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MULTILINEAR FEFFERMAN-STEIN INEQUALITY AND ITS GENERALIZATIONS

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Abstract. The Fefferman-Stein type inequalities are established for multilinear fractional maximal operators with a variable parameter defined with respect to the basis $\mathcal{B}$ on $\mathbb{R}^n$ which may be both either $\mathcal{Q}$ or $\mathcal{R}$, where $\mathcal{Q}$ (resp., $\mathcal{R}$) consists of all cubes (resp., of n-dimensional intervals) with sides parallel to the coordinate axes. Some related two-weight boundedness problems are also investigated.

1. Introduction

Let $\mathcal{B}$ in $\mathbb{R}^n$ be a basis which may be both either $\mathcal{Q}$ or $\mathcal{R}$, where $\mathcal{Q}$ (resp., $\mathcal{R}$) is a basis consisting of all cubes (resp., of $n-$dimensional intervals) with sides parallel to the coordinate axes. Further, let

$$\overrightarrow{f} := (f_1, \dots, f_m), \quad \overrightarrow{p} := (p_1, \dots, p_m), \quad \overrightarrow{w} = (w_1, \dots, w_m),$$

where p_i are the constants ($0 < p_i < \infty$) and w_i are a.e. positive functions defined on the Euclidean space. It will also be assumed that

$$\frac{1}{p} = \sum_{i=1}^{m} \frac{1}{p_i}. \tag{1}$$

For a given function $\alpha(\cdot)$ on $\mathbb{R}^n$, let

$$\alpha_- := \inf \alpha(\cdot), \quad \alpha_+ := \sup \alpha(\cdot).$$

In this paper we establish the following inequalities: $1 < p_i, q < \infty$, $i = 1, \dots, m$, and $1 < p < q < \infty$, where p is defined by (1). Then

(i)

$$\left\| \left(\mathcal{M}^{(\mathcal{B})}_{\alpha(\cdot)} \overrightarrow{f} \right) v \right\|_{L^q} \leq C \prod_{i=1}^{m} \left\| f_i \big(\widetilde{M}^{(\mathcal{B})}_{\alpha(\cdot), p_i, q} v_i \big)^{1/q} \right\|_{L^{p_i}}, \tag{2}$$

where $v(x) = \prod_{i=1}^{m} v_i^{p/p_i}(x)$, $\mathcal{M}^{(\mathcal{B})}_{\alpha(\cdot)}$ is a strong fractional maximal operator defined with respect to the basis $\mathcal{B}$ given by the formula

$$\mathcal{M}^{(\mathcal{B})}_{\alpha(x)}(\overrightarrow{f})(x) = \sup_{B \ni x, B \in \mathcal{B}} \prod_{i=1}^{m} \frac{1}{|B|^{1-\alpha(x)/(nm)}} \int_B |f_i(y_i)| dy_i, \quad 0 < \alpha_- \leq \alpha_+ < mn, \tag{3}$$

and $\widetilde{M}^{(\mathcal{B})}_{\alpha(\cdot), p_i, q}$, $i = 1, \dots, m$, are the appropriate fractional maximal operators (see the definition in Theorem 2.1).

(ii)

$$\left\| \mathcal{M}^{(\mathcal{B})}_{\alpha(\cdot), \mu} \overrightarrow{f} \right\|_{L^q_\mu} \leq C \prod_{i=1}^{m} \left\| f_i \big(\widetilde{M}^{(\mathcal{B})}_{\alpha(\cdot), p, q, \mu} d\mu \big)^{1/(mq)} \right\|_{L^{p_i}_\mu}, \ 0 < \alpha_- \leq \alpha_+ < mn,$$

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with $d\mu(x) = w(x)dx$, where w is a weight function satisfying the doubling condition, the maximal function $\mathcal{M}^{(\mathcal{B})}_{\alpha(x),\mu}$ is defined by

$$\mathcal{M}^{(\mathcal{B})}_{\alpha(x),\mu}(\overrightarrow{f})(x) = \sup_{B\ni x,\ B\in\mathcal{B}} \prod_{i=1}^{m} \frac{|B|^{\alpha(x)/(nm)}}{\mu(B)} \int\limits_B |f_i(y_i)|d\mu,\ 0<\alpha_- \le \alpha_+ < mn,\ 1<p<q<\infty,$$

and $\widetilde{M}^{(\mathcal{B})}_{\alpha(\cdot),p,q,\mu}$ is appropriate fractional maximal operator (see the definition in Theorem 2.2).

We claim that the these results are new even for the linear case ($m = 1$).

For two-weight inequalities and for strong fractional maximal operators with variable parameters we refer to the monograph [19], Chapter 6.

Recall that inequality (2) was derived in [14] for $v_1 = \cdots = v_m = v$ and $\alpha(\cdot) =$ const.

