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S-$\mathcal{I}$-CONVERGENCE OF SEQUENCES

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Abstract. In this article, we use the notions of a semi-open set and topological ideal, in order to define and study a new variant of the classical concept of convergence of sequences in topological spaces, namely, the S-$\mathcal{I}$-convergence. Some basic properties of S-$\mathcal{I}$-convergent sequences and their preservation under certain types of functions are investigated. Also, we study the notions related to compactness and cluster points by using semi-open sets and ideals. Finally, we explore the $\mathcal{I}$-convergence of sequences in the cartesian product space.

1. Introduction and Preliminaries

The ideal theory on a set was established in 1933 by Kuratowski [10]. This theory has recently been used in order to generalize several concepts of Mathematical Analysis and General Topology (see, e.g., see [4], [7], [8], [14]). In particular, in 2000, Kostyrko et al. [9] used ideals on the set $\mathbb{N}$ of the positive integer numbers to introduce the notion of $\mathcal{I}$-convergence on metric spaces, as a generalization of statistical convergence. In 2005, Lahiri and Das [11] extended the notion of $\mathcal{I}$-convergence to the context of topological spaces and established some basic properties. On the other hand, in l963, Levine [12] introduced the notion of semi-open set in topological spaces, which plays an important role in recently researches in General Topology. In this article, we use the notion of a semi-open set, in order to define and study a variant of the classical convergence in topological spaces, namely, the S-$\mathcal{I}$-convergence. Specifically, we investigate some basic properties of S-$\mathcal{I}$-convergent sequences and their preservation under certain types of functions. Also, we study the notions related to compactness and cluster points by using semi-open sets and ideals. In the final part of the work, we explore the $\mathcal{I}$-convergence of sequences in the product space.

Now we will give some definitions and results that will be useful to understand content.

Definition 1.1. Let X be a nonempty set, a family of sets $\mathcal{I} \subset 2^X$ is called an *ideal* [10] on X if the following properties are satisfied:

(1) $\emptyset \in \mathcal{I}$,
(2) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$,
(3) $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$.

An ideal $\mathcal{I}$ on X is called *nontrivial* if $\mathcal{I} \neq \{\emptyset\}$ and $X \notin \mathcal{I}$. A nontrivial ideal $\mathcal{I}$ on X is called *admissible* if $\mathcal{I} \supset \{\{x\} : x \in X\}$. Some examples of admissible ideals can be found in [9].

Throughout this work, (X, τ) stands for a topological space (written frequently as X) and $\mathcal{I}$ is a nontrivial ideal on $\mathbb{N}$, the set of all positive integer numbers.

Definition 1.2. A sequence $\{x_n\}$ in X is called *$\mathcal{I}$-convergent* [11] to a point x_0, if for every nonempty open set U containing x_0, $\{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{I}$.

Definition 1.3. A subset A of X is said to be *semi-open* [12], if there exists an open set U such that $U \subset A \subset Cl(U)$.

The collection of all semi-open sets of X is denoted by $SO(X)$. The complement of a semi-open set is called a *semi-closed* set. The *semi-closure* of a subset A of X, denoted by $sCl(A)$, is defined as

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the intersection of all semi-closed sets containing A [1]. Obviously, a point $x \in sCl(A)$ if and only if for every semi-open set U containing x, $U \cap A \neq \emptyset$.

In the following definition, we present some well-known in the literature types of functions in the literature, where X and Y are topological spaces.

Definition 1.4. A function $f : X \to Y$ is said to be:
(1) *semi-continuous* [12] if $f^{-1}(A) \in SO(X)$ for each open set A in Y;
(2)*irresolute* [2] if $f^{-1}(A) \in SO(X)$ for each $A \in SO(Y)$.

Theorem 1.5 ([12, Theorem 12])**.** *A function $f : X \to Y$ is semi-continuous if and only if for each $x \in X$ and each open set V in Y containing $f(x)$, there exists $U \in SO(X)$ such that $x \in U$ and $f(U) \subset V$.*

Theorem 1.6. *A function $f : X \to Y$ is irresolute if and only if for each $x \in X$ and each $V \in SO(Y)$ containing $f(x)$, there exists $U \in SO(X)$ such that $x \in U$ and $f(U) \subset V$.*

Definition 1.7. A topological space X is said to be *semi-Hausdorff* [13], if for each pair x, y of distinct points of X, there exist disjoint semi-open sets containing x and y, respectively.

