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ALMOST BICOMPLEX STRUCTURES

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Abstract. Bicomplex numbers exist in real $4n$-dimensions like quaternions. Also, quaternions are, as is known, associated to some tensorial structures defined in $4n$-dimensions, called almost quaternionic structures. In this paper we search the presence of such structures, which we call (almost) bicomplex structures, associated to bicomplex numbers. However, we can see that bicomplex numbers don't present a relation with the (almost) bicomplex structures because bicomplex numbers can be defined only in $4n$-dimensions even though $2n$-dimensions are sufficient to define the almost bicomplex structures. Two examples for 4- and 6-dimensions show clearly this result. Finally, the integrability conditions for these structures are investigated.

1. Introduction

Hypercomplex numbers [4,16] or division algebras [3] are of great importance in physics. Of course, quaternions play a pioneer role in this sense, i.e., the solutions of the $SU(2)$ Yang-Mills theory [1]. As is well known, the generators of the group $SU(2)$, that is, the Pauli matrices, present a quaternionic structure and therefore the $SU(2)$-valued gauge potentials (or connections) are indeed quaternions (or quaternion valued 1-forms). Other kind of these numbers is known as bicomplex numbers existing in the real $4n$-dimensions [14,15], and the system of bicomplex numbers is the first non-trivial Clifford commutative [2] complex.

For similar to the quaternions and corresponding (almost) quaternionic structures there arises the question: are there some tensorial structures associated to the bicomplex numbers? In this paper we search an answer to this question. Our result is that bicomplex numbers aren't associated to any tensorial structures like quaternions. So, bicomplex numbers can be defined only in $4n$-dimensions even though $2n$-dimensions are sufficient to define the (almost) bicomplex structures defined in this paper. The reason of this result is via Proposition 4.1 given by Obata [13] and Theorem 4.2 by Hoffmann and Kunze [6]. The bicomplex structures are easily seen in 4– and 6-dimensions in this paper. Finally, the integrability conditions for these structures are investigated.

2. Bicomplex Numbers

Consider complex numbers field $\mathbb{C}$ with imaginary unit $i = \sqrt{-1}$. Let $j = \sqrt{-1}$ be another imaginary unit satisfying commutative product rule $ij = ji = k$. Given a set of $\mathbb{R}$-linear tensor products $\mathbb{B} = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{R}^4$. Therefore, an element of this space is called a bicomplex number and is written as

$$q = z^1 + jz^2, \quad z^1, z^2 \in \mathbb{C}, \quad \text{or}$$
$$q = (x^1 + ix^2) + j(x^3 + ix^4), \quad x^1, x^2, x^3, x^4 \in \mathbb{R}.$$

Bicomplex numbers have the following multiplication rules:

$$ij = ji = k, \quad ik = ki = -j, \quad jk = kj = -i, \quad k^2 = +1$$

and the addition and subtraction operations are like in the real and complex numbers fields. Also, the zero and unit (or identity) elements of the bicomplex numbers are

$$0_{\mathbb{B}} = (0 + i0) + j(0 + i0) = 0,$$
$$1_{\mathbb{B}} = (1 + i0) + j(0 + i0) = 1.$$

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There is an important difference between $\mathbb{C}$ and $\mathbb{B}$: as the complex numbers form a field, the bicomplex numbers don't, since they contain the divisors of zero, i.e.,

$$(1+ij)(1-ij) = (1-ij)(1+ij) = 0.$$

Therefore bicomplex numbers space $\mathbb{B}$ is a commutative ring with unit and its algebraic properties can be seen in [14].

