k \triangle^q_{\xi,d} \mathcal{L}_M[\phi;\xi]| \\
&\quad\le \left(\frac{1}{|b|}\right)^{q-k} |A_k| \Bigg(\sum_{r=0}^{k} \int\limits_{\mathbb{R}} |K_M(x,\xi)| \frac{q!}{(q-r)!} |\psi(x)|^{q-r} |(\triangle^*_{x,a})^{(k-r)}\phi(x)|\,dx \Bigg) \\
&\quad= \left(\frac{1}{|b|}\right)^{q-k} |A_k| \Bigg(\sum_{r=0}^{k} \int\limits_{\mathbb{R}} |K_M(x,\xi)| \frac{q!}{(q-r)!} |\psi(x)|^{q-r} |(\triangle^*_{x,a})^{(k-r)}\phi(x)|\,dx \Bigg) \\
&\quad\le \left(\frac{1}{|b|}\right)^{q-k} |A_k| |K_M(x,\xi)| q! \Bigg(\sum_{r=0}^{q} \frac{q!}{r!(q-r)!} \int\limits_{\mathbb{R}} |\psi(x)|^{q-r} |(\triangle^*_{x,a})^{(k-r)}\phi(x)|\,dx \Bigg) \\
&\quad\le \left(\frac{1}{|b|}\right)^{q-k} |A_k| |K_M(x,\xi)| q! \Bigg(\sum_{r=0}^{q} \frac{q!}{r!(q-r)!} \int\limits_{\mathbb{R}} (1+|x|^2) |\psi(x)|^{q-r} \\
&\qquad\quad \times |(\triangle^*_{x,a})^{(k-r)}\phi(x)|\,\frac{dx}{(1+|x|^2)} \Bigg) \\
&\quad= \left(\frac{1}{|b|}\right)^{q-k} |A_k| |K_M(x,\xi)| q! \Bigg(\sum_{r=0}^{q} \frac{q!}{r!(q-r)!} \Bigg[\int\limits_{\mathbb{R}} |\psi(x)|^{q-r} |(\triangle^*_{x,a})^{(k-r)}\phi(x)| \\
&\qquad\quad \times \frac{dx}{(1+|x|^2)} + \int\limits_{\mathbb{R}} |\psi(x)|^{q+2-r} |(\triangle^*_{x,a})^{(k-r)}\phi(x)|\,\frac{dx}{(1+|x|^2)} \Bigg] \Bigg) \\
&\quad\le \left(\frac{1}{|b|}\right)^{q-k} |A_k| |K_M(x,\xi)| q! \Bigg(\sum_{r=0}^{k} \frac{q!}{r!(q-r)!} \int\limits_{\mathbb{R}} 2|\psi(x)|^{q+2-r} \\
&\qquad\quad \times |(\triangle^*_{x,a})^{(k-r)}\phi(x)|\,\frac{dx}{(1+|x|^2)} \Bigg) \\
&\quad= 2\left(\frac{1}{|b|}\right)^{q-k} |A_k| |K_M(x,\xi)| q! \Bigg(\sum_{r=0}^{q} \frac{q!}{(q-r)!} C_{q+2-r,\rho}(B+\rho)^{k-r} \\
&\qquad\quad \times (k-r)^{(k-r)\alpha} \int\limits_{\mathbb{R}} \frac{dx}{(1+|x|^2)} \Bigg) \\
&\quad\le 2\left(\frac{1}{|b|}\right)^{q-k} |A_k| |K_M(x,\xi)| q! \Bigg(\sum_{r=0}^{q} \frac{q!}{r!(q-r)!} \int\limits_{\mathbb{R}} C_{q+2-r,\rho}(B+\rho)^{k}
\end{aligned}
$$