Operator (3) for $\alpha(x) \equiv 0$ and $\mathcal{B} = \mathcal{R}$ was introduced in [10]. In this case we have multi(sub)linear strong maximal operator denoted by $\mathcal{M}^{(S)}$ and defined with respect to rectangles in $\mathbb{R}^k$ with sides parallel to the coordinate axes. In that paper the authors studied one– and two–weight problems for $\mathcal{M}^{(S)}$. In particular, they proved that the one-weight boundedness $\mathcal{M}^{(S)} : L^{p_1}_{w_1} \times \cdots \times L^{p_m}_{w_m} \mapsto L^p_{\nu_{\overrightarrow{w}}}$, $\nu_{\overrightarrow{w}} = \prod_{j=1}^m w_j^{p/p_j}$, holds if and only if $\overrightarrow{w}$ weight satisfies the strong $A_{\overrightarrow{p}}$ condition

$$\sup_{R\in\mathcal{R}} \left(\frac{1}{|R|}\int\limits_R \nu_{\overrightarrow{w}}(x)dx\right)^{1/p} \prod_{i=1}^m \left(\frac{1}{|R|}\int\limits_R w_i^{1-p_i'}(x)dx\right)^{1/p_i'} < \infty.$$

Historically, multilinear fractional integrals were introduced in their papers by L. Grafakos [8], C. Kenig and E. Stein [11], L. Grafakos and N. Kalton [9]. In particular, these works deal with the operator

$$B_\gamma(f,g)(x) = \int\limits_{\mathbb{R}^n} \frac{f(x+t)g(x-t)}{|t|^{n-\gamma}}dt,$$

where γ is a constant parameter satisfying the condition $0 < \gamma < n$.

In the above-mentioned papers it was proved that if $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{n}$, where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, then B_γ is bounded from $L^{p_1} \times L^{p_2}$ to L^q.

As a tool to understand B_γ, the operator

$$\mathcal{I}_\gamma(\overrightarrow{f})(x) = \int\limits_{(\mathbb{R}^n)^m} \frac{f_1(y_1)\cdots f_m(y_m)}{(|x-y_1|+\cdots+|x-y_m|)^{mn-\gamma}}d\overrightarrow{y},$$

where $x \in \mathbb{R}^n$, γ is constant satisfying the condition $0 < \gamma < nm$, $\overrightarrow{f} := (f_1,\dots,f_m)$, $\overrightarrow{y} := (y_1,\dots,y_m)$, was studied as well. The corresponding maximal operator is given by (see [22]) the formula

$$\mathcal{M}_\gamma(\overrightarrow{f})(x) = \sup_{Q\ni x} \prod_{i=1}^m \frac{1}{|Q|^{1-\frac{\gamma}{mn}}} \int\limits_Q |f_i(y_i)|dy_i,$$

and the supremum is taken over all cubes Q containing x.

For a variable parameter $\alpha(\cdot)$, let

$$\mathcal{I}_{\alpha(\cdot)}(\overrightarrow{f})(x) = \int\limits_{(\mathbb{R}^n)^m} \frac{f_1(y_1)\dots f_m(y_m)}{(|x-y_1|+\cdots+|x-y_m|)^{mn-\alpha(x)}}d\overrightarrow{y},$$

$$\mathcal{M}_{\alpha(\cdot)}(\overrightarrow{f})(x) = \sup_{Q\ni x} \prod_{i=1}^m \frac{1}{|Q|^{1-\frac{\alpha(x)}{mn}}} \int\limits_Q |f_i(y_i)|dy_i,$$

where $0 < \alpha_- \le \alpha_+ < nm$. The operator $\mathcal{M}_{\alpha(\cdot)}$ for $\alpha \equiv 0$ was introduced and studied in [21].

It can be immediately checked that

$$\mathcal{I}_{\alpha(x)}(\overrightarrow{f})(x) \ge c_{n,\alpha(\cdot)}\mathcal{M}_{\alpha(x)}(\overrightarrow{f})(x), \quad f_i \ge 0, \quad i = 1,\dots,m.$$

Throughout the paper, we use the notation $\mathcal{Q}$ to denote the family of all cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes.

Let $0 < r < \infty$ and let μ be a σ- finite measure on $\mathbb{R}^n$. We denote by $L^r_\mu(\mathbb{R}^n)$ the class of all $\mu-$ measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^r_\mu(\mathbb{R}^n)} := \left(\int\limits_{\mathbb{R}^n} |f(x)|^r d\mu(x) \right)^{1/r} < \infty.$$

If $d\mu(x) = w(x)dx$ with a weight function w, then we also use the symbol $L^r_w(\mathbb{R}^n)$ for $L^r_\mu(\mathbb{R}^n)$.

Definition 1.1 (Vector Muckenhoupt condition, [21])**.** Let $1 \le p_i < \infty$ for $i = 1, \dots, m$. Let w_i be weights on $\mathbb{R}^n$, $i = 1, \dots, m$. We say that $\overrightarrow{w} \in A_{\overrightarrow{p}}(\mathbb{R}^n)$ (or simply $\overrightarrow{w} \in A_{\overrightarrow{p}}$) if

$$\sup_{Q \in \mathcal{Q}} \left(\frac{1}{|Q|} \int\limits_Q \prod_{i=1}^m w_i^{p/p_i}(y) dy \right)^{1/p} \prod_{i=1}^m \left(\frac{1}{|Q|} \int\limits_Q w_i^{1-p_i'}(y) dy \right)^{1/p_i'} < \infty.$$

Remark 1.1. In the linear case ($m = 1$) the class $A_{\overrightarrow{p}}$ coincides with the well-known Muckenhoupt class A_p.

