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INVESTIGATION OF NONCLASSICAL TRANSMISSION PROBLEMS OF THE THERMO-ELECTRO-MAGNETO ELASTICITY THEORY FOR COMPOSED BODIES BY THE INTEGRAL EQUATION METHOD

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Abstract. We investigate multi-field problems for complex elastic anisotropic structures when in different adjacent components of the composed body different refined models of elasticity theory are considered. In particular, we analyse the case when we have the generalized thermo-electro-magneto elasticity model (GTEME model) in one region of the composed body and the generalized thermo-elasticity model (GTE model) in the other adjacent region. This type of mechanical problem is described mathematically by systems of partial differential equations with appropriate transmission and boundary conditions. In the GTEME model part we have six-dimensional unknown physical field (three components of the displacement vector, electric potential function, magnetic potential function, and temperature distribution function), while in the GTE model part we have four- dimensional unknown physical field (three components of the displacement vector and temperature distribution function). The diversity in dimensions of the interacting physical fields are taken into consideration in mathematical formulation and analysis of the corresponding boundary-transmission problems. We apply the potential method and the theory of pseudodifferential equations and prove the uniqueness and existence theorems of solutions to different type boundary-transmission problems in appropriate Sobolev spaces.

1. Introduction

Modern industrial and technological processes apply widely, on the one hand, composite materials with complex microstructure and, on the other hand, complex composed structures consisting of materials having essentially different physical properties (for example, piezoelectric, piezomagnetic, hemitropic materials, two- and multi-component mixtures, nano-materials, bio-materials, and solid structures constructed by composition of these materials, such as, e.g., Smart Materials and other meta-materials). Therefore the investigation and analysis of mathematical models describing the mechanical, thermal, electric, magnetic and other physical properties of such materials are of crucial importance for both fundamental research and practical applications.

In the study of active material systems, there is significant interest in the coupling effects between elastic, electric, magnetic and thermal fields. For example, piezoelectric materials (electro-elastic coupling) have been used as ultrasonic transducers and micro-actuators; pyroelectric materials (thermal-electric coupling) have been applied in thermal imaging devices; piezomagnetic materials (elastic-magnetic coupling) are pursued for health monitoring of civil structures (see [9, 12, 13, 15, 24–32, 39, 45–47, 50, 51, 53, 55], and the references therein).

Although natural materials rarely show full coupling between elastic, electric, magnetic, and thermal fields, some artificial materials do. In [54], it was reported that the fabrication of $BaTiO_3$-$CoFe_2O_4$ composite had the electro-magnetic effect not existing in either constituent. Other examples of similar complex coupling can be found in [3–6, 16–18, 34–36, 40, 41, 48, 56]. For more detailed historical and bibliographic data see [1, 7, 49].

In the present paper, we investigate multi-field problems for complex elastic anisotropic structures when in different adjacent components of the composed body different refined models of elasticity

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theory are considered. In particular, we analyse the case when we have the generalized thermo-electro-magneto elasticity model associated with Green-Lindsay's model (GTEME model) in one region of the composed body and the generalized thermo-elasticity model (GTE model) in the other adjacent region. The essential feature of the generalized models under consideration is that the heat propagation has a finite speed (see [2, 10, 11, 14, 19, 20, 49]). This type of mechanical problem is described mathematically by systems of second order partial differential equations with appropriate transmission and boundary conditions. In the GTEME model part we have six-dimensional unknown physical field (three components of the displacement vector, electric potential function, magnetic potential function, and temperature distribution function), while in the GTE model part we have four- dimensional unknown physical field (three components of the displacement vector and temperature distribution function). Other field characteristics (e.g., mechanical stresses, electric and magnetic fields, electric displacement vector, magnetic induction vector, heat flux vector and entropy density) can be then determined by the gradient equations and the constitutive equations.

Note that the basic and mixed initial-boundary value problems for the GTEME theory are investigated in the monograph [7]. The transmission problems for composed elastic structures are also studied when in all adjacent regions of piecewise homogeneous composite bodies the same type GTEME model is considered with different material constants.