There are three conjugations in bicomplex numbers. Here, $(\bar{\bullet})$ denotes the complex conjugation in the complex numbers field $\mathbb{C}$. Then, $\forall z_1, z_2 \in \mathbb{C}$, we have the following i, j and ij conjugations, respectively:

$$\bar{q} = \bar{z}_1 + j\bar{z}_2, \quad q^* = z^1 - jz^2, \quad q^\dagger = \bar{z}_1 - j\bar{z}_2,$$

Moduli in the bicomplex numbers are defined for two bicomplex numbers $w = z^1 + z^2 j = x^1 + ix^2 + jx^3 + ijx^4$ in two ways as real $\mathbb{B} \times \mathbb{B} \to \mathbb{R}$ and complex $\mathbb{B} \times \mathbb{B} \to \mathbb{C}$. They are written, respectively, as follows:

$$\begin{aligned} \|w\|^2 =& (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2, \\ \|w\|_*^2 =& w^* w = (z^1)^2 + (z^2)^2. \end{aligned}$$

More details of the algebra of these numbers and the analysis of bicomplex holomorphic functions can be found in Refs. [2, 14, 15].

3. Some Tensorial Structure on Manifolds

Let M be a smooth manifold of real even n-dimensions. If we write a smooth tensorial field I of rank $(1,1)$ on this manifold satisfying the relation

$$I^2 = \epsilon \mathbb{I},$$

where $\mathbb{I}$ is the identity matrix and ϵ is $\{-1, +1\}$, then we say that I is

- an almost complex structure for $\epsilon = -1$, or
- an almost product structure for $\epsilon = +1$.

Definition 3.1. Given three smooth tensorial fields I_1, I_2, I_3 of rank $(1,1)$ on an even dimensional manifold M which satisfy the following rules:

$$\begin{aligned} & I_1^2 = \epsilon_1 \mathbb{I}, \quad I_2^2 = \epsilon_2 \mathbb{I}, \quad I_3^2 = \epsilon_3 \mathbb{I}, \\ & I_2 I_1 = \epsilon I_1 I_2 = -\epsilon_3 I_3, \\ & I_3 I_2 = \epsilon I_2 I_3 = -\epsilon_1 I_1, \\ & I_1 I_3 = \epsilon I_3 I_1 = -\epsilon_2 I_2, \end{aligned}$$

where

$$\epsilon = \epsilon_1 \epsilon_2 \epsilon_3.$$

Therefore we mention the following cases from Ref. [7]:

I. If $\epsilon_1 = \epsilon_2 = \epsilon_3 = -1$, then the triplet (I_1, I_2, I_3) is called an almost quaternionic structure,

II. If $\epsilon_1 = \epsilon_2 = -1$ and $\epsilon_3 = +1$, we will say that the triplet (I_1, I_2, I_3) is ***an almost bicomplex structure***,

III. If $\epsilon_1 = \epsilon_2 = \epsilon_3 = +1$, then the triplet (I_1, I_2) is called an almost product structure.

If we denote the local coordinates on the manifold M by $\{x^\mu\} = (x^i, y^i) \in \mathbb{R}^{2n}$, where $i, j = 1, \ldots, n$, then the acting of an almost complex structure on their local coordinate bases is written as

$$I(\frac{\partial}{\partial x^i}) = -\frac{\partial}{\partial y^i}, \qquad I(\frac{\partial}{\partial y^i}) = \frac{\partial}{\partial x^i}.$$

Then, we can show the almost complex structure I on $\mathbb{R}^{2n}$ by a block matrix (or canonical) representation such that

$$I = \begin{pmatrix} \mathbf{0} & \mathbb{I}_{n\times n} \\ -\mathbb{I}_{n\times n} & \mathbf{0} \end{pmatrix},$$

where $\mathbb{I}_{n\times n}$ and $\mathbf{O}$ are the unit and zero $n \times n$ matrices, respectively.

4. Almost Bicomplex Structure

The Case II that we call bicomplex structure was handle by Hsu [7] and Liberman [11]. The mutual point of these authors is that this structure is considered as the distribution of a tangent bundle on a $4n$-dimensional manifold, since this structure is handled in the perspective of an almost quaternionic structure. If there exists a pair of two complex structures (I, J) commuting each other, $K = IJ = JI$, then we call it the almost bicomplex structure to the triple (I, J, K). Our claim mentioned above is whether this structure is associated to bicomplex numbers. Therefore, first we have to present Obata' s proposition.