Definition 1.2 (Vector Muckenhoupt–Wheeden condition, [22])**.** Let $1 \le p_i < \infty$ for $i = 1, \dots, m$. Suppose that $p < q < \infty$. We say that $\overrightarrow{w} = (w_1, \dots, w_m)$ satisfies $A_{\overrightarrow{p},q}(\mathbb{R}^n)$ condition ($\overrightarrow{w} \in A_{\overrightarrow{p},q}$) if

$$\sup_{Q \in \mathcal{Q}} \left(\frac{1}{|Q|} \int\limits_Q \prod_{i=1}^m w_i^{q}(y) dy \right)^{1/q} \prod_{i=1}^m \left(\frac{1}{|Q|} \int\limits_Q w_i^{-p_i'}(y) dy \right)^{1/p_i'} < \infty.$$

Theorem A ([21])**.** *Let $1 < p_i < \infty$, $i = 1, \dots, m$. Suppose that w_i are weights on $\mathbb{R}^n$. Then the operator M_0 is bounded from $L^{p_1}_{w_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}_{w_m}(\mathbb{R}^n)$ to $L^p_{\prod_{i=1}^m w_i^{p/p_i}}(\mathbb{R}^n)$ if and only if $\overrightarrow{w} \in A_{\overrightarrow{p}}(\mathbb{R}^n)$.*

Theorem B ([22])**.** *Let $1 < p_1, \dots, p_m < \infty$, $0 < \gamma < mn$, $\frac{1}{m} < p < \frac{n}{\gamma}$. Assume that q is an exponent satisfying the condition $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{n}$. Suppose that w_i are a.e. positive functions on $\mathbb{R}^n$ such that $w_i^{p_i}$ are weights. Then the inequality*

$$\left(\int\limits_{\mathbb{R}^n} \left(\big|N_\gamma(\overrightarrow{f})(x)\big| \prod_{i=1}^m w_i(x) \right)^q dx \right)^{1/q} \le C \prod_{i=1}^m \left(\int\limits_{\mathbb{R}^n} \big(|f_i(y)| w_i(x)\big)^{p_i} dx \right)^{1/p_i},$$

holds, where N_γ is either I_γ or M_γ, if and only if $\overrightarrow{w} \in A_{\overrightarrow{p},q}(\mathbb{R}^n)$.

Remark 1.2. The two-weight problem for linear fractional integral operators has been already solved. We mention the papers due to E. Sawyer [26] for the conditions involving the operator itself, due to M. Gabidzashvili and V. Kokilashvili [6] (see also [13]) and R. L. Wheeden [32] for integral type conditions.

Finally, we mention that the weighted inequalities for multilinear fractional integrals were also studied in [25], [4], [14], [15]. The study of the boundedness of multi(sub)linear fractional strong maximal operators was initiated in [10] and continued in [15], [2], [3], etc.

1.1. **Preliminaries.** By the symbol $\mathcal{DQ}(\mathbb{R}^n)$ (or shortly, $\mathcal{DQ}$) is denoted a countable collection of dyadic cubes that enjoy the following properties:

(i) $Q \in \mathcal{DQ} \Rightarrow l(Q) = 2^k$ for some $k \in \mathbb{Z}$;
(ii) $Q, P \in \mathcal{DQ} \Rightarrow Q \cap P \in \{\emptyset, P, Q\}$;
(iii) for each $k \in \mathbb{Z}$ the set $\mathcal{DQ}_k = \{Q \in \mathcal{DQ} : l(Q) = 2^k\}$ forms a partition of $\mathbb{R}^n$.

Definition 1.3. We say that a weight function ρ satisfies the dyadic reverse doubling condition with respect to the cubes ($\rho \in \mathcal{RDQ}^{(d)}(\mathbb{R})$) if there exists a constant $d > 1$ such that

$$d\rho(Q') \le \rho(Q),$$

for all $Q', Q \in \mathcal{DQ}$, where Q' is a child interval of Q, i.e., $Q' \subset Q$ and $|Q| = 2^n|Q'|$.

We shall also need the following Carleson–Hörmander type embedding theorem regarding the dyadic intervals.

Theorem C (see, e.g., [29], [31])**.** *Let $1 < r < q < \infty$ and let ρ be a weight function on $\mathbb{R}^n$ such that $\rho^{1-r'}$ satisfies the dyadic reverse doubling condition. Then the Carleson–Hörmander type inequality*

$$\sum_{Q\in\mathcal{DQ}}\bigg(\int\limits_Q \rho^{1-r'}(x)dx\bigg)^{-q/r'}\bigg(\int\limits_Q f(x)dx\bigg)^q \le c\bigg(\int\limits_{\mathbb{R}^n} f^r(x)\rho(x)dx\bigg)^{q/r}$$

holds for all non-negative $f\in L^r_\rho(\mathbb{R}^n)$.

We denote by $\mathcal{DR}$ the family of all dyadic rectangles in $\mathbb{R}^n$ given by the formula

$$\mathcal{DR} := \{2^{-k}(m+[0,1)) : k, m\in\mathbb{Z}\}^n.$$

Definition 1.4. We say that a weight function ρ satisfies the dyadic reverse doubling condition with respect to the rectangles ($\rho\in\mathcal{RDR}^{(d)}(\mathbb{R}^n)$) if there exists a constant $d>1$ such that

$$d\rho(R')\le\rho(R),$$

for all $R', R\in\mathcal{DR}$, where $R'\subset R$ and $|R| = 2|R'|$.