As we have already mentioned, the main goal of the present paper is investigation of the transmission problems when in different parts of adjacent regions of piecewise homogeneous composite bodies different models (in particular, GTEME and GTE models) are considered. The diversity in dimensions of the interacting physical fields essentially complicates mathematical formulation and analysis of the corresponding boundary-transmission problems. We apply the potential method and the theory of pseudodifferential equations and prove the uniqueness and existence theorems of solutions to different type basic boundary-transmission problems in appropriate Sobolev spaces. Properties of the layered potentials associated with the matrix differential operators of the GTEME and GTE models and the boundary operators generated by them are studied in [7, 21, 22, 42], and for the readers convenience, some results needed in our analysis are briefly presented in Appendix.

2. Basic Field Equation and Formulation of Boundary Transmission Problems

First we present the pseudo-oscillation equations of the GTEME and GTE models with corresponding Green's identities and afterwards we formulate the transmission problems. The pseudo-oscillation equations considered in the paper are obtained from the corresponding equations of dynamics by the Laplace transform and they contain a complex parameter $\tau = \sigma + i\omega$. Here we investigate the boundary-transmission problems for pseudo-oscillation equations. Solutions to the original dynamical initial-boundary-transmission problems can be then reconstructed by the inverse Laplace transform with respect to the parameter τ from solutions to the pseudo-oscillation problems. The detailed derivations of pseudo-oscillation equations from the dynamical constitutive relations can be found in [7] and [21]. In our analysis we will use essentially the results obtained in the monograph [7] and develop the potential method to complex boundary-transmission problems for anisotropic composed multilayered elastic structures.

2.1. **Field equations of the GTEME model and Green's formulas.** The basic linear system of pseudo-oscillation equations for the thermo-electro-magneto-elasticity theory associated with Green-Lindsay's model for homogeneous solids in matrix form reads as [7]

$$A(\partial_x, \tau)\, U(x,\tau) = \Phi(x,\tau)\,,$$

where $U = (u_1, u_2, u_3, \varphi, \psi, \vartheta)^\top := (u, \varphi, \psi, \vartheta)^\top$ is the sought for complex-valued vector function, $\Phi = (\Phi_1, \dots, \Phi_6)^\top$ is a given vector-function, and $A(\partial_x, \tau)$ is a matrix differential operator

$$A(\partial_x,\tau)=[A_{pq}(\partial_x,\tau)]_{6\times 6}:$$

$$:=\begin{bmatrix} [c_{rjkl}\partial_j\partial_l-\varrho\tau^2\delta_{rk}]_{3\times 3} & [e_{lrj}\partial_j\partial_l]_{3\times 1} & [q_{lrj}\partial_j\partial_l]_{3\times 1} & [-(1+\nu_0\tau)\lambda_{rj}\partial_j]_{3\times 1} \\ [-e_{jkl}\partial_j\partial_l]_{1\times 3} & \varkappa_{jl}\partial_j\partial_l & a_{jl}\partial_j\partial_l & -(1+\nu_0\tau)\,p_j\partial_j \\ [-q_{jkl}\partial_j\partial_l]_{1\times 3} & a_{jl}\,\partial_j\partial_l & \mu_{jl}\partial_j\partial_l & -(1+\nu_0\tau)m_j\partial_j \\ [-\tau\lambda_{kl}\partial_l]_{1\times 3} & \tau p_l\partial_l & \tau m_l\partial_l & \eta_{jl}\partial_j\partial_l-\tau^2 h_0-\tau d_0 \end{bmatrix}_{6\times 6}. \tag{2.1}$$