$$\begin{aligned}
&\quad \times k^{k\beta} \frac{dx}{(1+|x|^2)}\Bigg) \\
&\leq 2\left(\frac{1}{|b|}\right)^{q-k} |A_k||K_M(x,\xi)|q!\Bigg(\sum_{r=0}^{q} \frac{q!}{r!(q-r)!} C_{q,\rho}(B+\rho)^k \\
&\quad \times k^{k\beta} \int_{\mathbb{R}} \frac{dx}{(1+|x|^2)}\Bigg) \\
&\leq 2\left(\frac{1}{|b|}\right)^{q-k} |A_k||K_M(x,\xi)|q! C_{q,\rho}\Bigg(2^q(B+\rho)^k k^{k\beta} \int_{\mathbb{R}} \frac{dx}{(1+|x|^2)}\Bigg) \\
&= 2\pi\left(\frac{1}{|b|}\right)^{q-k} |A_k||K_M(x,\xi)|q! 2^q C_{q,\rho}(B+\rho)^k k^{k\beta} \\
&= 2\pi\left(\frac{1}{|b|}\right)^{q} |A_k||K_M(x,\xi)|q! 2^q C_{q,\rho}(|b||A_k|^{1/k}(B+\rho))^k k^{k\beta} \\
&= D_{q,\rho}(A+\delta)^k k^{k\beta}.
\end{aligned}$$

□

Similarly, the following can be proved. We skip the proof for conciseness.

Theorem 3.7. *Let* $\phi \in \mathcal{S}^{\Delta^*,\beta,B}_{\Delta^*,\alpha,A}$. *Then* $\mathcal{L}_M[\phi;\cdot] \in \mathcal{S}^{\Delta,\alpha,A'}_{\Delta,\beta,B'}$.

4. LCT Based Pseudo Differential Operator

Consider the linear differential operator given by

$$P(x, \triangle^*_{x,a}) = \sum_{r=0}^{m} a_r(x)(\triangle^*_{x,a})^r,$$

where $a_r(x)$ are the functions on $\mathbb{R}$. We also consider the polynomial given by

$$P_m(x,\xi) = \sum_{r=0}^{m} a_r(x)\left(\frac{-i\xi}{b}\right)^r$$

Let $\phi \in \mathcal{S}$. Then we have

$$\begin{aligned}
(P(x,\triangle^*_{x,a})\phi)(x) &= \sum_{r=0}^{m} a_r(x)(\triangle^*_{x,a})^r \phi(x) \\
&= \sum_{r=0}^{m} a_r(x)\mathcal{L}_{M^{-1}}\mathcal{L}_M[(\triangle^*_{x,a})^r\phi(x);\xi] \\
&= \sum_{r=0}^{m} a_r(x)\mathcal{L}_{M^{-1}}\left[\left(\frac{-i\xi}{b}\right)^r \mathcal{L}_M[\phi;\xi]\right] \\
&= \mathcal{L}_{M^{-1}}\left[\left(\sum_{r=0}^{m} a_r(x)\left(\frac{-i\xi}{b}\right)^r\right)\mathcal{L}_M[\phi;\xi]\right] \\
&= \mathcal{L}_{M^{-1}}\left[P(x,\xi)\mathcal{L}_M\phi(\xi)\right] \\
&= \int_{\mathbb{R}} K_{M^{-1}}(\xi,x)P_m(x,\xi)\mathcal{L}_M[\phi;\xi]\,d\xi.
\end{aligned}$$

We replace $P_m(x,\xi)$, the polynomial in $,\xi$ by a more general symbol $a(x,\xi)$, which need not to be a polynomial. This motivates the need to define a more general pseudo-differential operator which will be defined below. First, let us recall the following

Definition 4.1 ([31])**.** Let $m \in \mathbb{R}$. We define S^m to be the set of all functions $\sigma(x,\xi) \in C^\infty(\mathbb{R}\times\mathbb{R})$ such that for any two $k, q \in \mathbb{N}_0$, there is a positive constant c depending on k and q (which, without loss of generality, can be taken > 1) such that

$$\left|(D_x^k D_\xi^q)\sigma(x,\xi)\right| \quad \leq \quad c^{k+q}(1+|\xi|)^{m-q}. \tag{4.1}$$

It is customary to call the function $\sigma \in S^m$ a symbol. Now, we define the following

Definition 4.2. Let σ be a symbol. Define the $\Delta-$pseudo-differential operator $T_{\sigma,M}$ associated with σ by

$$(T_{\sigma,M}\phi)(x) = \int\limits_{\mathbb{R}} K_{M^{-1}}(\xi,x)\sigma(x,\xi)\mathcal{L}_M(\phi)(\xi)\,d\xi, \quad \phi \in \mathcal{S}.$$

Remark 4.3. The mapping properties of pseudo-differential operators between Schwartz spaces are well-known in the literature (see, e.g., [31]). Recently, in [19], pseudo-differential operators have been studied in the framework of a fractional Fourier transform and in [23] they have been studied in the spaces $\mathcal{S}_\Delta$ and in the corresponding space of tempered distribution $\mathcal{S}'_\Delta$. Below, we prove the mapping properties of the operator $T_{\sigma,M}$ between the generalized Gelfand-Shilov type spaces defined in Section 3.