We denote by $\mathcal{DB}(\mathbb{R}^n)$ (or simply, $\mathcal{DB}$) the dyadic grid which is $\mathcal{DQ}$ for $\mathcal{B}=\mathcal{Q}$ and $\mathcal{DR}$ for $\mathcal{B}=\mathcal{R}$.

In the sequel, under the symbol $\mathcal{DRB}^{(d)}(\mathcal{R}^n)$ (or simply, $\mathcal{DRB}^{(d)}$) we mean the class of weights satisfying the dyadic reverse doubling condition in the sense of cubes if $\mathcal{B}=\mathcal{Q}$, and in the sense of rectangles if $\mathcal{B}=\mathcal{R}$. Further, for $B\in\mathcal{B}$ and $c>0$ we denote by cB the set in $\mathbb{R}^n$ with the same center but with c times the side-length of B. We say that a measure μ defined on $\mathbb{R}^n$ satisfies the doubling condition with respect to $\mathcal{Q}$ ($\mu\in\mathcal{DCQ}$) if there is a positive constant b_μ such that for all $B\in\mathcal{Q}$ the inequality

$$\mu(2B)\le b_\mu\mu(B) \tag{4}$$

holds; further, we say that μ satisfies the doubling condition with respect to $\mathcal{R}$ ($\mu\in\mathcal{DCR}$) if (4) holds for all $B\in\mathcal{R}$. We write $\mu\in\mathcal{DCB}$ if $\mu\in\mathcal{DCQ}$ for a basis $\mathcal{Q}$, and $\mu\in\mathcal{DCR}$ for the basis $\mathcal{R}$.

Definition 1.5. We say that a measure μ satisfies the reverse doubling condition with respect to $\mathcal{R}$ ($\mu\in\mathcal{RDR}$) if there is a constant $\beta>1$ such that $\beta\mu(R')\le\mu(R)$ for any $R, R'\in\mathcal{R}$, where R' is the two-equal division of R. Further, μ satisfies the reverse doubling condition with respect to $\mathcal{Q}$ ($\mu\in\mathcal{RDQ}$) if there is a constant $\beta>1$ such that $\beta\mu(Q')\le\mu(Q)$ for any $Q, Q'\in\mathcal{Q}$, where R' is the 2^n-equal division of Q. We say that $\mu\in\mathcal{RDB}$ if $\mu\in\mathcal{RDR}$ for $\mathcal{B}=\mathcal{DR}$, and $\mu\in\mathcal{RDQ}$ for $\mathcal{B}=\mathcal{DQ}$.

The following fact was noticed in [28]:

Remark 1.3 ([28])**.** The condition $\mu\in\mathcal{DCB}$ is equivalent to the condition $\mu\in\mathcal{RDB}$.

Proposition 1.1 ([2])**.** *Let $1<r<q<\infty$ and let ρ be a weight function on $\mathbb{R}^n$ such that $\rho^{1-r'}\in\mathcal{RDR}(\mathbb{R}^n)$. Then there is a positive constant C such that the inequality*

$$\sum_{R\in\mathcal{DR}}\bigg(\int\limits_R \rho^{1-r'}(x)dx\bigg)^{-q/r'}\bigg(\int\limits_R f(x)dx\bigg)^q \le C\bigg(\int\limits_{\mathbb{R}^n} f^r(x)\rho(x)dx\bigg)^{q/r}$$

holds for all non-negative $f\in L^r_\rho(\mathbb{R}^n)$.

Proposition 1.1 for the weight $\rho^{(n)}$ having the form $\rho^{(n)}(x_1,\dots,\rho_n)=\rho_1(x_1)\times\cdots\times\rho_n(x_n)$ can also be derived by a simple proof based on the mathematical induction. Indeed, the statement is true owing to Theorem C for $n=1$. Suppose that it is true for $n-1$–dimensional dyadic rectangles and a weight of the form $\rho^{(n-1)}(x_1,\dots,\rho_{n-1})=\rho_1(x_1)\times\cdots\times\rho_n(x_{n-1})$. We set $R:=I_1\times\cdots\times I_n$, $R_{n-1}:=I_1\times\cdots\times I_{n-1}$.

We have

$$\sum_{\rho^{(n)}(R)\in\mathcal{DR}(\mathbb{R}^n)} |R|^{\frac{q}{r'}}\Bigg(\int\limits_R f(x_1,\dots,x_n)\prod_{i=1}^n \rho_i(x_i)dx_1\dots dx_n\Bigg)^q$$

$$\le \sum_{I_n\in\mathcal{DR}(\mathbb{R})} \rho_n(I_n)^{\frac{q}{r'}} \sum_{R_{n-1}\in\mathcal{DR}(\mathbb{R}^{n-1})} \Bigg(\prod_{i=1}^{n-1}\rho_i(x_i)\Bigg)^{-\frac{q}{r'}}$$

$$\times\Bigg(\int\limits_{R_{n-1}}\Bigg(\int\limits_{I_n} f(x_1,\dots,x_n)\rho(x_n)dx_n\Bigg)dx_1\dots dx_{n-1}\rho_1(x_1)\times\dots\times\rho_n(x_{n-1})\Bigg)^q$$