The superscript $(\cdot)^\top$ denotes transposition operation, $\tau=\sigma+i\omega$ is a complex parameter, the summation over the repeated indices is meant from 1 to 3; $\partial=\partial_x=(\partial_1,\partial_2,\partial_3)$, $\partial_j=\partial/\partial x_j$. The components of the vector function U have the following physical sense: the first three components correspond to the elastic displacement vector $u=(u_1,u_2,u_3)^\top$, the forth and fifth ones, φ and ψ are, respectively, the electric and magnetic potentials, and the sixth component ϑ stands for the temperature distribution; c_{rjkl} are the elastic constants, e_{jkl} are the piezoelectric constants, q_{jkl} are the piezomagnetic constants, $\varkappa_{jk}$ are the dielectric (permittivity) constants, μ_{jk} are the magnetic permeability constants, a_{jk} are the coupling coefficients connecting electric and magnetic fields, p_j and m_j are the constants characterizing the relation between thermodynamic processes and electromagnetic effects, λ_{rj} are the thermal strain constants, η_{jk} are the heat conductivity coefficients, ϱ denotes the mass density, ν_0 and h_0 are two relaxation times, d_0 is a constitutive coefficient. These constants satisfy the symmetry conditions:

$$\begin{gathered} c_{rjkl}=c_{jrkl}=c_{klrj},\ \ e_{klj}=e_{kjl},\ \ q_{klj}=q_{kjl}, \\ \varkappa_{kj}=\varkappa_{jk},\ \ \lambda_{kj}=\lambda_{jk},\ \ \mu_{kj}=\mu_{jk},\ \ a_{kj}=a_{jk},\ \ \eta_{kj}=\eta_{jk},\ \ r,j,k,l=1,2,3. \end{gathered} \tag{2.2}$$

From physical considerations it follows that (see, e.g., [2, 14, 33, 44, 49]):

$$\begin{gathered} c_{rjkl}\,\xi_{rj}\,\xi_{kl}\geq\delta_0\,\xi_{kl}\,\xi_{kl},\ \ \varkappa_{kj}\,\xi_k\,\xi_j\geq\delta_1\,|\xi|^2,\ \ \mu_{kj}\,\xi_k\,\xi_j\geq\delta_2\,|\xi|^2,\ \ \eta_{kj}\,\xi_k\,\xi_j\geq\delta_3\,|\xi|^2, \\ \text{for all }\ \xi_{kj}=\xi_{jk}\in\mathbb{R}\text{ and for all }\ \xi=(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^3, \end{gathered} \tag{2.3}$$

$$\nu_0>0,\quad h_0>0,\quad d_0\nu_0-h_0>0, \tag{2.4}$$

where δ_0, δ_1, δ_2, and δ_3 are positive constants depending on material parameters.

Due to the symmetry conditions (2.2), with the help of (2.3) one can easily derive the inequalities

$$\begin{gathered} c_{rjkl}\,\zeta_{rj}\,\overline{\zeta_{kl}}\geq\delta_0\,\zeta_{kl}\,\overline{\zeta_{kl}},\ \ \varkappa_{kj}\,\zeta_k\,\overline{\zeta_j}\geq\delta_1\,|\zeta|^2,\ \ \mu_{kj}\,\zeta_k\,\overline{\zeta_j}\geq\delta_2\,|\zeta|^2,\ \ \eta_{kj}\,\zeta_k\,\overline{\zeta_j}\geq\delta_3\,|\,\zeta|^2, \\ \text{for all }\ \zeta_{kj}=\zeta_{jk}\in\mathbb{C}\text{ and for all }\ \zeta=(\zeta_1,\zeta_2,\zeta_3)\in\mathbb{C}^3, \end{gathered} \tag{2.5}$$

where the over bar denotes complex conjugation. The positive definiteness of the potential energy and the laws of thermodynamics imply that the following 8×8 matrix

$$M=[M_{kj}]_{8\times 8}:=\begin{bmatrix} [\varkappa_{jl}]_{3\times 3} & [a_{jl}]_{3\times 3} & [p_j]_{3\times 1} & [\nu_0 p_j]_{3\times 1} \\ [a_{jl}]_{3\times 3} & [\mu_{jl}]_{3\times 3} & [m_j]_{3\times 1} & [\nu_0 m_j]_{3\times 1} \\ [p_j]_{1\times 3} & [m_j]_{1\times 3} & d_0 & h_0 \\ [\nu_0 p_j]_{1\times 3} & [\nu_0 m_j]_{1\times 3} & h_0 & \nu_0 h_0 \end{bmatrix}_{8\times 8} \tag{2.6}$$