Definition 1.8. Let X be a topological space and A be a subset of X. A point $x \in X$ is said to be a *semi-limit point* [3] of A if for every semi-open set U containing x, $A \cap (U - \{x\}) \neq \emptyset$.

Definition 1.9. A topological space X is said to be:
(1) *semi-compact* [5] if every cover of X by semi-open sets has a finite subcover;
(2) *semi-Lindelöf* [6] if every cover of X by semi-open sets has a countable subcover.

2. The S-$\mathcal{I}$-convergence and its Basic Properties

In this section, we introduce the concept of an S-$\mathcal{I}$-convergent sequence to a point of a topological space and study its relevant properties.

Definition 2.1. A sequence $\{x_n\}$ in X is said to be *S-$\mathcal{I}$-convergent* to a point $x_0 \in X$ if for every nonempty semi-open set U containing x_0, $\{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{I}$. In this case, x_0 is called the S-$\mathcal{I}$-limit of $\{x_n\}$ and is denoted by S-$\mathcal{I}$-$\lim x_n = x_0$.

Lemma 2.2. *The S-$\mathcal{I}$-convergence implies $\mathcal{I}$-convergence for any nontrivial ideal $\mathcal{I}$ on $\mathbb{N}$.*

Proof. The proof is immediate from the fact that any open set is semi-open and the definition of S-$\mathcal{I}$-convergence. □

The following example shows that the converse of Lemma 2.2 is not necessarily true.

Example 2.3. Let $\mathbb{R}$ be the set of real numbers with the usual topology, $\mathcal{I}$ be an admissible ideal and the sequence $\{x_n\}$ be defined as $x_n = a^n$, where $0 < a < 1$. Observe that the sequence $x_n = a^n$ is $\mathcal{I}$-convergent to 0, since for any open set W containing 0, the set $\{n \in \mathbb{N} : x_n \notin W\}$ is finite. Now consider the semi-open set $U = (-1, 0]$. It is easy to see that the set $\{n \in \mathbb{N} : x_n \notin U\}$ is equal to the set of natural numbers and then the sequence $x_n = a^n$ is not S-$\mathcal{I}$-convergent to 0.

Remark 2.4. If $\mathcal{I}$ is an admissible ideal, then an ordinary convergence implies $\mathcal{I}$-convergence and, in addition, if $\mathcal{I}$ does not contain any infinite set, both concepts coincide (see [11]).

An immediate consequence of Remark 2.4 is the following result.

Proposition 2.5. *If $\mathcal{I}$ is an admissible ideal not containing any infinite set, then S-$\mathcal{I}$-convergence implies convergence.*

The following example shows that the converse of Proposition 2.5 is not necessarily true.

Example 2.6. Let $\mathbb{R}$ be the set of real numbers with the usual topology and the sequence $\{x_n\}$ be defined as $x_n = \frac{1}{n}$. Observe that $\{x_n\}$ converges to 0. Consider the semi-open set $U = (-1, 0]$ and note that $0 \in U$, but $\{n \in \mathbb{N} : x_n \notin U\} = \mathbb{N}$. Therefore $\{n \in \mathbb{N} : x_n \notin U\} \notin \mathcal{I}$ (for any nontrivial ideal $\mathcal{I}$) and so $\{x_n\}$ is not S-$\mathcal{I}$-convergent to 0.

Proposition 2.7. *Let X be a discrete topological space and $\mathcal{I}$ be an admissible ideal, then convergence implies the S-$\mathcal{I}$-convergence.*

Proof. The proof follows from the fact that in the discrete topology the collections of open sets and semi-open sets are the same. □

Example 2.8. Consider $X = \mathbb{R}$ with the usual topology and $\{x_n\}$ the sequence in X defined as $x_n = (-1)^n$. It is clear that $\{x_n\}$ do not converge to any point of X. Now, let $M = \{2j-1 : j \in \mathbb{N}\}$ and take $\mathcal{I} = 2^M$. Then $\mathcal{I}$ is a nontrivial ideal on $\mathbb{N}$, and $\{x_n\}$ is S-$\mathcal{I}$-convergent (also $\mathcal{I}$-convergent) to -1.

