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ON SOME NEW DECOMPOSITION THEOREMS IN MULTIFUNCTIONAL HERZ ANALYTIC FUNCTION SPACES IN BOUNDED PSEUDOCONVEX DOMAINS

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Abstract. Under certain integral condition we present new sharp decomposition theorems for multifunctional Herz spaces in the unit ball and pseudoconvex domains expanding the known results from the unit ball. This expands completely the well-known atomic decomposition theorem for one functional Bergman space in the unit ball.

Introduction and Main Results

The problem we consider is well-known for functional spaces in $\mathbb{R}^n$ (the problem of equivalent norms) (see, e.g., [9]). Let $X, (X_j)$ be a function space in a fixed product domain and (or) in $\mathbb{C}^n$ (normed or quasinormed); our aim is to find an equivalent expression for $\|f_1 \dots f_m\|_X$; $m \in \mathbb{N}$. (Note that they are closely connected with spaces on the product domains, since often if $f(z_1 \dots z_m) = \prod\limits_{j}^{m} f_j(z_j)$, then $\|f\|_X = \prod\limits_{j=1}^{m} \|f_j\|_{X_j}$). The obtained results also, as we'll see below, extend some well-known assertions to the atomic decomposition of A^p_α type spaces.

To study such a group of functions it is natural, for example, to ask about the structure of each $\{f_j\}_{j=1}^m$ of this group.

This can be done, for example, if we turn to the following question of finding conditions on $\{f_1, \dots, f_m\}$ such that $\|f_1, \dots, f_m\|_X \asymp \prod\limits_{i=1}^{m} \|f_j\|_{X_j}$ decomposition is valid. In this case we find that if for some positive constant c, $\prod\limits_{i=1}^{m} \|f_j\|_{X_j} \le c\|f_1 \dots f_m\|_X$, then each $f_j, f_j \in X_j$; $j = 1, \dots, m$, where X_j is a new normed (or quasinormed) function space in the $\mathbb{D}$ domain and, hence, we can now easily get properties of $\{f_j\}$ based on the facts of already known one functional function space theory. (For example, to use the known theorems for each $f_j \in X_j$, $j = 1, \dots, m$ on atomic decompositions). This idea has been applied to Bergman spaces in the unit ball and then to bounded pseudoconvex domains with a smooth boundary (see the recent paper [5]. In this paper, we extend these results in various directions by using modification of the known proof.

We denote as usual by $A^p_\alpha(B)$ the Bergman space in the unit ball B (see [5,8]), where $0 < p < \infty$, $\alpha > -1$.

We showed in [5] that $\|f_1 \dots f_m\|_{A^p_\tau} \asymp \prod\limits_{i=1}^{m} \|f_j\|_{A^p_{\alpha_j}}$ is valid under certain integral (A) condition (see below) if $p \le 1$ and if $\tau = \tau(p, \alpha_1, \dots, \alpha_m, m)$.

From our discussion above we can see that of interest is to show that

$$\prod_{i=1}^{m} \|f_j\|_{A^p_{\alpha_j}(B)} \le c_1 \|f_1 \dots f_m\|_{A^p_\tau(B)},$$

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since the reverse follows directly from the uniform estimate (see [8, 10])

$$|f(z)|(1-|z|)^{\frac{\alpha_j+n+1}{p}} \leq c\|f\|_{A^p_{\alpha_j}};\ \ 0 \leq p < \infty;\ \ \alpha_j > -1;\ \ j = 1, \dots, m$$

and ordinary induction. This leads easily to the fact that τ can be calculated as

$$\tau = (n+1)(m-1) + \left(\sum_{u=1}^{m} \alpha_j\right);\ \ \alpha_j > -1;\ \ 0 \leq p < \infty;$$

It should be noted that similar very simple proof based only on various known uniform estimates can be used in all our proofs below for analogous inequalities in various spaces. So, we are, mainly, concentrated on the reverse estimates (see [8] for various uniform estimates).

