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# THE STRONG UNIQUENESS PROPERTY OF INVARIANT MEASURES IN INFINITE DIMENSIONAL TOPOLOGICAL VECTOR SPACES

MARIKA KHACHIDZE[1] AND ALEKS KIRTADZE[1,2]

**Abstract.** It is shown that there exists a normalized $\sigma$-finite invariant Borel measure in the topological vector space $\mathbf{R}^\omega$, having the strong uniqueness property on $\mathbf{R}^\omega$.

It is well known that the uniqueness property of invariant measures plays an important role in various questions of modern analysis, probability theory, and geometry. The main purpose of the present paper is to consider the uniqueness property of invariant measures from the general point of view and to investigate some interrelations between this and other properties of invariant measures.

Throughout this article, we use the following standard notation:

$\mathbf{N}$ is the set of all natural numbers;

$\mathbf{R}$ is the set of all real numbers;

$\omega$ is the first infinite cardinal number (i.e., $\omega = \mathrm{card}(\mathbf{N})$;

$\mathbf{c}$ is the cardinality of the continuum (i.e., $\mathbf{c} = 2^\omega$);

$\mathrm{dom}(\mu)$ is the domain of a given measure $\mu$;

$\mathbf{R}^\omega$ is the space of all real-valued sequences;

$\mathbf{B}(\mathbf{R}^\omega)$ is the Borel $\sigma$-algebra on $\mathbf{R}^\omega$.

Let $E$ be a nonempty set, $G$ be a group of transformations of $E$, and let $M$ be a class of $\sigma$-finite $G$-invariant measures on $E$ (note here that the domains of measures from $M$ may differ from each other).

We say that a set $X$ has the uniqueness property with respect to $M$ if for every two measures $\mu_1 \in M$ and $\mu_2 \in M$ such that $X \in \mathrm{dom}(\mu_1)$ and $X \in \mathrm{dom}(\mu_2)$ we have the following equality:

$$\mu_1(X) = \mu_2(X).$$

We present several simple examples illustrating the above notions.

**Example 1.** Let $M$ be a class of nonzero $\sigma$-finite $G$-measures in $\mathbf{R^n}$. Then the unit coordinate cube in $\mathbf{R^n}$ has the uniqueness property with respect to $M$.

**Example 2.** If $X \subset \mathbf{R^n}$ is an absolutely negligible subset in the class of all $G$-measures, then $X$ has the uniqueness property with respect to the same class of measures (see [5], [8]).

**Example 3.** Every subset $X \subset \mathbf{R^n}$, measurable with respect to the classical Jordan measure, has the uniqueness property in the class of $\pi_n$-volumes, where $\pi_n$ denotes the group of all translations in $\mathbf{R^n}$.

Let again $E$ be a nonempty set, $G$ be a group of transformations of $E$, and $M$ be a class of $\sigma$-finite $G$-invariant measures on $E$.

A measure $\mu \in M$ possesses the strong uniqueness property with respect to $M$ if $\mathrm{dom}(\mu)$ contains only those elements that have the uniqueness property with respect to the same class of measures.

In other words, a measure $\mu \in M$ possesses the strong uniqueness property with respect to $M$ if for every $X \in \mathrm{dom}(\mu)$ and for every two measures $\mu_1 \in M$ and $\mu_2 \in M$ such that $X \in \mathrm{dom}(\mu_1)$ and $X \in \mathrm{dom}(\mu_2)$ we have the following equality:

$$\mu_1(X) = \mu_2(X).$$

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For the above-mentioned definitions, see [5], [8].

From the above-mentioned definitions it follows that if a measure $\mu$ has the strong uniqueness property with respect to $M$, then $\mu$ has the uniqueness property with respect to the same class of measures, too.

**Example 4.** From the definition of $G$-volumes it follows that each volume from the class of all $G$-volumes on $\mathbf{R^n}$ has the strong uniqueness property.

It is known that in the infinite-dimensional vector spaces there are no analogues of the classical Lebesgue measure. In other words, the above-mentioned spaces do not admit nontrivial $\sigma$-finite translation-invariant Borel measure. In this context it should be noted that A. Kharazishvili has constructed a normalized $\sigma$-finite metrically transitive Borel measure $\chi$ in $\mathbf{R}^\omega$ which is invariant with respect to the dense everywhere vector subspace $G$, where

$$G = \{x : x \in \mathbf{R}^\omega, \mathrm{card}\{i : i \in \mathbf{N} : x_i \neq 0\} < \omega\}.$$

For a detailed information of measure $\chi$, see [6] or [7].

Let $s_0$ be the central symmetry of $\mathbf{R}^\omega$ with respect to the origin and let $S_\omega$ be the group generated by $s_0$ and $G$. It is clear that each element of $S_\omega$ can be represented in the form $s_0 \circ g$ or $g \circ s_0$.