Theorem 2.9. *Let X be a semi-Hausdorff space. If $\{x_n\}$ is a S-$\mathcal{I}$-convergent sequence in X, then the point of S-$\mathcal{I}$-convergence is unique.*

Proof. Consider $\{x_n\}$, a sequence that is S-$\mathcal{I}$-convergent in a semi-Hausdorff space X. Suppose that the sequence $\{x_n\}$ has two distinct points of S-$\mathcal{I}$-convergence, say x_0 and y_0. Since X is a semi-Hausdorff space, there exist $U, V \in SO(X)$ such that $x_0 \in U$, $y_0 \in V$ and $U \cap V = \emptyset$. On the other hand, by the definition of the S-$\mathcal{I}$-convergence, we have $\{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{I}$ and $\{n \in \mathbb{N} : x_n \notin V\} \in \mathcal{I}$, which implies that

$$\{n \in \mathbb{N} : x_n \in (U \cap V)^c\} = \{n \in \mathbb{N} : x_n \in U^c\} \cup \{n \in \mathbb{N} : x_n \in V^c\} \in \mathcal{I}.$$

As $\mathcal{I}$ is a nontrivial ideal, then $\{n \in \mathbb{N} : x_n \in (U \cap V)^c\} \neq \mathbb{N}$ and hence there exists $n_0 \in \mathbb{N}$ such that $n_0 \notin \{n \in \mathbb{N} : x_n \in (U \cap V)^c\}$, and so $x_{n_0} \in (U \cap V)$, which is a contradiction. This shows that the point of S-$\mathcal{I}$-convergence is unique. □

Corollary 2.10. *Let X be a Hausdorff space. If $\{x_n\}$ is a S-$\mathcal{I}$-convergent sequence in X, then the point of S-$\mathcal{I}$-convergence is unique.*

Theorem 2.11. *If $\mathcal{I}$ is an admissible ideal and if there exists a sequence $\{x_n\}$ of distinct elements in a subset A of X which is S-$\mathcal{I}$-convergent to $x_0 \in X$, then x_0 is a semi-limit point of A.*

Proof. Let U be any semi-open subset of X containing the point x_0. Since $\{x_n\}$ is S-$\mathcal{I}$-convergent to x_0, therefore $\{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{I}$ and so $\{n \in \mathbb{N} : x_n \in U\} \notin \mathcal{I}$, otherwise it would be $\{n \in \mathbb{N} : x_n \notin U\} \cup \{n \in \mathbb{N} : x_n \in U\} = \mathbb{N} \in \mathcal{I}$, which contradicts that $\mathcal{I}$ is nontrivial. As $\mathcal{I}$ is an admissible ideal, it follows that $\{n \in \mathbb{N} : x_n \in U\}$ is an infinite set, otherwise,

$$\{n \in \mathbb{N} : x_n \in U\} = \bigcup_{x_n \in U} \{n\} \in \mathcal{I},$$

which is $\{n \in \mathbb{N} : x_n \in U\}$ would be a finite union of unitary sets, which is a contradiction because $\{n \in \mathbb{N} : x_n \in U\} \notin \mathcal{I}$. Choose $n_0 \in \{n \in \mathbb{N} : x_n \in U\}$ such that $x_{n_0} \neq x_0$, then $x_{n_0} \in A \cap (U - \{x_0\})$ and so, $A \cap (U - \{x_0\}) \neq \emptyset$. This shows that for any semi-open set U containing the point x_0, we have $A \cap (U - \{x_0\}) \neq \emptyset$. □

Corollary 2.12. *If $\mathcal{I}$ is an admissible ideal and if there exists a sequence $\{x_n\}$ of distinct elements in a subset $A \subset X$ which is S-$\mathcal{I}$-convergent to $x_0 \in X$, then $x_0 \in sCl(A)$.*

Corollary 2.13. *If $\mathcal{I}$ is an admissible ideal and if there exists a sequence $\{x_n\}$ of distinct elements in a subset $A \subset X$ which is S-$\mathcal{I}$-convergent to $x_0 \in X$, then x_0 is a limit point of A.*

Definition 2.14. Let X be a topological space and $\{x_n\}$ be a sequence in X. We say that a point $x \in X$ is a *semi-cluster point* of the sequence $\{x_n\}$ if for every semi-open set U containing x, there exist infinitely many natural numbers n such that $x_n \in U$.