Note also that this argument likewise allows get easily even more general version with $|f_1|^{p_1} \dots |f_m|^{p_m}$ instead of $|f_1|^p \dots |f_m|^p$ (that has been discussed above for $0 < p_j < \infty$, $j = 1, \dots, m$).

We denote by dV (or $d\delta$) Lebesgue measure on the unit ball B and by C, C_α, C_1, C_2 various positive constants below. By $H(B)$ we denote the space of all analytic functions on B and by $D(a_k, r)$ or $B(z, r)$ Bergman's ball in $\mathbb{B}$ (see [8, 10]).

Assume that

$$f_1(w_1)\dots f_m(w_m) = c\alpha \int_B \frac{(f_1(z)\dots f_m(z))(1-|z|)^\alpha dV(z)}{\prod_{j=1}^{m}(1-<z, w_j>)^{\frac{n+1+\alpha}{m}}} \tag{A}$$

$$\alpha > \alpha_0,\ \ w_j \in B,\ \ j = 1, \dots, m,$$

where α_0 is large enough.

We define some direct extensions of classical Bergman A^p_α function spaces in the unit ball in Herz spaces.

We fix an r-lattice $\{a_k\}$ in the ball (see [10]) till the end of the paper.

Let

$$K^{p,q}_\alpha = \left\{f \in H(B) : \int_B \left(\int_{B(z,r)} |f(\tilde{z})|^p (1-|\tilde{z}|)^\alpha dV(\tilde{z})\right)^{\frac{q}{p}} dV(z) < \infty\right\};$$

$$M^{p,q}_\alpha = \left\{f \in H(B) : \sum_{k \geq 0} \left(\int_{D(a_k,r)} |f(w)|^p (1-|w|)^\alpha dV(w)\right)^{\frac{q}{p}} < \infty\right\};\quad 0 < p, q < \infty, \alpha > -1;$$

(Note $M^{p,p}_\alpha = A^p_\alpha$, $0 < p < \infty$, $\alpha > -1$)

$$K^{p,\infty}_\alpha = \left\{f \in H(B) : \int_B \left(\sup_{z \in B(w,r)}\right)|f(z)|^p (1-|z|)^\alpha dV(w) < \infty\right\};$$

$$M^{p,\infty}_\alpha = \left\{f \in H(B) : \sum_{k \geq 0} \left(\sup_{z \in D(a_k,r)}\right)|f(z)|^p (1-|z|)^\alpha < \infty\right\}.$$

$$0 < p, q < \infty, \quad \alpha \geq 0;$$

These are Banach space for $\min(p, q) \geq 1$ and complete metric spaces for other values.

Theorem 1. *Let X be one of these spaces and $0 < q \leq p \leq 1$; (or $0 < p \leq 1$; $q = \infty$). Then for $f_1, \dots, f_m \in H(B)$; $\alpha_j > -1$, (or $\alpha_j \geq 0$); $j = 1, \dots, m$, we have $\|f_1 \dots f_m\|_{X^{p,q}_\tau} \asymp \prod_{i=1}^{m} \|f_i\|_{X^{p,q}_\alpha}$, if for some $\beta; \beta > \beta_0$ and some $\tau, \tau > -1$ (or $\tau \geq 0$),*

$$f_1(w_1)\dots f_m(w_m) = \int_B \frac{(f_1(z)\dots f_m(z)) \times (1-|z|)^\beta}{\prod_{i=1}^{m}(<1-z, w_j>)^{\frac{\beta+n+1}{m}}} dV(z);\ w_j \in B,\ j = 1, \dots, m, \tag{S}$$

where $\tau = \tau(p, q, n, m, \alpha_1, \dots, \alpha_m)$.