Theorem 4.4. *The $\Delta-$pseudo-differential operator $T_{\sigma,M}$ maps $\mathcal{S}_{\Delta^*,\alpha,A}$ into $\mathcal{S}_{\Delta^*,1+\alpha,A'}$ for some $A' > 0$ depending on A.*

Proof. Let $\phi \in \mathcal{S}_{\Delta^*,\alpha,A}$. Then for each $\delta > 0$,

$$|(x^k(\triangle^*_{x,a})^q\phi)| \leq C_{q,\delta}(A+\delta)^k k^{k\alpha}, \qquad k,q \in \mathbb{N}_0.$$

Now,

$$\begin{aligned}
&|x^k(\triangle^*_{x,a})^q(T_{\sigma,M}\phi)(x)|\\
&= \left|x^k(\triangle^*_{x,a})^q\int\limits_{\mathbb{R}} K_{M^{-1}}(\xi,x)\sigma(x,\xi)\mathcal{L}_M[\phi](\xi)\,d\xi\right|\\
&= \left|x^k\int\limits_{\mathbb{R}}(\triangle^*_{x,a})^q[K_{M^{-1}}(\xi,x)\sigma(x,\xi)]\mathcal{L}_M[\phi](\xi)\,d\xi\right|\\
&= \left|x^k\int\limits_{\mathbb{R}}\left(\sum_{r=0}^{q}A_{q,r}(\triangle^*_{x,a})^r K_{M^{-1}}(\xi,x)D_x^{q-r}\sigma(x,u)\right)\mathcal{L}_M[\phi](\xi)\,d\xi\right|\\
&= \left|\int\limits_{\mathbb{R}}\left(\sum_{r=0}^{q}A_{q,r}x^k\left(\frac{-i\xi}{b}\right)^r K_{M^{-1}}(\xi,x)D_x^{q-r}\sigma(x,\xi)\right)\mathcal{L}_M[\phi](\xi)\,d\xi\right|\\
&= \left|\left(\frac{-i}{b}\right)^{-k}\int\limits_{\mathbb{R}}\left(\sum_{r=0}^{q}A_{q,r}\left(\frac{-i\xi}{b}\right)^r(\triangle^*_{\xi,d})^k K_{M^{-1}}(\xi,x)D_x^{q-r}\sigma(x,\xi)\right)\mathcal{L}_M[\phi](\xi)\,d\xi\right|\\
&= \left|\left(\frac{-i}{b}\right)^{-k}\sum_{r=0}^{q}A_{q,r}\left(\frac{-i}{b}\right)^r\int\limits_{\mathbb{R}}(\triangle^*_{\xi,d})^k K_{M^{-1}}(\xi,x)D_x^{q-r}\sigma(x,\xi)\xi^r\mathcal{L}_M[\phi](\xi)\,d\xi\right|\\
&= \left|\sum_{r=0}^{q}A_{q,r}\left(\frac{-i}{b}\right)^{-k+r}\int\limits_{\mathbb{R}}K_{M^{-1}}(\xi,x)(\triangle_{\xi,d})^k[D_x^{q-r}\sigma(x,\xi)\xi^r\mathcal{L}_M[\phi](\xi)]\,d\xi\right|\\
&= \left|\sum_{r=0}^{q}A_{q,r}\left(\frac{-i}{b}\right)^{-k+r}\int\limits_{\mathbb{R}}K_{M^{-1}}(\xi,x)\sum_{j=0}^{k}B_{k,j}D_\xi^j D_x^{q-r}\sigma(x,\xi)(\triangle_{\xi,d})^{(k-j)}[\xi^r\mathcal{L}_M[\phi](\xi)]\,d\xi\right|\\
&= \Bigg|\sum_{r=0}^{q}A_{q,r}\left(\frac{-i}{b}\right)^{-k+r}\int\limits_{\mathbb{R}}K_{M^{-1}}(\xi,x)\sum_{j=0}^{k}B_{k,j}D_\xi^j D_x^{q-r}\sigma(x,\xi)
\end{aligned}$$