is positive definite. Moreover, it follows that the matrices

$$\Lambda^{(1)}:=\begin{bmatrix} [\varkappa_{kj}]_{3\times 3} & [a_{kj}]_{3\times 3} \\ [a_{kj}]_{3\times 3} & [\mu_{kj}]_{3\times 3} \end{bmatrix}_{6\times 6},\qquad \Lambda^{(2)}:=\begin{bmatrix} d_0 & h_0 \\ h_0 & \nu_0 h_0 \end{bmatrix}_{2\times 2} \tag{2.7}$$

are positive definite as well, i.e.,

$$\begin{gathered} \varkappa_{kj}\,\zeta'_k\,\overline{\zeta'_j}+a_{kj}\,(\zeta'_k\,\overline{\zeta''_j}+\overline{\zeta'_k}\,\zeta''_j)+\mu_{kj}\,\zeta''_k\,\overline{\zeta''_j}\geq\kappa_1\,(|\zeta'|^2+|\zeta''|^2)\qquad\forall\zeta',\zeta''\in\mathbb{C}^3, \\ d_0\,|z_1|^2+h_0\,(z_1\,\overline{z_2}+\overline{z_1}\,z_2)+\nu_0 h_0\,|z_2|^2\geq\kappa_2\,(|z_1|^2+|z_2|^2)\qquad\forall z_1,z_2\in\mathbb{C}, \end{gathered}$$

with some positive constants κ_1 and κ_2 depending on the material parameters involved in matrices (2.7).

Further, let us introduce the generalized stress operator $\mathcal{T}(\partial_x,n,\tau)$ associated with the pseudo-oscillation operator $A(\partial_x,\tau)$,

$$\mathcal{T}(\partial_x,n,\tau)=\big[\,\mathcal{T}_{pq}(\partial_x,n,\tau)\,\big]_{6\times 6}:$$

$$:=\begin{bmatrix} [c_{rjkl}\,n_j\partial_l]_{3\times 3} & [e_{lrj}\,n_j\partial_l]_{3\times 1} & [q_{lrj}\,n_j\partial_l]_{3\times 1} & [-(1+\nu_0\tau)\lambda_{rj}n_j]_{3\times 1}\\ [-e_{jkl}\,n_j\partial_l]_{1\times 3} & \varkappa_{jl}\,n_j\partial_l & a_{jl}\,n_j\partial_l & -(1+\nu_0\tau)p_jn_j\\ [-q_{jkl}\,n_j\partial_l]_{1\times 3} & a_{jl}\,n_j\partial_l & \mu_{jl}\,n_j\partial_l & -(1+\nu_0\tau)m_jn_j\\ [0]_{1\times 3} & 0 & 0 & \eta_{jl}n_j\partial_l \end{bmatrix}_{6\times 6}. \tag{2.8}$$

For a six vector $U=(u,\varphi,\psi,\vartheta)^\top$ we can calculate the so-called generalized stress vector $\mathcal{T}U$,

$$\begin{aligned}\mathcal{T}(\partial_x,n,\tau)\,U(x,\tau)=&\big(\sigma_{1j}(x,\tau)n_j(x),\ \sigma_{2j}(x,\tau)n_j(x),\ \sigma_{3j}(x,\tau)n_j(x),\\ &-D_j(x,\tau)n_j(x),\ -B_j(x,\tau)n_j(x),\ -T_0^{-1}q_j(x,\tau)n_j(x)\big)^\top.\end{aligned} \tag{2.9}$$

Due to (2.9), the components of the stress vector have the following physical sense: the first three components correspond to the mechanical stress vector in the theory of generalized thermo-electro-magneto-elasticity, the forth and the fifth components correspond to the normal components of the electric displacement vector and the magnetic induction vector, respectively, with opposite sign, and finally, the sixth component is $(-T_0^{-1})$ times the normal component of the heat flux vector; here $n=(n_1,n_2,n_3)$ stands for the unit normal vector to the corresponding surface element, σ_{ij} are the components of the mechanical stress tensor, T_0 is the initial reference temperature, that is the temperature in the natural state in the absence of deformation and electromagnetic fields, $D=(D_1,D_2,D_3)^\top$ is the electric displacement vector and $B=(B_1,B_2,B_3)^\top$ is the magnetic induction vector.