Theorem 2.15. *If $\mathcal{I}$ is an admissible ideal and $\{x_n\}$ is a sequence having a S-$\mathcal{I}$-convergent subsequence, then $\{x_n\}$ has a semi-cluster point.*

Proof. By the hypothesis, $\{x_n\}$ has a subsequence $\left\{x_{k(n)}\right\}$ which is S-$\mathcal{I}$-convergent, say to x_0. We will show that x_0 is a semi-cluster point of $\{x_n\}$. Let U be any semi-open set containing x_0, then $\left\{n \in \mathbb{N} : x_{k(n)} \notin U\right\} \in \mathcal{I}$, and since $\mathcal{I}$ is an admissible ideal, we have $\left\{n \in \mathbb{N} : x_{k(n)} \in U\right\}$ is an infinite set. Thus, U has infinite terms of the subsequence $\left\{x_{k(n)}\right\}$ and hence, of the sequence $\{x_n\}$. This shows that x_0 is a semi-cluster point of $\{x_n\}$. □

Theorem 2.16. *If $B \subset X$ is a semi-closed set, then for any sequence in B which is S-$\mathcal{I}$-convergent to x_0, we have $x_0 \in B$.*

Proof. Suppose that $B \subset X$ is a semi-closed set and $\{x_n\}$ is any sequence in B which is S-$\mathcal{I}$-convergent to the point x_0, but $x_0 \notin B$. Since B is semi-closed, we have $sCl(B) = B$ and thus, $x_0 \notin sCl(B)$. Then there exists a semi-open set U containing x_0 such that $B \cap U = \emptyset$. By the hypothesis, we have $\{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{I}$ and $\{n \in \mathbb{N} : x_n \in U\} \notin \mathcal{I}$, which imply that $\{n \in \mathbb{N} : x_n \in U\} \neq \emptyset$. Thus, there exists $n_0 \in \{n \in \mathbb{N} : x_n \in U\}$, which is $x_{n_0} \in U$. Since $\{x_n\}$ is a sequence in B, hence $x_{n_0} \in B$, as well. Therefore, $x_{n_0} \in B \cap U$ and so $B \cap U \neq \emptyset$, which is a contradiction. □

Corollary 2.17. *If $B \subset X$ is a closed set, then for any sequence in B which is S-$\mathcal{I}$-convergent to x_0, we have $x_0 \in B$.*

Theorem 2.18. *Let $f : X \to Y$ be a semi-continuous function. If $\{x_n\}$ is a sequence in X which is S-$\mathcal{I}$-convergent to $x_0 \in X$, then $\{f(x_n)\}$ is an $\mathcal{I}$-convergent sequence to $f(x_0)$.*

Proof. Assume that $\{x_n\}$ is a sequence in X which is S-$\mathcal{I}$-convergent to $x_0 \in X$ and let V be an open set in Y containing the point $f(x_0)$. By Theorem 1.5, there exists $U \in SO(X)$ containing x_0 such that $f(U) \subset V$. We claim that $\{n \in \mathbb{N} : f(x_n) \notin V\} \subset \{n \in \mathbb{N} : x_n \notin U\}$. In effect, if $n_0 \in \{n \in \mathbb{N} : f(x_n) \notin V\}$, then $f(x_{n_0}) \notin V$ and so $f(x_{n_0}) \notin f(U)$, it follows that $x_{n_0} \notin U$ and hence $n_0 \in \{n \in \mathbb{N} : x_n \notin U\}$. Since $\{x_n\}$ is S-$\mathcal{I}$-convergent to x_0, we have $\{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{I}$ and, consequently, $\{n \in \mathbb{N} : f(x_n) \notin V\} \in \mathcal{I}$. This shows that $\{f(x_n)\}$ is $\mathcal{I}$-convergent to $f(x_0)$. □

It is clear that the condition that $f : X \to Y$ is semi-continuous does not guarantee that if $\{x_n\}$ is an S-$\mathcal{I}$-convergent sequence in X, then $\{f(x_n)\}$ is an S-$\mathcal{I}$-convergent sequence in Y. In the following theorem, we show that the S-$\mathcal{I}$-convergence is preserved by irresolute functions.

Theorem 2.19. *Let $f : X \to Y$ be an irresolute function. If $\{x_n\}$ is a sequence in X which is S-$\mathcal{I}$-convergent to $x_0 \in X$, then $\{f(x_n)\}$ is an S-$\mathcal{I}$-convergent sequence to $f(x_0)$.*

Proof. The proof is similar to that of Theorem 2.18. Just the use is made of the characterization of an irresolute function given in Theorem 1.6. □

Example 2.20. Let $\mathcal{I}$ be the collection of all finite subsets of $\mathbb{N}$, $X = \mathbb{R}$ with the usual topology, $Y = \{0, 1\}$ with the Sierpinski topology, $f : X \to Y$ the function defined by $f(x) = 0$ and $\{x_n\}$ the sequence in X defined as $x_n = (-1)^n$. Note that f is a semi-continuous (resp. irresolute) function such that $\{f(x_n)\}$ is $\mathcal{I}$-convergent (resp. S-$\mathcal{I}$-convergent) to $0 \in Y$, but $\{x_n\}$ do not S-$\mathcal{I}$-converge to any point of X.