$$\le \sum_{I_n\in\mathcal{DR}(\mathbb{R}^{n-1})} |I_n|^{-\frac{q}{r'}}\Bigg(\int\limits_{R_{n-1}}\Bigg(\int\limits_{I_n} f(x_1,\dots,x_{n-1})\rho_{x_n}dx_1\Bigg)^q$$

$$\times\,\rho_1(x_1)\dots\rho_n(x_{n-1})dx_1\dots dx_{n-1}\Bigg)^p dx_n\Bigg)^{q/r}$$

$$\le \sum_{I_n\in\mathcal{DR}(\mathbb{R}^{n-1})} |I_n|^{-\frac{q}{p'}}\Bigg(\int\limits_{I_n}\Bigg(f^r(x_1,\dots,x_{n-1})\rho_1(x_1)\dots\rho_n(x_n)dx_1\dots dx_{n-1}\Bigg)^{\frac{1}{r}}\rho_n(x_n)dx_n\Bigg)^q$$

$$\le C\Bigg(\int\limits_{\mathbb{R}^n} f^r(x_1,\dots x_n)\rho(x_1,\dots,x_n)dx_1\dots dx_n\Bigg)^{\frac{q}{r}}.$$

2. Main Results

Now we formulate our main results.

Theorem 2.1. *Let $1<p_i<\infty$, $i=1,\dots,m$. Suppose that $p<q<\infty$ and $0<\alpha_-\le\alpha_+<mn$. Let v_i's be weights on $\mathbb{R}^n$, $i=1,\cdots,m$. We set $v(x)=\prod_{i=1}^m v_i^{p/p_i}(x)$. Then the inequality*

$$\|\mathcal{M}^{(\mathcal{B})}_{\alpha(\cdot)}(\overrightarrow{f})\|_{L^q_v(\mathbb{R}^n)}\le C\prod_{i=1}^m \|f_i\big(\widetilde{M}^{(\mathcal{B})}_{\alpha(\cdot),p_i,q}v_i\big)^{1/q}\|_{L^{p_i}(\mathbb{R}^n)}$$

holds, where

$$\widetilde{M}^{(\mathcal{B})}_{\alpha(x),p_i,q}v_i(x)=\sup_{B\ni x,B\in\mathcal{B}}\prod_{i=1}^n\Bigg(\frac{1}{|B|^{q/p}}\int\limits_B |B|^{\frac{\alpha(y)q}{n}}v_i(y)dy\Bigg)^{p/p_i},\quad i=1,\dots,m.$$

The next two corollaries were proved in [15] for $\alpha(\cdot)\equiv\alpha=\text{const}$.

Corollary 2.1. *Let $1<p_i<\infty$, $i=1,\dots,m$. Suppose that $p<q<\infty$ and $0<\alpha_-\le\alpha_+<mn$. Let v be a weight on $\mathbb{R}^n$. Then the following inequality*

$$\|\mathcal{M}^{(\mathcal{B})}_{\alpha(\cdot)}(\overrightarrow{f})\|_{L^q_v(\mathbb{R}^n)}\le C\prod_{i=1}^m \|f_i\big(\widetilde{M}^{(\mathcal{B})}_{\alpha(\cdot),p,q}v\big)^{1/q}\|_{L^{p_i}(\mathbb{R}^n)},$$

holds, where

$$\widetilde{M}^{(\mathcal{B})}_{\alpha(x),p,q}v(x)=\sup_{B\ni x,B\in\mathcal{B}}\frac{1}{|B|^{q/p}}\int\limits_B |B|^{\frac{\alpha(y)q}{n}}v(y)dy.$$

Corollary 2.2. *Let the conditions of Corollary 2.1 be satisfied. Then the inequality*

$$\|\mathcal{M}^{(\mathcal{B})}_{\alpha(\cdot)}(\overrightarrow{f})\|_{L^q_v(\mathbb{R}^n)}\le C\prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R}^n)}\tag{5}$$

holds if and only if

$$\sup_{B\in\mathcal{B}}\frac{1}{|B|^{q/p}}\int\limits_B |B|^{\frac{\alpha(y)q}{n}}v(y)dy<\infty.$$

Theorem 2.2. *Let $1 < p_i < \infty$, $i = 1, \dots, m$. Suppose that $p < q < \infty$ and $0 < \alpha_- \le \alpha_+ < mn$. Suppose that a measure μ is doubling, $d\mu(x) = v(x)dx$, where v is a weight on $\mathbb{R}^n$. Then the inequality*

$$\|\mathcal{M}^{(\mathcal{B})}_{\alpha(\cdot),\mu}(\overrightarrow{f})\|_{L^q_\mu(\mathbb{R}^n)} \le C \prod_{i=1}^m \Big\| f_i \Big(\widetilde{M}^{(\mathcal{B})}_{\alpha(\cdot),p,q,\mu}(d\mu)\Big)^{1/(mq)} \Big\|_{L^{p_i}_\mu(\mathbb{R}^n)}$$

holds, where

$$\widetilde{M}^{(\mathcal{B})}_{\alpha(x),p,q,\mu}(d\mu)(x) = \sup_{B \ni x, B \in \mathcal{B}} \frac{1}{\mu(B)^{q/p}} \int_B |B|^{\frac{\alpha(y)q}{n}} d\mu(y). \tag{6}$$