$$\times \sum_{i=0}^{(k-j)} C_{k-j,i} D_\xi^i \xi^r (\triangle^*_{\xi,d})^{(k-j)-i} \mathcal{L}_M[\phi](\xi)]\, d\xi \Bigg|$$

$$\leq \sum_{r=0}^{q} |A_{q,r}||b|^{k-r} \int_{\mathbb{R}} |K_{M^{-1}}(x,\xi)| \sum_{j=0}^{k} |B_{k,j}||D_\xi^j D_x^{q-r} \sigma(x,\xi)|$$

$$\times \sum_{i=0}^{(k-j)} |C_{k-j,i}||D_\xi^i \xi^r||(\triangle_{\xi,d})^{(k-j)-i} \mathcal{L}_M[\phi](\xi)|\, d\xi. \tag{4.2}$$

Writing

$$|A_q| = \sup_r |A_{q,r}|, \quad |B_k| = \sup_j |B_{k,j}|, \quad |C_k| = \sup_{i,j} |C_{k-j,i}|, \tag{4.3}$$

the last estimate by using (4.1) gives

$$|x^k (\triangle^*_{x,a})^q (T_{\sigma,M}\phi)(x)|$$

$$\leq |A_q||B_k||C_k| \sum_{r=0}^{q} |b|^{k-r} \int_{\mathbb{R}} \sum_{j=0}^{k} |c^{(q-r)+j}|(1+|\xi|)^{m-j}$$

$$\times \sum_{i=0}^{(k-j)} |D_\xi^i \xi^r||(\triangle_{\xi,d})^{(k-j)-i} \mathcal{L}_M[\phi](\xi)|\, d\xi$$

$$\leq |A_q||B_k||C_k| \sum_{r=0}^{q} |b|^{k-r} \int_{\mathbb{R}} \sum_{j=0}^{k} |c^{(q-r)+j}|(1+|\xi|)^{m-j}$$

$$\times \sum_{i=0}^{(k-j)} \frac{r!}{(r-i)!} |\xi^{r-i}||(\triangle_{\xi,d})^{(k-j)-i} \mathcal{L}_M[\phi](\xi)|\, d\xi$$

$$\leq |A_q||B_k||C_k| \sum_{r=0}^{q} |b|^{k-r} \int_{\mathbb{R}} \sum_{j=0}^{k} c^{q+k}(1+|\xi|)^{m-j}$$

$$\times \sum_{i=0}^{(k-j)} \frac{r!}{(r-i)!} C_{r-i,\delta}(A+\delta)^{k-j-i}(k-j-i)^{(k-j-i)\alpha}\, d\xi$$

$$\leq |A_q||B_k||C_k| \sum_{r=0}^{q} |b|^{k-r} \int_{\mathbb{R}} \sum_{j=0}^{k} c^{q+k}(1+|\xi|)^{m-j}$$

$$\times (k-j)! \sum_{i=0}^{(k-j)} \frac{r!}{i!(r-i)!} C_{r-i,\delta}(A+\delta)^k (k)^{k\alpha}\, d\xi$$

$$\leq C_{q,\delta}|A_q||B_k||C_k| \sum_{r=0}^{q} |b|^{k-r} \int_{\mathbb{R}} \sum_{j=0}^{k} c^{q+k}(1+|\xi|)^{m-j}(k-j)! 2^r\, d\xi$$

$$\times (A+\delta)^k (k)^{k\alpha}$$

$$\leq C_{q,\delta}|A_q||B_k||C_k||b|^k \sum_{r=0}^{q} |b|^{-r} k! 2^q c^{q+k} \sum_{j=0}^{k} \int_{\mathbb{R}} (1+|\xi|)^{m-j}\, d\xi$$

$$\times (A+\delta)^k (k)^{k\alpha}$$

$$\leq 2^q c^q C_{q,\delta}|A_q| \left(\sum_{r=0}^{q} |b|^{-r} \right) |B_k||C_k||b|^k k! c^k I_k (A+\delta)^k (k)^{k\alpha},$$