Recall that $E=(E_1,E_2,E_3)^\top=-\operatorname{grad}\varphi$ and $H=(H_1,H_2,H_3)^\top=-\operatorname{grad}\psi$ are electric and magnetic fields, respectively, σ_{ij} are the components of the mechanical stress tensor, $\varepsilon_{kj}=2^{-1}(\partial_k u_j+\partial_j u_k)$ are the components of the mechanical strain tensor, $q=(q_1,q_2,q_3)^\top$ is the heat flux vector, and the corresponding constitutive equations read as

$$\begin{aligned}\sigma_{rj}(x,\tau)&=c_{rjkl}\varepsilon_{kl}(x,\tau)+e_{lrj}\partial_l\varphi(x,\tau)+q_{lrj}\partial_l\psi(x,\tau)-(1+\nu_0\tau)\lambda_{rj}\vartheta(x,\tau),\\ D_j(x,\tau)&=e_{jkl}\,\varepsilon_{kl}(x,\tau)-\varkappa_{jl}\,\partial_l\varphi(x,\tau)-a_{jl}\,\partial_l\psi(x,\tau)+(1+\nu_0\tau)p_j\vartheta(x,\tau),\\ B_j(x,\tau)&=q_{jkl}\,\varepsilon_{kl}(x,\tau)-a_{jl}\,\partial_l\varphi(x,\tau)-\mu_{jl}\partial_l\psi(x,\tau)+(1+\nu_0\tau)m_j\vartheta(x,\tau),\\ q_j(x,\tau)&=-T_0\,\eta_{jl}\partial_l\vartheta(x,\tau)\,.\end{aligned}$$

Let $\Omega=\Omega^+$ be a bounded domain of $\mathbb{R}^3$ with a sufficiently smooth boundary $S=\partial\Omega$ and $\Omega^-=\mathbb{R}^3\setminus\overline{\Omega}$. For simplicity, in what follows, we assume that $S\in C^\infty$, if not otherwise stated.

By $C^k(\overline{\Omega})$ we denote the subspace of functions from $C^k(\Omega)$ whose derivatives up to the order k are continuously extendable to S from $\Omega^\pm$; $C^{k,\alpha}(\overline{\Omega^\pm})$ denotes the subspace of functions from $C^k(\overline{\Omega^\pm})$ whose kth order derivatives are Hölder continuous in $\Omega^\pm$ with exponent $\alpha\in(0,1]$. By L_p, $L_{p,\mathrm{loc}}$, $L_{p,\mathrm{comp}}$, W_p^r, $W_{p,\mathrm{loc}}^r$, $W_{p,\mathrm{comp}}^r$, H_p^s, and $B_{p,q}^s$ (with $r\geq 0$, $s\in\mathbb{R}$, $1<p<\infty$, $1\leq q\leq\infty$) we denote the well-known Lebesgue, Sobolev-Slobodetskii, Bessel potential, and Besov function spaces, respectively (see, e.g., [37, 52]). Recall that $H_2^r=W_2^r=B_{2,2}^r$, $H_2^s=B_{2,2}^s$, $W_p^t=B_{p,p}^t$, and $H_p^k=W_p^k$, for any $r\geq 0$, for any $s\in\mathbb{R}$, for any positive and non-integer t, and for any non-negative integer k.