3. Compactness and S-$\mathcal{I}$-convergence

Proposition 3.1. *Let X be a topological space and $\mathcal{I}$ be an admissible ideal that does not contain infinite sets. If any sequence $\{x_n\}$ in X has a subsequence which is S-$\mathcal{I}$-convergent, then (X, τ) is a sequentially compact space.*

Proof. This is an immediate consequence of Proposition 2.5. □

Proposition 3.2. *Let X be a topological space and $\mathcal{I}$ be an admissible ideal. If for any infinite subset A of X, there exists a sequence $\{x_n\}$ of distinct elements in A, which is S-$\mathcal{I}$-convergent in X, then (X, τ) is a limit point compact space.*

Proof. This is an immediate consequence of Corollary 2.10. □

Recall that a point p of a topological space X is said to be an *ω-accumulation point* of $A \subset X$ if for every open set U containing p, $U \cap A$ is an infinite set. On the other hand, a point $p \in X$ is said to be an *$\mathcal{I}$-cluster* point [11] of a sequence $\{x_n\}$ in X if for every open set U containing p, $\{n \in \mathbb{N} : x_n \in U\} \notin \mathcal{I}$. In the following two definitions we introduce some modifications of these concepts using semi-open sets.

Definition 3.3. Let X be a topological space and $\{x_n\}$ be a sequence in X. A point $p \in X$ is called a *S-$\mathcal{I}$-cluster* point of $\{x_n\}$ if for any semi-open set U containing p, $\{n \in \mathbb{N} : x_n \in U\} \notin \mathcal{I}$.

Definition 3.4. Let X be a topological space and $A \subset X$. We say that $p \in X$ is a *semi-ω-accumulation* point of A if for every semi-open set U containing p, $U \cap A$ is an infinite set.

Theorem 3.5. *Let X be a topological space and $\mathcal{I}$ be an admissible ideal. If every sequence $\{x_n\}$ in X has an S-$\mathcal{I}$-cluster point, then every infinite subset of X has a semi-ω-accumulation point. The converse is true if $\mathcal{I}$ does not contain infinite sets.*

Proof. Suppose that every sequence in X has an S-$\mathcal{I}$-cluster point and let A be an infinite subset of X, then there exists a sequence $\{x_n\}$ of distinct points in A. Let p be an S-$\mathcal{I}$-cluster point of $\{x_n\}$ and U be any semi-open set U containing p, then $\{n \in \mathbb{N} : x_n \in U\} \notin \mathcal{I}$. Using the fact that $\mathcal{I}$ is an admissible ideal, it follows that $\{n \in \mathbb{N} : x_n \in U\}$ is an infinite subset; as a consequence, U contains infinitely many points of $\{x_n\}$ and hence of A, that is, $U \cap A$ is an infinite set. This shows that p is a semi-ω-accumulation point of A.

Conversely, suppose that every infinite subset of X has a semi-ω-accumulation point. Let $\{x_n\}$ be any sequence in X and let A be the range of $\{x_n\}$. If A is infinite, then by the hypothesis, A has a point of semi-ω-accumulation, say p. Let U be any semi-open set U containing p, then $U \cap A$ is an infinite set, and it follows that U has infinitely many points of A and hence, of the sequence $\{x_n\}$, which implies that $\{n \in \mathbb{N} : x_n \in U\}$ is an infinite set. Since $\mathcal{I}$ is an admissible ideal that does not contain infinite sets, we conclude that $\{n \in \mathbb{N} : x_n \in U\} \notin \mathcal{I}$, that is, p is an S-$\mathcal{I}$-cluster point of $\{x_n\}$. On the other hand, if A is finite, then there exists a point $p \in X$ such that $x_n = p$ for infinitely many subindexes n. Therefore, for every semi-open set U containing p, the set $\{n \in \mathbb{N} : x_n \in U\}$ is infinite and so, $\{n \in \mathbb{N} : x_n \in A\} \notin \mathcal{I}$, which implies that p is an S-$\mathcal{I}$-cluster point of $\{x_n\}$. ☐

Corollary 3.6. *Let X be a topological space and $\mathcal{I}$ be an admissible ideal. If every sequence $\{x_n\}$ has an S-$\mathcal{I}$-cluster point, then every infinite subset of X has an ω-accumulation point.*

Theorem 3.7. *Let X be a topological space and $\mathcal{I}$ be an admissible ideal. If X is a semi-Lindelöff space such that every sequence in X has an S-$\mathcal{I}$-cluster point, then X is a semi-compact space.*