Corollary 2.3. *Let the conditions of Theorem 2.2 hold. Then the inequality*

$$\|\mathcal{M}^{(\mathcal{B})}_{\alpha(\cdot),\mu}(\overrightarrow{f})\|_{L^q_\mu(\mathbb{R}^n)} \le C \prod_{i=1}^m \|f_i\|_{L^{p_i}_\mu(\mathbb{R}^n)}$$

holds if and only if

$$\sup_{B \in \mathcal{B}} \frac{1}{\mu(B)^{q/p}} \int_B |B|^{\frac{\alpha(y)q}{n}} d\mu(y) < \infty.$$

Let us introduce the following strong fractional maximal operator defined with respect to a measure μ:

$$\mathcal{N}^{(\mathcal{B})}_{\alpha,\mu}(\overrightarrow{f})(x) = \sup_{B \ni x, B \in \mathcal{B}} \prod_{i=1}^m \frac{1}{\mu(B)^{1-\alpha/m}} \int_B |f_i(y)| d\mu(y),$$

where α is a constant such that $0 < \alpha < nm$.

We have also proved the following statement.

Theorem 2.3. *Let μ be an infinite measure on $\mathbb{R}^n$ without atoms such that $\mu \in \mathcal{DCB}$, $1 < p_i < \infty$, $i = 1, \dots, m$. Let α be a constant such that $0 < \alpha < n/p$. Then the inequality*

$$\|\mathcal{N}^{(\mathcal{B})}_{\alpha,\mu}(\overrightarrow{f})\|_{L^q_\mu(\mathbb{R}^n)} \le C \prod_{i=1}^m \|f_i\|_{L^{p_i}_\mu(\mathbb{R}^n)}$$

holds if and only if $q = \frac{np}{n-\alpha p}$.

It should be mentiond that the necessary and sufficient condition governing the boundedness of the multilinear fractional integral operator

$$T_{\gamma,\mu}\vec{f}(x) = \int_{X^m} \frac{f_1(y_1)\dots f(y_m)}{\big(d(x,y_1) + \cdots + d(x,y_m)\big)^{m-\gamma}} d\mu(\vec{y}), \quad d\mu(\vec{y}) := d\mu(y_1)\dots d\mu(y_m)$$

defined with respect to a measure μ on a σ-algebra of Borel sets of quasi-metric space (X, d, μ) from the product $L^{p_1}(X,\mu) \times \cdots \times L^{p_m}(X,\mu)$ to $L^q(X,\mu)$ has been established recently in [16].

3. Proofs of the Main Results

In this section we give the proofs of the main results of this paper.

First of all, we will need the following statement.

Lemma 3.1 ([20])**.** *There exist 2^n shifted dyadic grids*

$$\mathcal{D}^\beta := \{2^{-k}([0,1)^n + m + (-1)^k\beta) : k \in \mathbb{Z}, m \in \mathbb{Z}^n\}, \quad \beta \in \{0, 1/3\}^n,$$

such that for any given cube Q there are a β and a $Q_\beta \in \mathcal{D}^\beta$ with $Q \subset Q_\beta$ and $l(Q_\beta) \le 6l(Q)$.

As a consequence of this lemma, one has the following pointwise estimate

$$\mathcal{M}_{\alpha(\cdot)}(\overrightarrow{f})(x) \le C \sum_{\beta \in \{0,1/3\}^n} \mathcal{M}^{(d),\mathcal{D}^\beta}_{\alpha(\cdot)}(\overrightarrow{f})(x), \tag{7}$$

where $\mathcal{M}^{(d),\mathcal{D}^\beta}_{\alpha(\cdot)}$ is the dyadic multi(sub)linear fractional maximal operator corresponding to the dyadic grid $\mathcal{D}^\beta$ defined by

$$(\mathcal{M}^{(d),\mathcal{D}^\beta}_{\alpha(\cdot)}\overrightarrow{f})(x)=\sup_{\mathcal{D}^\beta\ni Q,Q\ni x}\prod_{i=1}^m\frac{1}{|Q|^{1-\alpha(\cdot)/(nm)}}\int\limits_Q|f_i(y_i)|dy_i,\quad 0<\alpha_-\le\alpha_+<mn,$$

and the constant C depending only on n, m and α.

Remark 3.1. It can be checked that estimates similar to (7) are also true for the operators $\mathcal{M}^{(\mathcal{B})}_{\alpha(\cdot)}$, $\mathcal{M}^{(\mathcal{B})}_{\alpha(\cdot),\mu}$ and $\mathcal{N}^{(\mathcal{B})}_{\alpha(\cdot),\mu}$ provided that $\mu\in\mathcal{DCB}$.

