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ON THE DOUBLE FOURIER–WALSH–PALEY SERIES OF CONTINUOUS FUNCTIONS

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Abstract. The following theorem is proved: There exists a continuous function F on $[0,1]^2$ such that the modulus of continuity

$$\omega(F;\delta) = O\left(\frac{1}{\sqrt{\log_2 \frac{1}{\delta}}}\right), \quad \delta \to 0+,$$

and the double Fourier-Walsh-Paley series of F diverges on a set of positive measure by rectangles.

1. Introduction

Kolmogorov [6] proved that there exists a function $f \in L([0,2\pi])$ with the trigonometric Fourier series that diverges almost everywhere. Later, he constructed a function with everywhere divergent trigonometric Fourier series [7].

Stein [8] established that there exists a function $f \in L([0,1])$ with the Fourier–Walsh–Paley series that diverges almost everywhere.

Carleson [3] proved that if $f \in L^2([0,2\pi])$, then its trigonometric Fourier series converges almost everywhere. Billard [2] showed that if $g \in L^2([0,1])$, then its Fourier–Walsh–Paley series converges almost everywhere.

Fefferman [4] proved that in the contrast to the Carleson's theorem, there exists a continuous function of two variables on $[0,2\pi]^2$ with the double trigonometric Fourier series that diverges almost everywhere by rectangles.

In [5], we have shown that there exists a continuous function on $[0,1]^2$ the with double Fourier–Walsh–Paley series that diverges almost everywhere by rectangles.

The system of Rademacher functions $\{r_n(x)\}_{n=0}^{\infty}$ on $[0,1)$ is defined as follows. Set

$$r_0(x) = \begin{cases} 1 & \text{for } 0 \le x < \frac{1}{2}, \\ -1 & \text{for } \frac{1}{2} \le x < 1. \end{cases}$$

We extend the function $r_0(x)$ on $(-\infty,\infty)$ with period 1. For $n \ge 1$, set

$$r_n(x) = r_0(2^n x).$$

The Walsh-Paley system of functions is defined as follows. Set $W_0(x) = 1$, for all $x \in [0,1)$. For $n = 2^{m_1} + 2^{m_2} + \cdots + 2^{m_q}$, $(n \ge 1,\, m_1 > m_2 \cdots > m_q \ge 0,)$ set

$$W_n(x) = r_{m_1}(x) r_{m_2}(x) \dots r_{m_q}(x) \quad x \in [0,1).$$

Let $f \in L([0,1])^2$ and let

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{i,j}(f) W_i(x) W_j(y)$$

be the Fourier series of f with respect to the double Walsh–Paley system $\{W_i(x)W_j(y)\}_{i,j=0}^{\infty}$ on $[0,1)^2$.

The moduls of continuity $\omega(F;\delta)$ of a continuous function F on $[0,1]^2$ is defined by

$$\omega(F;\delta) = \sup_{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \le \delta} \{|f(x_1,y_1) - f(x_2,y_2)|,\, (x_1,y_1),\, (x_2,y_2) \in [0,1]^2\}.$$

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We have established that we are able to achieve a certain smoothness in the divergence of the double Fourier–Walsh–Paley series of continuous functions. More precisely the following statement is true.

Theorem 1. *There exists a continuous function F on $[0,1]^2$ such that*

$$\omega(F;\delta) = O\left(\frac{1}{\sqrt{\log_2 \frac{1}{\delta}}}\right), \quad \delta \to 0+,$$

and the double Fourier–Walsh–Paley series of F diverges on a set of positive measure by rectangles.

Note that for the double trigonometric system, Bakhbokh and Nikishin [1] established stronger result where instead of $O\left(\frac{1}{\sqrt{\log_2 \frac{1}{\delta}}}\right)$, $\delta \to 0+$, one has $O\left(\frac{1}{\log_2 \frac{1}{\delta}}\right)$, $\delta \to 0+$. We note also that our method of proof in [5] allowed to achieve only a smoothness of order $O\left(\frac{1}{\log_2 \log_2 \frac{1}{\delta}}\right)$, $\delta \to 0+$.

It is important to remark also that in the case of the trigonometric system the n-th kernel can be written as follows

$$\sin nx \cot \frac{t}{2} + \cos nx,$$

that is, as a sum of two terms, each of which is a product of a function in the trigonometric system ($\sin nx$ and $\cos nx$) multiplied by a function that does not depend on n ($\cot \frac{t}{2}$ and 1). This is not the case for the Walsh-Paley system and this fact complicates the proofs of the divergence results for this system.

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