where

$$I_k = \sum_{j=0}^{k} \int_{\mathbb{R}} (1+|\xi|)^{m-j}\, d\xi$$

in which the integral converges by choosing $m-k+1<0$. Thus we have

$$\begin{aligned}
&|x^k(\triangle^*_{x,a})^q(T_{\sigma,M}\phi)(x)| \\
&\quad \le 2^q c^q C_{q,\delta}|A_q|\left(\sum_{r=0}^{q}|b|^{-r}\right)|B_k||C_k||b|^k k^k C^k I_k (A+\delta)^k (k)^{k\alpha} \\
&\quad = E_{q,\delta}(A'+\delta')^k k^{(k+1)\alpha} \\
&\quad = E_{q,\delta'}(A'+\delta')^k k^{k(1+\alpha)},
\end{aligned}$$

where

$$E_{q,\delta} = 2^q c^q C_{q,\delta}|A_q|\left(\sum_{r=0}^{q}|b|^{-r}\right),\ A' = (|B_k||C_k||b|^k c^k I_k)^{1/k} A.$$

□

Theorem 4.5. *The Δ−pseudo-differential operator $T_{\sigma,M}$ maps $\mathcal{S}^{\Delta^*,\beta,B}$ into $\mathcal{S}^{\Delta^*,1+\beta,B'}$ for some $B'>0$ depending on B.*

Proof. Let $\phi \in \mathcal{S}_{\Delta^*,\alpha,A}$. Then for each $\beta>0$,

$$|(x^k(\triangle^*_{x,a})^q\phi)| \le C_{k,\beta}(B+\beta)^q q^{q\beta}, \quad k,q\in\mathbb{N}_0.$$

Now, using (4.1) and (4.2), we have

$$\begin{aligned}
&|x^k(\triangle^*_{x,a})^q(T_{\sigma,M}\phi)(x)| \\
&\quad \le \sum_{r=0}^{q}|A_{q,r}||b|^{k-r}\int_{\mathbb{R}}|K_{M^{-1}}(x,\xi)|\sum_{j=0}^{k}|B_{k,j}||D^j_\xi D^{q-r}_x\sigma(x,\xi)| \\
&\qquad \times \sum_{i=0}^{(k-j)}|C_{k-j,i}||D^i_\xi \xi^r||(\triangle_{\xi,d})^{(k-j)-i}\mathcal{L}_M[\phi](\xi)|\, d\xi \\
&\quad \le |A_q||B_k||C_k|\sum_{r=0}^{q}|b|^{k-r}\int_{\mathbb{R}}\sum_{j=0}^{k}|c^{(q-r)+j}|(1+|\xi|)^{m-j} \\
&\qquad \times \sum_{i=0}^{(k-j)}|D^i_\xi \xi^r||(\triangle_{\xi,d})^{(k-j)-i}\mathcal{L}_M[\phi](\xi)|\, d\xi \\
&\quad \le |A_q||B_k||C_k|\sum_{r=0}^{q}|b|^{k-r}\int_{\mathbb{R}}\sum_{j=0}^{k}|c^{(q-r)+j}|(1+|\xi|)^{m-j} \\
&\qquad \times \sum_{i=0}^{(k-j)}\frac{r!}{(r-i)!}|\xi^{r-i}||(\triangle_{\xi,d})^{(k-j)-i}\mathcal{L}_M[\phi](\xi)|\, d\xi \\
&\quad \le |A_q||B_k||C_k|\sum_{r=0}^{q}|b|^{k-r}\int_{\mathbb{R}}\sum_{j=0}^{k}c^{q+k}(1+|\xi|)^{m-j} \\
&\qquad \times \sum_{i=0}^{(k-j)}\frac{r!}{(r-i)!}C_{k-j-i,\beta}(B+\beta)^{r-i}(r-i)^{(r-i)\beta}\, d\xi \\
&\quad \le |A_q||B_k||C_k|\sum_{r=0}^{q}|b|^{k-r}\int_{\mathbb{R}}\sum_{j=0}^{k}c^{q+k}(1+|\xi|)^{m-j}
\end{aligned}$$

$$\times\, C_{k,\beta} \sum_{i=0}^{(k-j)} \frac{r!}{(r-i)!} (B+\beta)^{r-i} (r-i)^{(r-i)\beta}\, d\xi$$