Proof of Theorem 2.1. First we show that the two-weight inequality

$$\|\mathcal{M}^{(d),\mathcal{B}}_{\alpha(\cdot)}(\overrightarrow{f})\|_{L^q_v(\mathbb{R}^n)}\le C\prod_{i=1}^m\|f_i\big(\widetilde{M}^{(d),(\mathcal{B})}_{\alpha(\cdot),p_i,q}v_i\big)^{1/q}\|_{L^{p_i}(\mathbb{R}^n)}$$

holds, where $\mathcal{M}^{(d),(\mathcal{B})}_{\alpha(\cdot)}$ is an appropriate to $\mathcal{M}^{\mathcal{B}}_{\alpha(\cdot)}$ dyadic maximal operator and

$$\widetilde{M}^{(\mathcal{B})}_{\alpha(\cdot),p_i,q}v_i(x)=\sup_{B\ni x,B\in(\mathcal{B})}\left(\frac{1}{|B|^{q/p}}\int\limits_B|B|^{\frac{\alpha(y)q}{n}}v_i(y)dy\right)^{p/p_i},\ \ i=1,\dots,m.$$

For every $x\in\mathbb{R}^n$, let us take $B_x\in\mathcal{DB}$ such that $B_x\ni x$ and

$$\big(\mathcal{M}^{(d),(\mathcal{B})}_{\alpha(\cdot)}\overrightarrow{f}\big)(x)\le\frac{2}{|B_x|^{m-\alpha(x)/n}}\prod_{i=1}^m\int\limits_{B_x}|f_i(y_i)|dy_i. \tag{8}$$

Without loss of generality, we can assume, for example, that f_i, $i=1,\dots,m$ are non–negative, bounded and have compact supports.

Let us introduce a set

$$F_B=\{x\in\mathbb{R}^n: x\in B \text{ and } (8)\ \ holds\ \ for\ \ B\}.$$

It is obvious that $F_B\subset B$ and $\mathbb{R}^n=\cup_{B\in\mathcal{DB}}F_B$.

Now, applying Hölder's inequality, we have

$$\begin{aligned}
I=\int\limits_{\mathbb{R}^n}\big(\mathcal{M}^{(d),(\mathcal{B})}_{\alpha(x)}\overrightarrow{f})(x)\big)^q\left(\prod_{i=1}^m v_i^{p/p_i}(x)\right)dx&\le\sum_{B\in\mathcal{DB}}\int\limits_{F_B}\big(\mathcal{M}^{(d),(\mathcal{B})}_{\alpha(x)}\overrightarrow{f}(x)\big)^q\left(\prod_{i=1}^m v_i^{p/p_i}(x)\right)dx\\
&\le 2^q\sum_{B\in\mathcal{DB}}|B|^{-mq}\left(\int\limits_B|B|^{\alpha(x)q/n}\left(\prod_{i=1}^m v_i^{p/p_i}(x)\right)dx\right)\left(\prod_{i=1}^m\int\limits_B f_i(y_i)dy_i\right)^q\\
&\le 2^q\sum_{B\in\mathcal{DB}}|B|^{-mq}\prod_{i=1}^m\left(\int\limits_B|B|^{\alpha(x)q/n}v_i(x)dx\right)^{\frac{p}{p_i}}\left(\prod_{i=1}^m\int\limits_B f_i(y_i)dy_i\right)^q\\
&\le 2^q\sum_{B\in\mathcal{DB}}\prod_{i=1}^m|B|^{-q/p_i'}\left(\frac{1}{|B|^{q/p}}\int\limits_B|B|^{\alpha(x)q/n}v_i(x)dx\right)^{\frac{p}{p_i}}\left(\int\limits_B f_i(y_i)dy_i\right)^q\\
&\le 2^q\sum_{B\in\mathcal{DB}}\prod_{i=1}^m|B|^{-q/p_i'}\left(\int\limits_B f_i(y_i)\Big(\widetilde{M}^{(\mathcal{B})}_{\alpha(\cdot),p_i,q}v_i(y_i)\Big)^{1/q}dy_i\right)^q.
\end{aligned}$$

Further, by using Hölder's inequality in the form

$$\sum_k a_k^{(1)}\times\cdots\times a_k^{(m)}\le\prod_{j=1}^m\left(\sum_k(a_k^{(j)})^{p_j/p}\right)^{p/p_j}$$

for positive sequences $\{a_k^{(j)}\}$, $j=1,\dots,m$, we have

$$I \le 2^q \Big[\sum_{B\in\mathcal{DB}} |B|^{-(qp_1)/(pp_1')}\Big(\int_B f_1(y_1)(\widetilde{M}^{(\mathcal{B})}_{\alpha(y_1),p_1,q}v_1(x))^{1/q}dy_1\Big)^{qp_1/p}\Big]^{p/p_1}$$

$$\times\cdots\times\Big[\sum_{B\in\mathcal{DB}} |B|^{-(qp_m)/(pp_m')}\Big(\int_B f_m(y_m)(\widetilde{M}^{(\mathcal{B})}_{\alpha(y_m),p_m,q}v_m(x))^{1/q}dy_m\Big)^{qp_m/p}\Big]^{p/p_m}.$$

Finally, Theorem C (for $\mathcal{B}=\mathcal{Q}$) and Proposition 1.1 (for $\mathcal{B}=\mathcal{R}$) for the exponents $(p_i, qp_i/p)$, $i=1,\dots,m$, and weight $\rho\equiv 1$, yield that

$$I \le c\prod_{i=1}^m \|f_i\big(\widetilde{M}^{(\mathcal{B})}_{\alpha(x),p_i,q}v_1(x)\big)^{1/q}\|^q_{L^{p_i}(\mathbb{R}^n)}.$$