For arbitrary vector functions

$$U=(u_1,u_2,u_3,\varphi,\psi,\vartheta)^\top\in\big[C^2(\overline{\Omega})\big]^6 \text{ and } U'=(u_1',u_2',u_3',\varphi',\psi',\vartheta')^\top\in\big[C^2(\overline{\Omega})\big]^6,$$

the following first Green identity

$$\int\limits_{\Omega}\Big[A(\partial_x,\tau)\,U\cdot U'+\mathcal{E}_\tau(U,\overline{U'})\Big]dx=\int\limits_{\partial\Omega}\{\mathcal{T}(\partial_x,n,\tau)U\}^+\cdot\{U'\}^+dS \tag{2.10}$$

holds, where the central dot denotes the scalar product of two vectors in the complex vector space $\mathbb{C}^N$, i.e., $a\cdot b\equiv(a,b):=\sum_{j=1}^N a_j\,\overline{b_j}$ for $a,b\in\mathbb{C}^N$, the symbols $\{\cdot\}^\pm$ denote the one sided limits (the trace operators) on $\partial\Omega^\pm$ from $\Omega^\pm$, the operators $A(\partial_x,\tau)$ and $\mathcal{T}(\partial_x,n,\tau)$ are given by (2.1) and (2.8), respectively, and

$$
\begin{aligned}
&\mathcal{E}_\tau(U,\overline{U'}) := c_{rjkl}\,\partial_l u_k\,\overline{\partial_j u'_r} + \varrho\,\tau^2\,u_r\,\overline{u'_r} + e_{lrj}\,(\partial_l\varphi\,\overline{\partial_j u'_r} - \partial_j u_r\,\overline{\partial_l\varphi'})\\
&+ q_{lrj}\,(\partial_l\psi\,\overline{\partial_j u'_r} - \partial_j u_r\,\overline{\partial_l\psi'}) + \varkappa_{jl}\,\partial_l\varphi\,\overline{\partial_j\varphi'} + a_{jl}\,(\partial_l\varphi\,\overline{\partial_j\psi'} + \partial_j\psi\,\overline{\partial_l\varphi'})\\
&+ \mu_{jl}\partial_l\psi\overline{\partial_j\psi'} + \lambda_{kj}[\tau\overline{\vartheta'}\partial_j u_k - (1+\nu_0\tau)\vartheta\overline{\partial_j u'_k}] - p_l[\tau\overline{\vartheta'}\partial_l\varphi + (1+\nu_0\tau)\vartheta\overline{\partial_l\varphi'}]\\
&- m_l\,[\tau\,\overline{\vartheta'}\,\partial_l\psi + (1+\nu_0\tau)\vartheta\,\overline{\partial_l\psi'}] + \eta_{jl}\,\partial_l\vartheta\,\overline{\partial_j\vartheta'} + \tau\,(h_0\tau + d_0)\,\vartheta\,\overline{\vartheta'}\,. \qquad (2.11)
\end{aligned}
$$

Note that Green's formula (2.10) by a standard limiting procedure can be generalized to the Lipschitz domains and to vector functions $U \in [W^1_2(\Omega)]^6$ with $A(\partial_x,\tau)U \in [L_2(\Omega)]^6$ and $U' \in [W^1_2(\Omega)]^6$. Using Green's first identity, we can correctly determine a *generalized trace of the stress vector* $\{\mathcal{T}(\partial_x,n,\tau)U\}^+ \in [H_2^{-1/2}(\partial\Omega)]^6$ for a function $U \in [W^1_2(\Omega)]^6$ with $A(\partial_x,\tau)U \in [L_2(\Omega)]^6$ by the following relation (cf. [7,38,43])

$$
\big\langle\,\{\mathcal{T}(\partial_x,n,\tau)U\}^+\,,\,\{U'\}^+\big\rangle_{\partial\Omega} := \int\limits_\Omega [\,A(\partial_x,\tau)\,U\cdot U' + \mathcal{E}_\tau(U,\overline{U'})\,]\,dx, \qquad (2.12)
$$

where $U' \in [W^1_2(\Omega)]^6$ is an arbitrary vector function. Here the symbol $\langle\,\cdot\,,\,\cdot\,\rangle_{\partial\Omega}$ denotes the duality pairing of $[H_2^{-1/2}(\partial\Omega)]^6$ with $[H_2^{1/2}(\partial\Omega)]^6$ which extends the usual L_2 inner product for the complex-valued vector functions,

$$
\langle\,f\,,\,g\,\rangle_{\partial\Omega} = \int\limits_{\partial\Omega}\sum_{j=1}^{6} f_j(x)\,\overline{g_j(x)}\,dS \ \text{ for } \ f,g \in [L_2(\partial\Omega)]^6.
$$