Proposition 4.1 (Obata [13]). *Let $\tilde{J} \in GL(n, \mathbb{C})$ be a non-singular complex matrix such that*

$$\tilde{J} = A + \mathrm{i}B,$$

where $A, B \in GL(n, \mathbb{R})$. Then the correspondence

$$\tilde{J} \to J = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in GL(2n, \mathbb{R})$$

is an isomorphism. The matrix J is commutated by a matrix, independent of n odd or even such that

$$I = \begin{pmatrix} \mathbf{0} & \mathbb{I}_{n\times n} \\ -\mathbb{I}_{n\times n} & \mathbf{0} \end{pmatrix} \in GL(2n, \mathbb{R}),$$

so,

$$IJ = JI.$$

If A is unitary, B is orthogonal, and vice versa.

Since an almost complex structure is a diagonalizable matrix, Hoffman and Kunze' s theorem is valid.

Theorem 4.2 (Hoffman-Kunze [6]). *A set of commuting diagonalizable matrices are simultaneously diagonalizable.*

Therefore, we can say that the matrix J is also a diagonalizable matrix, and so we have the following

Corollary 4.3. *Let I and J be two almost complex structures commuting each other on a smooth manifold of real $2n$-dimensions, hence*

$$IJ = JI = K, \quad IK = KI = -J, \quad JK = KJ = -I, \quad K^2 = +\mathbb{I}_{2n\times 2n}. \tag{1}$$

Therefore, if the $n \times n$ matrices A and B satisfy the relations

$$AB + BA = \mathbf{0}_{n\times n},$$
$$A^2 - B^2 = -\,\mathbb{I}_{n\times n},$$

then I, J, K are constructed as independent of n odd or even as follows:

$$I = \begin{pmatrix} \mathbf{0} & \mathbb{I}_{n\times n} \\ -\mathbb{I}_{n\times n} & \mathbf{0} \end{pmatrix}, \quad J = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}, \quad K = \begin{pmatrix} -B & A \\ -A & -B \end{pmatrix}. \tag{2}$$

For this result we can give two examples in 4– and 6-dimensions to this result. In 4-dimensions, with respect to equation (2), we get

$$I = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 0 & 0 & p \\ 0 & 0 & p & 0 \\ 0 & -p & 0 & 0 \\ -p & 0 & 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & -p & 0 & 0 \\ -p & 0 & 0 & 0 \\ 0 & 0 & 0 & -p \\ 0 & 0 & -p & 0 \end{pmatrix}, \tag{3}$$

where $p^2 = 1$. On the other hand, in 6-dimensions, we have

$$I = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \; J = \begin{pmatrix} 0 & 0 & s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q & 0 \\ -s & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s \\ 0 & -q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -s & 0 & 0 \end{pmatrix}, \; K = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & s \\ 0 & -q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -s & 0 & 0 \\ 0 & 0 & -s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -q & 0 \\ s & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4}$$

where $q^2 = s^2 = 1$.

5. Integrability of Almost Bicomplex Structures

In this paper, we use the following geometrical preliminaries . Let M be a smooth manifold of real even dimensions n with local coordinates $\{x^\mu\} \in \mathbb{R}^n$, $(\mu = 1, \ldots, n)$. Given a connection on a tangent bundle TM by the map $\nabla : \mathcal{C}^\infty(TM) \to \Lambda^1(TM)$ together with the covariant derivative

$$\nabla = \mathrm{d} + [\Gamma, \cdot],$$

where $\Gamma \in \Lambda^1(\mathrm{End}(TM))$ is the 1-form connection and d is an exterior derivative operator. The curvature of this connection is

$$R = \nabla\Gamma = \mathrm{d}\Gamma + \Gamma \wedge \Gamma \in \Lambda^2(\mathrm{End}(TM)).$$

Suppose that let M be an almost complex manifold with the almost complex structure I. Given two vector fields X, Y on this manifold. The torsion tensor of the almost complex structure I, called also Nijenhuis tensor, is defined as follows:

$$N_I(X, Y) = [I, I](X, Y) = 2\left\{[IX, IY] - [X, Y] - I[X, IY] - I[IX, Y]\right\},$$

where $[X, Y] = X(Y) - Y(X)$ is the Lie bracket.