At last, taking into account Remark 3.1, we can pass from $\mathcal{M}^{(d),\mathcal{B}}_{\alpha(x)}$ to $\mathcal{M}^{(\mathcal{B})}_{\alpha(x)}$. □

Proof of Corollary 2.2. The proof of the *sufficiency* is a direct consequence of Theorem 2.1. For the *necessity* we take test functions: $f_j=\chi_B$, with $B\in\mathcal{B}$. By applying inequality (5) for these functions, we get the desired condition. □

Proof of Theorem 2.2. Following the proof of Theorem 2.1 we get the inequality

$$\Big\|\mathcal{M}^{(d),(\mathcal{B})}_{\alpha(\cdot),\mu}(\overrightarrow{f})\Big\|_{L^q_\mu(\mathbb{R}^n)} \le C\prod_{i=1}^m\Big\|f_i\Big(\widetilde{M}^{(\mathcal{B})}_{\alpha(\cdot),p,q,\mu}(d\mu)\Big)^{1/(qm)}\Big\|_{L^{p_i}(\mathbb{R}^n)}$$

where $\mathcal{M}^{(d),(\mathcal{B})}_{\alpha(\cdot),\mu}$ is the dyadic analogue of $\mathcal{M}^{(\mathcal{B})}_{\alpha(\cdot),\mu}$ and $\widetilde{M}^{(\mathcal{B})}_{\alpha(\cdot),p,q,\mu}(d\mu)(x)$ is defined by (6).

Indeed, observe that

$$I=\int_{\mathbb{R}^n}\big(\mathcal{M}^{(d),(\mathcal{B})}_{\alpha(x),\mu}\overrightarrow{f})(x)\big)^q d\mu(x) \le \sum_{B\in(\mathcal{DB})}\int_{F_B}\big(\mathcal{M}^{(d),(\mathcal{B})}_{\alpha(x)}\overrightarrow{f}(x)\big)^q d\mu(x)$$

$$\le 2^q\sum_{B\in\mathcal{DB}}\Big(\int_B |B|^{(\alpha(x)q)/n}d\mu(x)\Big)\prod_{j=1}^m\Big(\frac{1}{\mu(B)}\int_B f_j(y_j)d\mu(y_j)\Big)^q$$

$$=2^q\sum_{B\in\mathcal{DB}}\prod_{j=1}^m \mu(B)^{-q/p_j'}\Big(\int_B f_j(y_j)\Big(\mu(B)^{-q/p}\int_B |B|^{\frac{\alpha(x)q}{n}}d\mu(x)\Big)^{1/(mq)}d\mu(y_j)\Big)^q$$

$$\le C\Big[\sum_{B\in\mathcal{DB}}\mu(B)^{-qp_1/(pp_1')}\Big[\int_B f_1(y_1)\Big(\widetilde{M}^{(\mathcal{B})}_{\alpha(\cdot),p,q,\mu}(d\mu)(y_1)\Big)^{1/(mq)}d\mu(y_1)\Big]^{qp_1/p}\Big]^{p/p_1}$$

$$\times\cdots\times\Big[\sum_{B\in\mathcal{DB}}\mu(B)^{-qp_m/(pp_m')}\Big[\int_B f_j(y_m)\Big(\widetilde{M}^{(\mathcal{B})}_{\alpha(\cdot),p,q,\mu}(d\mu)(y_m)\Big)^{1/(mq)}d\mu(y_m)\Big]^{qp_m/p}\Big]^{p/p_m}.$$

Now, applying Theorem C (for $\mathcal{B}=\mathcal{Q}$) and Proposition 1.1 (for $\mathcal{B}=\mathcal{R}$) for the weight $\rho\equiv v$ and exponents $(p_i, qp_i/p)$, $i=1,\dots,m$, we can conclude that

$$I\le C\prod_{j=1}^m\Big\|f_j\Big(\widetilde{M}^{(\mathcal{B})}_{\alpha(\cdot),p,q,\mu}(d\mu)\Big)^{1/(mq)}\Big\|^q_{L^{p_j}_\mu(\mathbb{R}^n)}. \qquad \square$$

Proof of Theorem 2.3. The sufficiency follows in the same manner as in the previous theorems by considering dyadic version of the operator $\mathcal{N}^{(\mathcal{B})}_{\alpha,\mu}$ and Remark 3.1; that is why we are focused on the necessity. Let $f_i(x)=\chi_B(x)$. Then the following inequality

$$\|\mathcal{N}^{(\mathcal{B})}_{\alpha,\mu}\overrightarrow{f}\|_{L^q_\mu(\mathbb{R}^n)}\ge \mu(B)^{1/q+\alpha/n}$$

holds.

On the other hand, we notice that

$$\prod_{i=1}^{m} \|f_i\|_{L^{p_i}_\mu(\mathbb{R}^n)} = \mu(B)^{1/p},$$

therefore,

$$\mu(B)^{1/q+\alpha/n-1/p} \leq C.$$

Since $\mu(\mathbb{R}^n) = \infty$ and μ is a measure without atoms, we conclude that

$$q = \frac{pn}{n - \alpha p}. \qquad \square$$

Remark 3.2. Thus from the above proof we can conclude that in the necessity part of Theorem 2.3 no doubling condition is needed.

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