Using the method of [6], we will construct a $\sigma$-finite Borel measure on $\mathbf{R}^\omega$ which is invariant with respect to the group $S_\omega$.

In particular, we put

$$A_n = \mathbf{R}_1 \times \mathbf{R}_2 \times \cdots \times \mathbf{R}_n \times \Big(\prod_{i>n} \triangle_i\Big),$$

where $n \in \mathbf{N}$ and

$$(\forall i)(i \in \mathbf{N} \Rightarrow \mathbf{R}_i = \mathbf{R} \wedge \triangle_i = [-1, 1]).$$

For an arbitrary natural number $i \in \mathbf{N}$, consider the Lebesgue measure $\mu_i$ defined on the space $\mathbf{R}_i$ and satisfying the condition $\mu_i(\triangle_i) = 1$. Let us denote by $\lambda_i$ the Lebesgue measure defined on the $\triangle_i$ such that $\lambda_i(\triangle_i) = 1$.

For an arbitrary $n \in \mathbf{N}$, let us denote by $\chi_n$ the measure defined by

$$\chi_n = \Big(\prod_{1 \le i \le n} \mu_i\Big) \times \Big(\prod_{i>n} \lambda_i\Big),$$

and by $\overline{\chi_n}$ the Borel measure in the space $\mathbf{R}^\omega$ defined by

$$\overline{\chi_n} = \chi_n(X \cap A_n),\ X \in \mathbf{B}(\mathbf{R}^\omega).$$

**Lemma 1.** *For an arbitrary Borel set $X \in \mathbf{B}(\mathbf{R}^\omega)$ there exists a limit*

$$\chi(X) = \lim_{n \to \infty} \overline{\chi_n}(X).$$

*Moreover, the functional $\chi$ is a nonzero $\sigma$-finite measure on the Borel $\sigma$-algebra $\mathbf{B}(\mathbf{R}^\omega)$ which is invariant with respect to the group $S_\omega$.*

Let $\chi_1$ denote the completion of measure $\chi$. In other words, $\chi_1$ is the complete $S_\omega$-measure in $\mathbf{R}^\omega$ and has the uniqueness property with respect to the class of all $\sigma$-finite $S_\omega$-measures. The following statement is valid.

**Theorem 1.** *There exists a partition $\{A, B\}$ of $\mathbf{R}^\omega$ satisfying the following three conditions:*

(1) $(\forall F)$ *($F \subset \mathbf{R}^\omega$, $F$ is a closed subset, $\chi_1(F) > 0 \Rightarrow \mathrm{card}(A \cap F) = \mathrm{card}(B \cap F) = \mathbf{c}$);*

(2) $(\forall g)$ $(g \in G \Rightarrow \mathrm{card}(A \triangle g(A)) < \mathbf{c}, \mathrm{card}(B \triangle g(B)) < \mathbf{c})$*;*

(3) $(\forall h)$ *($h \in s_0 \Rightarrow h(B) = A \cup \{0\}$, where $\{0\}$ is the neutral element of the additive group $\mathbf{R}^\omega$).*

Analogous partitions can be found in [2], [5], [8].

By using these two sets, the measure $\chi_1$ can be extended to a measure which fails to have a strong uniqueness property. For this purpose, it suffices to define a $\sigma$-algebra $S$ generated by the union

$$\{A, B\} \cup \mathrm{dom}(\chi_1) \cup \mathbf{F}(\mathbf{R}^\omega),$$

where

$$\mathbf{F}(\mathbf{R}^\omega) = \{Z : Z \subset \mathbf{R}^\omega, \mathrm{card}(Z) < \mathbf{c}\}.$$

It is clear that any element from $S$ can be represented in the form

$$((A \cap X) \cup (B \cap Y) \cup X_1) \setminus X_2,$$

where $X \in \mathrm{dom}(\chi_1)$, $Y \in \mathrm{dom}(\chi_1)$, $X_1 \in \mathbf{F}(\mathbf{R}^\omega)$ and $X_2 \in \mathbf{F}(\mathbf{R}^\omega)$.

Define on the $S$ the functional $\mu$ by the formula

$$\mu(((A \cap X) \cup (B \cap Y) \cup X_1) \setminus X_2) = \frac{1}{2}(\chi_1(X) + \chi_1(Y)).$$

This definition of $\mu$ is correct and $\mu$ is a measure extending $\chi_1$.

The following Theorem is valid.

**Theorem 2.** *In the space $\mathbf{R}^\omega$, there exists a $\sigma$-finite $S_\omega$-measure $\mu$ having the strong uniqueness property.*

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[1]Georgian Technical University, 77 Kostava Str., Tbilisi 0175, Georgia

[2]A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 6 Tamarashvili Str., Tbilisi 0177, Georgia

*E-mail address*: m.khachidze1995@gmail.com

*E-mail address*: kirtadze2@yahoo.com