Kobayashi and Nomizu [10] say that every almost complex manifold M admits an almost complex affine connection such that its torsion T is given by $N = 8T$, where N is the torsion of the almost complex structure I on M. Then, an almost complex structure is said to be integrable, $\mathrm{d}I = 0$, if its torsion vanishes $(N = 0)$ and is parallel, $\nabla I = 0$, with respect to the connection ∇. Thus a complex structure on $\mathbb{R}^{n=2m}$ is equivalent to a torsion-free $GL(m, C)$-structure [8].

Definition 5.1. Let I_1 and I_2 be two tensor fields of $(1, 1)$ type on an even dimensional manifold satisfying $I_1^2 = \epsilon_1\mathbb{I}$, $I_2^2 = \epsilon_2\mathbb{I}$ and $I_1 I_2 = \epsilon I_3$ for some constants $\epsilon, \epsilon_1, \epsilon_2$. They are covariant constant tensors with respect to the connection ∇ if

$$\nabla I_1 = 0, \qquad \nabla I_2 = 0, \qquad (also\ \nabla I_3 = 0).$$

Thus ∇ is called the (I_1, I_2)-connection (and, consequently, I_3-connection in view of $I_1 I_2 = \epsilon_3 I_3$).

Suppose for a short time that we have an almost quaternionic structure induced by three almost complex structures I_1, I_2, I_3 such that $I_1 I_2 = -I_2 I_1 = I_3$. The integrability conditions of the almost quaternionic structures are shortly given by six vanishing Lie brackets $[I_i, I_j] = 0$, $(i, j = 1, 2, 3)$ and vanishing curvature tensor of symmetric affine connection, that is, Levi-Civita, [12, 17]. Indeed, we can generalize this for a unique almost complex structure by the following

Theorem 5.2. *Let I be an almost complex structure which is a tensor field of* $(1, 1)$ *type on a manifold. If this tensor field* (*or almost complex structure*) *is parallel with respect to a connection* ∇ *on this manifold, i.e., $\nabla I = 0$, then this connection is likewise flat.*

Proof. Let $\nabla = \mathrm{d} + [\Gamma, \dot{]}$ be the covariant derivative of the connection ∇. If $\nabla I = 0$, then $\mathrm{d}I + \Gamma I - I\Gamma = 0$. The exterior derivative of this expression reads as $RI = IR$, where $R = \mathrm{d}\Gamma + \Gamma \wedge \Gamma$ is the curvature of the connection. Also, we can write $R = I^{-1}RI = I^{-1}IR = \pm R$ from $I^2 = \pm\mathbb{I}$. Therefore, if $I^2 = -\mathbb{I}$, then the curvature of a connection, compatible by (or parallel to) almost complex structure I, is flat: $R = 0$. □

From all the above and Theorem 5.2, we can give for the integrability of almost complex structure the following

Corollary 5.3. *An almost complex manifold M admits a torsion free almost complex affine connection if and only if an almost complex structure has no torsion* [10]*. On an almost complex manifold there exists an affine connection whose almost complex structure is a covariant constant* [5, 13]*, and any connection which is compatible by this almost complex structure is flat.*

We have defined the almost bicomplex structure for two almost complex structures I and J in real $2n$-dimensions which commute each other as follows:

$$IJ = JI = K, \quad IK = KI = -J, \quad JK = KJ = -I, \quad K^2 = +\mathbb{I}_{2n\times 2n}.$$

One can easily see from Definition 5.1 that the almost bicomplex structure (I, J) is parallel with respect to an affine connection on the manifold. As a natural consequence, $K = IJ = JI$ is also parallel with respect to the same connection. When this connection is symmetric, that is, Levi Civita, this almost bicomplex structure is integrable.

On the other hand, in order to investigate another integrability condition of this structure we need the Lie brackets $[I, J]$, $[I, K]$, $[J, K]$ and $[K, K]$ as well as the Nijenhuis tensors $[I, I]$ and $[J, J]$ of the almost complex structures I and J. Therefore, we consider the following proposition due to Kobayashi and Nomizu.