2.2. **Field equations of the GTE model and Green's formulas.** The basic linear system of pseudo-oscillation equations for the thermo-elasticity theory associated with Green-Lindsay's model for homogeneous solids in matrix form reads as (see [7,8,21–23])

$$
A(\partial_x,\tau)\,U(x,\tau) = \Phi(x,\tau)\,,
$$

where $U = (u_1,u_2,u_3,\vartheta)^\top := (u,\vartheta)^\top$ is a complex valued unknown vector function with $u = (u_1,u_2,u_3)^\top$ being the elastic displacement vector and ϑ the temperature distribution, $\Phi = (\Phi_1,\Phi_2,\Phi_3,\Phi_4)^\top$ is a given vector function,

$$
A(\partial_x,\tau) = [A_{pq}(\partial_x,\tau)]_{4\times4} := \begin{bmatrix} [c_{rjkl}\partial_j\partial_l - \varrho\tau^2\delta_{rk}]_{3\times3} & [-(1+\nu_0\tau)\lambda_{rj}\partial_j]_{3\times1}\\ [-\tau\lambda_{kl}\partial_l]_{1\times3} & \eta_{jl}\partial_j\partial_l - \tau^2 h_0 - \tau d_0 \end{bmatrix}_{4\times4}. \qquad (2.13)
$$

The corresponding constitutive relations are

$$
\begin{aligned}
\sigma_{rj}(x,\tau) &= c_{rjkl}\varepsilon_{kl}(x,\tau) - (1+\nu_0\tau)\lambda_{rj}\vartheta(x,\tau),\\
q_j(x,\tau) &= -\,T_0\,\eta_{jl}\partial_l\vartheta(x,\tau)\,.
\end{aligned}
$$

The stress operator in the theory of thermo-elasticity has the form

$$
\mathcal{T}(\partial_x,n,\tau) = [\,\mathcal{T}_{pq}(\partial_x,n,\tau)\,]_{4\times4} := \begin{bmatrix} [c_{rjkl}\,n_j\partial_l]_{3\times3} & [-(1+\nu_0\tau)\lambda_{rj}n_j]_{3\times1}\\ [0]_{1\times3} & \eta_{jl}n_j\partial_l \end{bmatrix}_{4\times4} \qquad (2.14)
$$

and the corresponding generalized thermo-stress vector is written as

$$
\mathcal{T}(\partial_x,n,\tau)\,U(x,\tau) = \big(\sigma_{1j}(x,\tau)n_j(x),\ \sigma_{2j}(x,\tau)n_j(x), \sigma_{3j}(x,\tau)n_j(x), -T_0^{-1}q_j(x,\tau)n_j(x)\big)^\top,
$$

where the first three components correspond to the mechanical stress vector in the theory of generalized thermo-elasticity and the fourth component is $(-T_0^{-1})$ times the normal component of the heat flux vector; here again, $n = (n_1,n_2,n_3)$ stands for the unit normal vector to the corresponding surface element.

For arbitrary vector functions $U = (u_1,u_2,u_3,\vartheta)^\top \in \big[C^2(\overline{\Omega})\big]^4$ and $U' = (u'_1,u'_2,u'_3,\vartheta')^\top \in \big[C^2(\overline{\Omega})\big]^4$, we have Green's first identity for the thermo-elasticity case

$$
\int\limits_\Omega \Big[A(\partial_x,\tau)\,U\cdot U' + \mathcal{E}_\tau(U,\overline{U'})\Big]dx = \int\limits_{\partial\Omega}\{\mathcal{T}(\partial_x,n,\tau)U\}^+\cdot\{U'\}^+ dS\,, \qquad (2.15)
$$