$$\le |A_q||B_k||C_k| \sum_{r=0}^{q} |b|^{k-r} \int_{\mathbb{R}} \sum_{j=0}^{k} c^{q+k} (1+|\xi|)^{m-j}$$

$$\times\, C_{k,\beta}\, r! \sum_{i=0}^{r} \frac{r!}{i!(r-i)!} (B+\beta)^{r-i} q^{q\beta}\, d\xi$$

$$\le |A_q||B_k||C_k| \sum_{r=0}^{q} |b|^{k-r} \sum_{j=0}^{k} c^{q+k} \int_{\mathbb{R}} (1+|\xi|)^{m-j}$$

$$\times\, C_{k,\beta}\, r! \sum_{i=0}^{r} \frac{r!}{i!(r-i)!} (B+\beta)^{r-i} q^{q\beta}\, d\xi$$

$$= |A_q||B_k||C_k| \sum_{r=0}^{q} |b|^{k-r} c^{q+k} \sum_{j=0}^{k} \int_{\mathbb{R}} (1+|\xi|)^{m-j}$$

$$\times\, C_{k,\beta}\, r! (1+B+\beta)^r q^{q\beta}\, d\xi$$

$$\le |A_q||B_k||C_{k,\beta}||C_k| \sum_{r=0}^{q} |b|^{k-r} c^{q+k} I_k q! (1+B+\beta)^q q^{q\beta}\, d\xi,$$

where

$$I_k = \sum_{j=0}^{k} \int_{\mathbb{R}} (1+|\xi|)^{m-j}\, d\xi$$

in which the integral converges by choosing $m-k+1<0$. Thus we have

$$\begin{aligned}
&|x^k(\triangle^*_{x,a})^q(T_{\sigma,M}\phi)(x)| \\
&\quad \le |b|^k C^k I_k |B_k||C_{k,\beta}||C_k||A_q| \left(\sum_{r=0}^{q} |b|^{-r} \right) c^q q^q (1+B+\beta)^q q^{q\beta} \\
&\quad = E_{k,\beta}(B'+\beta')^q q^{q(1+\beta)} \\
&\quad = E_{k,\beta'}(B'+\beta')^q q^{q(1+\beta)},
\end{aligned}$$

where

$$E_{k,\beta} = |b|^k C^k I_k |B_k||C_{k,\beta}||C_k|,\ B' = \left(|A_q| \left(\sum_{r=0}^{q} |b|^{-r} \right) c^q \right)^{1/q} A$$

and

$$\beta' = \left(|A_q| \left(\sum_{r=0}^{q} |b|^{-r} \right) c^q \right) \beta.$$

□

In a similar way we can prove the following

Theorem 4.6. *The $\Delta-$pseudo-differential operator $T_{\sigma,M}$ maps $\mathcal{S}^{\Delta^*,\beta,B}_{\Delta^*,\alpha,A}$ into $\mathcal{S}^{\Delta^*,1+\beta,B'}_{\Delta^*,1+\alpha,A'}$.*

5. Concluding Remark

Harmonic oscillators occupy an important place in several science and engineering fields. Many mechanical systems such as vibrating string with small amplitude about an equilibrium point can be modeled as a simple harmonic oscillator. In [18], the author points out that one of the most important features of an harmonic oscillator is its energy eigenstates which can be described in terms of their coordinate space wave functions $\psi_n(x)$ as

$$\frac{d^2\psi_n}{dX^2} + \frac{2M}{h^2}\left(E_n - \frac{M\Omega^2 X^2}{2}\right)\psi_n = 0, \tag{5.1}$$

where X is the spatial coordinate, E_n is the energy of the nth stationary state of the oscillator, M is the mass and Ω is the frequency. The author in [18] solved equation (4.1) by using a simple harmonic transformation, although many classical approaches already exist (see, e.g., [4], [27]).

Similarly, in [11], the authors describe the relationship of fractional Fourier transform and certain variants of equation (4.1). Since the LCT is more general than the fractional Fourier transform, it is of interest to work out the relationship between LCT and some further generalized equations as dealt with in [18] or [11].

In the recent paper [3], the authors have defined and studied a transform more general than the LCT, the so-called Special Affine Fourier Transform (SAFT). It will be of interest if the results of the present paper are extended in the framework of SAFT.

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