Our theorem extends the known result to the atomic decomposition of Bergman multifunctional space A^p_α (see [5]). For $p = q$, in the unit disk, ball we have $M^{p,p}_\alpha = A^p_\alpha$, $K^{p,p}_\alpha = A^p_\beta$, $0 < p < \infty$ for some $\beta = \beta(p, q)$ (see [2]). If, in addition, $m = 1$, then the integral condition(s) vanishes and we can apply now the atomic decomposition theorem for A^p_α class in the ball, disk (see [10]).

Remark 1. For each space, τ is a special number which can be fixed.

Remark 2. For the mixed norm of $A^{p,q}_\alpha, F^{p,q}_\alpha$ spaces we have found the very similar almost sharp results:

$$A^{p,q}_\alpha = \Big\{ f \in H(B) : \int_0^1 \Big(\int_S |f(z)|^p d\sigma(\xi) \Big)^{\frac{q}{p}} (1 - |z|)^\alpha d|z| < \infty \};$$
$$0 < p, \quad q < \infty, \quad \alpha > -1,$$

where $S = |z| = 1, and d\sigma$ is a Lebesgue measure on S, and

$$F^{p,q}_\alpha = \Big\{ f \in H(B) : \int_S \Big(\int_0^1 |f(z)|^p (1 - |z|)^\alpha d|z| \Big)^{\frac{q}{p}} d\sigma(\xi) < \infty \Big\};$$
$$F^{p,\infty}_\beta = \Big\{ f \in H(B) : \int_S \Big(\sup_{0<r<1} \Big) |f(r\xi)|^p (1 - r)^\beta d\sigma(\xi) < \infty \Big\};$$
$$A^{p,\infty}_\alpha = \Big\{ f \in H(B) : \int_0^1 (M_\infty(f, r)^p) (1 - r)^\alpha dr < \infty \Big\};$$
$$0 < q < \infty, \quad 0 < p < \infty, \quad \alpha > -1, \quad \beta \geq 0.$$

Note now if each (f_i) from one functional (X_i) space can be decomposed into atoms and then, since $\|f_1, \dots, f_m\|_X \asymp \prod_{i=1}^m \|f_i\|_{X_i}$, we can also decompose each (f_i) also as soon as $\|f_1 \dots f_m\|_X < \infty, m > 1$ because integral condition we posed is valid for spaces with infinite or finite indices.

Now we turn to the case of more general spaces on the bounded pseudoconvex domains with a smooth boundary on Ω, using Kobayashi balls $B(z, r)$.

First, we define the spaces and then formulate our theorems.

For the basic definitions ofthe function theory in Ω, we refer to [1], [5], [7], [3].

Let, further, (for some of these spaces see, for example, [3])

$$(A^{p,q}_\alpha)(\Omega) = \Big\{ f \in H(\Omega) : \int_0^\rho \Big(\int_{\partial D_r} |f(\omega)|^p d\sigma(\omega) \Big)^{\frac{q}{p}} \times (r^\alpha) dr < \infty \Big\};$$
$$\alpha > -1; \quad 0 < p, \quad q \leq \infty.$$

We refer to [5] for the ∂D_r domain and $d\sigma$ is a Lebesgue measure on ∂D_r, where $H(\Omega)$ is a space of all analytic functions on Ω, $\delta(w) = dist(w, \partial\Omega)$ (for these $A^{p,q}_\alpha$ spaces our result is almost sharp).

We fix an r-lattice in pseudoconvex domains (see [5]).

Let also

$$(M^{p,q}_\alpha)(\Omega) = \Big\{ f \in H(\Omega) : \int_\Omega \Big(\int_{B(z,r)} |f(\omega)|^p (\delta^\alpha(\omega)) dV(\omega) \Big)^{\frac{q}{p}} dV(z) < \infty \Big\};$$
$$\alpha > -1; \quad 0 < p, \quad q < \infty;$$
$$(K^{p,q}_\alpha)(\Omega) = \Big\{ f \in H(\Omega) : \sum_{k \geq 0} \Big(\int_{D(a_k,r)} |f(\omega)|^p \times (\delta^\alpha(\omega)) dV(\omega) \Big) < \infty \Big\};$$
$$\alpha > -1; \quad 0 < p, \quad q < \infty;$$

$$(K_{\alpha}^{p,\infty})(\Omega) = \left\{ f \in H(\Omega) : \int_{\Omega} \left(\sup_{z \in B(w,r)} |f(z)|^p \right) (\delta(z))^{\alpha} \, dV(\omega) < \infty \right\};$$
$$0 < p \le \infty;$$

$M^{p,\infty}$ can be defined similarly as in the ball.