Proposition 5.4 (Kobayashi [9])**.** *Let A and B be tensor fields of type* $(1, 1)$ *and $X, Y \in \Gamma(M)$ vector fields on the manifold M. Set*

$$\begin{aligned} S(X,Y) = [P,Q](X,Y) =& [PX,QY] + [QX,PY] - P[X,QY] - P[QX,Y] \\ & - Q[X,PY] - Q[PX,Y] + (PQ+QP)[X,Y]. \end{aligned}$$

Then the mapping $S : \Gamma(M)\times\Gamma(M) \to \Gamma(M)$ is a skew-symmetric tensor field of type $(1, 2)$*, $S(X, Y) = -S(Y, X)$.*

Using Proposition 5.4 and following [17], we investigate the integrability properties of almost bicomplex structures. Similar theorems were obtained for almost quaternionic manifolds by Yano [17].

On the other hand, we write the following relation:

$$\begin{aligned} [P,QR](X,Y) =& [PX,QRY] + [QRX,PY] - P([X,QRY] + [QRX,Y]) \\ & - QR([X,PY] + [PX,Y]) - (PQR + QRP)[X,Y]. \end{aligned} \tag{5}$$

i) If we choose $P = Q = I$ and $R = J$, considering the commutation relation $IJ = K$ from the equation (5), we get

$$[I,K](X,Y) = I[I,J](X,Y) + \frac{1}{2}([I,I](JX,Y) + [I,I](X,JY]).$$

Similarly, for $P = Q = J$ and $R = I$, we have

$$[J,K](X,Y) = J[J,I](X,Y) + \frac{1}{2}([J,J](IX,Y) + [J,J](X,IY]).$$

Therefore, if $[I, I] = 0$, then

$$[I,K](X,Y) = [I,IJ](X,Y) = I[I,J](X,Y), \tag{6}$$

and if $[J, J] = 0$, then

$$[J,K](X,Y) = [J,JI](X,Y) = J[J,I](X,Y), \tag{7}$$

ii) If we choose $P = I$, $Q = J$ and $R = K$, we get

$$\begin{aligned} -[I,I](X,Y) - [J,J](X,Y) =& [I,J](KX,Y) + [I,J](X,KY) \\ & + I[J,K](X,Y) + J[I,K](X,Y) \end{aligned}$$

If $[I, I] = 0$ and $[J, J] = 0$ simultaneously, because of equations (6) and (7), then we get

$$[I,J](KX,Y) + [I,J](X,KY) = 0,$$

or shortly,

$$[I,J] = [J,I] = [K,K] = IJ - JI = K - K = 0.$$

Then we can see that the almost complex structures I and J must simultaneously be integrable. Thus, we have following theorem:

Theorem 5.5. *If I and J are two almost complex structures on a smooth manifold of real $2n$-dimensions which commute each other, $IJ = JI = K$, then I, J and K must simultaneously be integrable as follows:*

$$[I,I] = [J,J] = 0, \quad [I,J] = [J,I] = [K,K] = 0, \quad [I,K] = [J,K] = 0.$$

6. Conclusion

When one compares quaternions and bicomplex numbers handled in this paper, although these numbers live in the real $4n$ dimensions, the associated almost complex structures to these numbers behave different concept. So, almost quaternionic structure has to be defined in $4n$-dimensions, but any even dimension is sufficient for the almost bicomplex structure because of Proposition 4.1 given by Obata [13] and Theorem 4.2 by Hoffmann and Kunze [6]. This means that the almost bicomplex structures in the concept of this paper don't relate to the bicomplex numbers. In the quaternions two anticommuting almost complex structures induce the third almost complex structure, however, two commuting almost complex structures cannot induce a third almost complex structure. As is shown from equation (1), if I and J are two almost complex structures having commutations relationship $IJ = JI = K$, then $K^2 = +\mathbb{I}$, that is K isn't an almost complex structure. Thus the triplet (I,J,K) cannot be associated to the bicomplex numbers. Although, in this case, we have used the term "almost bicomplex structure" for this triplet. We have shown clearly this situation on the almost complex structures obtained in 4– and 6-dimensions given in equations (3) and (4), respectively. Consequently, by Theorem (5.5) we have presented the integrability of the bicomplex structure.

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