Proof. Suppose that X is a semi-Lindelöff space such that every sequence in X has an S-$\mathcal{I}$-cluster point and let $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ be a semi-open cover of X. Since X is a semi-Lindelöff space, $\mathcal{U}$ contains a countable subcover, say $\mathcal{U}' = \{U_1, U_2, \ldots, U_m, \ldots\}$. Proceeding by induction, let $A_1 = U_1$ and for each $m > 1$, let A_m be the first member of the sequence of U's which is not covered by $U_1 \cup U_2 \cup \cdots \cup U_{m-1}$. We claim that in the above selection, there exists m_0 such that for all $m > m_0$ it is impossible to continue with the algorithm. In effect, if in the above selection it is possible to do this for all $n > 1$, we choose a point $a_n \in A_n$ for all $n \in \mathbb{N}$ such that $a_n \notin A_k$ for $k < n$. Now, consider the sequence $\{a_m\}$ and let p be an S-$\mathcal{I}$-cluster point of $\{a_n\}$. Then $p \in A_j$ for some j. By the definition of an S-$\mathcal{I}$-cluster point and the admissibility of the ideal $\mathcal{I}$, we have $\{n \in \mathbb{N} : a_n \in A_j\} \notin \mathcal{I}$ and $\{n \in \mathbb{N} : a_n \in A_j\}$ must be an infinite set of $\mathbb{N}$. Thus, there exists $r > j$ such that $r \in \{n \in \mathbb{N} : a_n \in A_j\}$; that is, there exists some $r > j$ such that $a_r \in A_j$, which is a contradiction. As a consequence, there exists m_0 such that for all $m > m_0$ it is impossible to continue the algorithm and, therefore, $\{A_1, A_2, \ldots, A_{m_0}\}$ is a finite subcover of X. ☐

Corollary 3.8. *Let X be a topological space and $\mathcal{I}$ be an admissible ideal. If X is a semi-Lindelöff space such that every sequence in X has an S-$\mathcal{I}$-cluster point, then X is a compact space.*

4. The $\mathcal{I}$-convergence in the Product Space

Theorem 4.1. *Let $\{(X_\alpha, \tau_\alpha) : \alpha \in \Delta\}$ be an indexed family of topological spaces, $\prod_{\alpha\in\Delta} X_\alpha$ be the product space and $\{x_\alpha(n)\}$ be a sequence in X_α for all $\alpha \in \Delta$. Then $\{x_\alpha(n)\}$ is $\mathcal{I}$-convergent to p_α for all $\alpha \in \Delta$ if and only if $\{(x_\alpha(n))_{\alpha\in\Delta}\}$ is $\mathcal{I}$-convergent to $(p_\alpha)_{\alpha\in\Delta}$.*

Proof. Let A be an open set in $\prod_{\lambda\in\Lambda} X_\alpha$ containing the point $(p_\alpha)_{\alpha\in\Delta}$, then there exists a basic open set $B = \prod_{\alpha\in\Delta} B_\alpha$ such that $(p_\alpha)_{\alpha\in\Delta} \in B \subset A$. It follows that $p_\alpha \in B_\alpha$ for all $\alpha \in \Delta$. Since $\prod_{\alpha\in\Delta} B_\alpha$ is a basic open set in the product space $\prod_{\alpha\in\Delta} X_\alpha$, it follows that $B_\alpha = X_\alpha$, except for a finite number of indexes, say $\alpha_1, \dots, \alpha_k$. Thus, $p_{\alpha_i} \in B_{\alpha_i}$ for $i \in \{1, \dots, k\}$ and $p_\alpha \in X_\alpha$ for $\alpha \neq \alpha_1, \dots, \alpha_k$. Since $\{x_\alpha(n)\}$ is $\mathcal{I}$-convergent to p_α for all $\alpha \in \Delta$, therefore $\{n \in \mathbb{N} : x_\alpha(n) \notin B_\alpha\} \in \mathcal{I}$ for all $\alpha \in \Delta$ and hence

$$\bigcup_{\alpha\in\Delta} \{n \in \mathbb{N} : x_\alpha(n) \notin B_\alpha\} = \bigcup_{i=1}^{k} \{n \in \mathbb{N} : x_{\alpha_i}(n) \notin B_{\alpha_i}\} \in \mathcal{I}.$$

We claim that $\{n \in \mathbb{N} : (x_\alpha(n))_{\alpha\in\Delta} \notin B\} \subset \bigcup_{i=1}^{k} \{n \in \mathbb{N} : x_{\alpha_i}(n) \notin B_{\alpha_i}\}$.