For these general spaces and domains we, however, impose one additional condition on the weighted Bergman Kernel $K(z, \omega)$ in Ω domain to get a new sharp result.

Theorem 2 (for pseudoconvex domains)**.** *Let $f_i \in H(\Omega)$, $i = 1, \ldots, m$.*

Let X be one of $K_{\alpha}^{p,q}$, or $M_{\alpha}^{p,q}$ type spaces defined above. Then for some τ, we have

$$\|f_1, \ldots, f_m\|_{X_{\tau}^{p,q}} \asymp \prod_{i=1}^{m} \|f_i\|_{X_{\alpha_i}^{p,q}},$$

for $0 < q \le p \le 1$ or $p \le 1, q = \infty, \alpha_i > -1$ or $\alpha_i \ge 0$ for $i = 1, \ldots, m$, if

$$(f_1(z_1), \ldots, f_m(z_m)) = \mathbb{C} \int_{\Omega} (f_1(\omega), \ldots, f_m(\omega)) \left(\prod_{j=1}^{m} K_{\frac{\tau+n+1}{m}}(z_j, \omega) \right) \delta^{\tau}(\omega) dV(\omega)$$
$$\tau > \tau_0, \;\; z_i \in \Omega, \;\; i = 1, \ldots, m,$$

under one additional condition on Bergman Kernel of t type $K_t(z, w)$

$$\int_{B(\tilde{z},r)} |K_t(z, \omega)|^p \, \delta^{\tilde{\alpha}}(z) dv(z) \le \mathbb{C} \left| K_{tp+\tilde{\alpha}+n+1}(w, \tilde{z}) \right|, \tilde{z}, w \in \Omega$$

for every Kobayashi ball $B(\tilde{z}, r), \tilde{z} \in B, \tilde{\alpha} > -1$, $t > 0$, $r > 0$ (with modification for $p = \infty$).

Theorems 1 and 2 are, probably, valid for $p, q \ge 1$, and we will turn to this problem in our other paper.

Remark 1′. Similar results with very similar proofs are valid for analytic spaces on tubular domains over symmetric cones. Such type spaces in unbounded domains have been studied recently by many authors. (see, for example, [6–8] and various references therein).

Proofs are essentially the same and we will present them in the other separate paper devoted, mainly, to the spaces in such a type of general unbounded domains in $\mathbb{C}^n$.

For example, for $(A_{\tau}^{p,q})$ spaces in a tubular domain T_{Ω}, $\|f_1 \ldots f_m\|_{A_S^p(T_{\Omega})} \asymp \prod_{i=1}^{m} \|f_i\|_{A_{\tau_i}^p(T_{\Omega})}$ is valid for $1 < p < \infty$; $\tau_j > -1$; $S = S(\tau_1, \ldots, \tau_m, p, q, m)$; if

$$f_1(w_1) \ldots f_m(w_m) = \int_{T_{\Omega}} \frac{(f_1(z) \ldots f_m(z) \bigtriangleup^{\tau} (I_m(z)))}{\prod_{i=1}^{m} \bigtriangleup^{\frac{\tau + \frac{2n}{r}}{m}} \left(\frac{z - w_i}{i}\right)} dV(z)$$

for $w_j \in T_{\Omega}, \tau > \tau_0, \tau_0$ is large enough, $j = 1, \ldots, m$, where $\bigtriangleup^{\tau}$ is a determinant function of T_{Ω} (see [6], [8]), dv is a Lebesgue measure on T_{Ω}.

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