In effect, let $n_0 \in \{n \in \mathbb{N} : (x_\alpha(n))_{\alpha\in\Delta} \notin B\}$, then we have $(x_\alpha(n_0))_{\alpha\in\Delta} \notin B = \prod_{\alpha\in\Delta} B_\alpha$, which implies that there exists $\alpha_0 \in \Delta$ such that $x_{\alpha_0}(n_0) \notin B_{\alpha_0}$ and since $B_\alpha = X_\alpha$ for $\alpha \neq \alpha_1, \dots, \alpha_k$, necessarily $\alpha_0 \in \{\alpha_1, \dots, \alpha_k\}$, otherwise there would be a contradiction; now, as $x_{\alpha_0}(n_0) \notin B_{\alpha_0}$, we have

$$n_0 \in \{n \in \mathbb{N} : x_{\alpha_0}(n) \notin B_{\alpha_0}\} \subset \bigcup_{i=1}^{k} \{n \in \mathbb{N} : x_{\alpha_i}(n) \notin B_{\alpha_i}\}.$$

Therefore, $\{n \in \mathbb{N} : (x_\alpha(n))_{\alpha\in\Delta} \notin B\} \subset \bigcup_{i=1}^{k} \{n \in \mathbb{N} : x_{\alpha_i}(n) \notin B_{\alpha_i}\}$. On the other hand, the fact that $B = \prod_{\alpha\in\Delta} B_\alpha \subset A$ implies that

$$\begin{aligned} \{n \in \mathbb{N} : (x_\alpha(n))_{\alpha\in\Delta} \notin A\} &\subset \{n \in \mathbb{N} : (x_\alpha(n))_{\alpha\in\Delta} \notin B\} \\ &\subset \bigcup_{i=1}^{k} \{n \in \mathbb{N} : x_{\alpha_i}(n) \notin B_{\alpha_i}\}. \end{aligned}$$

Since $\bigcup_{i=1}^{k} \{n \in \mathbb{N} : x_{\alpha_i}(n) \notin B_{\alpha_i}\} \in \mathcal{I}$, it follows that

$$\{n \in \mathbb{N} : (x_\alpha(n))_{\alpha\in\Delta} \notin A\} \in \mathcal{I},$$

which shows that $\{(x_\alpha(n))_{\alpha\in\Delta}\}$ is $\mathcal{I}$-convergent to $(p_\alpha)_{\alpha\in\Delta}$.

Conversely, let β be an arbitrary element of Δ and consider the set $\{n \in \mathbb{N} : x_\beta(n) \notin B_\beta\}$, where B_β is an arbitrary open set of X_β containing the point $p_\beta \in X_\beta$. Now, let $B = \prod_{\alpha\in\Delta} B_\alpha$ a basic open set in the product space $\prod_{\alpha\in\Delta} X_\alpha$ containing the point $(p_\alpha)_{\alpha\in\Delta}$ such that $\pi_\beta\big(\prod_{\alpha\in\Delta} B_\alpha\big) = B_\beta$. By the hypothesis, the set $\{n \in \mathbb{N} : (x_\alpha(n))_{\alpha\in\Delta} \notin B\} \in \mathcal{I}$. On the other hand, since $B = \prod_{\alpha\in\Delta} B_\alpha$ is a basic open set, therefore $B_\alpha = X_\alpha$ except for a finite number of indexes, say $\alpha_1, \dots, \alpha_k$. Suppose that $\beta = \alpha_j$ for some $1 \leq j \leq k$ (if $\beta \neq \alpha_j$ for all $1 \leq j \leq k$, the result is trivial). We claim that

$$\bigcup_{i=1}^{k} \{n \in \mathbb{N} : x_{\alpha_i}(n) \notin B_{\alpha_i}\} \subset \{n \in \mathbb{N} : (x_\alpha(n))_{\alpha\in\Delta} \notin B\}.$$

In effect, let $n_0 \in \bigcup_{i=1}^{k} \{n \in \mathbb{N} : x_{\alpha_i}(n) \notin B_{\alpha_i}\}$, then there exists $\alpha_0 \in \Delta$ such that $n_0 \in \{n \in \mathbb{N} : x_{\alpha_0}(n) \notin B_{\alpha_0}\}$, which implies that $x_{\alpha_0}(n_0) \notin B_{\alpha_0}$ and so, $(x_\alpha(n_0))_{\alpha\in\Delta} \notin B = \prod_{\alpha\in\Delta} B_\alpha$, hence $n_0 \in \{n \in \mathbb{N} : (x_\alpha(n))_{\alpha\in\Delta} \notin B\}$. As $\{n \in \mathbb{N} : (x_\alpha(n))_{\alpha\in\Delta} \notin B\} \in \mathcal{I}$, then $\bigcup_{i=1}^{k} \{n \in \mathbb{N} : x_{\alpha_i}(n) \notin B_{\alpha_i}\} \in \mathcal{I}$ and, consequently, $\{n \in \mathbb{N} : x_\beta(n) \notin A_\beta\} \in \mathcal{I}$. This shows that $\{x_\beta(n)\}$ is $\mathcal{I}$-convergent to p_β, and since $\beta \in \Delta$ is arbitrary, the proof is complete. $\square$

Corollary 4.2. *Let* $\{(X_\alpha, \tau_\alpha) : \alpha \in \Delta\}$ *be an indexed family of topological spaces,* $\prod_{\alpha\in\Delta} X_\alpha$ *be the product space and* $\{x_\alpha(n)\}$ *be a sequence in* X_α *for all* $\alpha \in \Delta$*. If* $\{x_\alpha(n)\}$ *is* S-$\mathcal{I}$*-convergent to* p_α *for all* $\alpha \in \Delta$*, then* $\{(x_\alpha(n))_{\alpha\in\Delta}\}$ *is* $\mathcal{I}$*-convergent to* $(p_\alpha)_{\alpha\in\Delta}$*.*

Corollary 4.3. *Let* $\{(X_\alpha, \tau_\alpha) : \alpha \in \Delta\}$ *be an indexed family of topological spaces,* $\prod_{\alpha\in\Delta} X_\alpha$ *be the product space and* $\{x_\alpha(n)\}$ *be a sequence in* X_α *for all* $\alpha \in \Delta$*. If* $\{(x_\alpha(n))_{\alpha\in\Delta}\}$ *is* S-$\mathcal{I}$*-convergent to* $(p_\alpha)_{\alpha\in\Delta}$*, then* $\{x_\alpha(n)\}$ *is* S-$\mathcal{I}$*-convergent to* p_α *for all* $\alpha \in \Delta$*.*

Recall that if $\{I_\alpha\}_{\alpha\in\Delta}$ is a chain of ideals on X, then $\bigcup_{\alpha\in\Delta} I_\alpha$ is an ideal on X [15]. Next, we give two immediate consequences related to a chain of ideals on $\mathbb{N}$.

Corollary 4.4. *Let* $\{\mathcal{I}_\alpha\}_{\alpha\in\Delta}$ *be a chain of nontrivial ideals on* $\mathbb{N}$*,* $\prod_{\alpha\in\Delta} X_\alpha$ *be the product space of a family of topological spaces* $\{(X_\alpha, \tau_\alpha) :_{\alpha\in\Delta}\}$ *and* $\{x_\alpha(n)\}$ *be a sequence in* X_α *for all* $\alpha \in \Delta$*. If* $\{x_\alpha(n)\}$ *is* $\mathcal{I}_\alpha$*-convergent to* p_α *for all* $\alpha \in \Delta$*, then* $\{(x_\alpha(n))_{\alpha\in\Delta}\}$ *is* $\mathcal{I}$*-convergent to* $(p_\alpha)_{\alpha\in\Delta}$*, where* $\mathcal{I} = \bigcup_{\alpha\in\Delta} \mathcal{I}_\alpha$*.*

Corollary 4.5. *Let* $\{\mathcal{I}_\alpha\}_{\alpha\in\Delta}$ *be a chain of nontrivial ideals on* $\mathbb{N}$*,* $\prod_{\alpha\in\Delta} X_\alpha$ *be the product space of a family of topological spaces* $\{(X_\alpha, \tau_\alpha) :_{\alpha\in\Delta}\}$ *and* $\{x_\alpha(n)\}$ *be a sequence in* X_α *for all* $\alpha \in \Delta$*. If* $\{(x_\alpha(n))_{\alpha\in\Delta}\}$ *is* $\mathcal{I}_\alpha$*-convergent to* $(p_\alpha)_{\alpha\in\Delta}$*, then* $\{x_\alpha(n)\}$ *is* $\mathcal{I}$*-convergent to* p_α *for all* $\alpha \in \Delta$*, where* $\mathcal{I} = \bigcup_{\alpha\in\Delta} \mathcal{I}_\alpha$*.*

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