where

$$\mathcal{E}_\tau(U,\overline{U'}) := c_{rjkl}\,\partial_l u_k\,\overline{\partial_j u'_r} + \varrho\,\tau^2\,u_r\,\overline{u'_r} + \lambda_{kj}[\tau\overline{\vartheta'}\partial_j u_k - (1+\nu_0\tau)\vartheta\overline{\partial_j u'_k}]$$
$$+ \eta_{jl}\,\partial_l\vartheta\,\overline{\partial_j\vartheta'} + \tau\,(h_0\tau + d_0)\,\vartheta\,\overline{\vartheta'}\,. \tag{2.16}$$

By a standard limiting procedure, Green's first formula (2.15) can be extended to the Lipschitz domains and to the vector functions $U \in \big[W_2^1(\Omega)\big]^4$ and $U' \in \big[W_2^1(\Omega)\big]^4$ possessing the property $A(\partial_x,\tau)\,U \in \big[L_2(\Omega)\big]^4$,

$$\int\limits_{\Omega} \big[A(\partial_x,\tau)\,U\cdot U' + \mathcal{E}_\tau(U,\overline{U'})\big]dx = \big\langle \{\mathcal{T}(\partial_x,n,\tau)U\}^+\,,\,\{U'\}^+\big\rangle_S\,.$$

Green's first formula holds also for the exterior unbounded domain Ω^- in the class of functions decaying at infinity.

Definition 2.1. We say that a vector function $U = (u_1,u_2,u_3,\vartheta)^\top \in [W^1_{2,\,\mathrm{loc}}(\Omega^-)]^4$ belongs to the class $\mathbf{Z}_\tau(\Omega^-)$ if the components of U satisfy the following decay conditions at infinity:

$$\begin{aligned} &u_k(x) = \mathcal{O}(|x|^{-2}), && \partial_j u_k(x) = \mathcal{O}(|x|^{-2}),\\ &\vartheta(x) = \mathcal{O}(|x|^{-2}), && \partial_j\vartheta(x) = \mathcal{O}(|x|^{-2}), \quad k,j = 1,2,3. \end{aligned} \tag{2.17}$$

Evidently, $\mathbf{Z}_\tau(\Omega^-) \subset [W_2^1(\Omega^-)]^4$.

For arbitrary vector functions $U = (u_1,u_2,u_3,\vartheta)^\top \in [C^1(\overline{\Omega^-})]^4 \cap [C^2(\Omega^-)]^4 \cap \mathbf{Z}_\tau(\Omega^-)$ and $U' = (u'_1,u'_2,u'_3,\vartheta')^\top \in [C^1(\overline{\Omega^-})]^4 \cap \mathbf{Z}_\tau(\Omega^-)$ with $A(\partial_x,\tau)\,U \in [L_{2,\mathrm{comp}}(\Omega^-)]^4$ the following Green's first identity for the exterior domain Ω^-

$$\int\limits_{\Omega^-} \Big[A(\partial_x,\tau)\,U\cdot U' + \mathcal{E}_\tau(U,\overline{U'})\Big]dx = -\int\limits_{\partial\Omega^-} \{\mathcal{T}(\partial_x,n,\tau)U\}^-\cdot\{U'\}^- dS \tag{2.18}$$

holds true. By a standard limiting procedure, Green's formula (2.18) can be extended to the Lipschitz domains and to the vector functions $U \in \big[W^1_{2,\,\mathrm{loc}}(\Omega^-)\big]^4$ and $U' \in \big[W^1_{2,\,\mathrm{loc}}(\Omega^-)\big]^4$ satisfying the decay conditions at infinity (2.17) and possessing the property $A(\partial_x,\tau)\,U \in \big[L_{2,\,\mathrm{comp}}(\Omega^-)\big]^4$,

$$\int\limits_{\Omega^-} \big[A(\partial_x,\tau)\,U\cdot U' + \mathcal{E}_\tau(U,\overline{U'})\big]dx = -\big\langle \{\mathcal{T}(\partial_x,n,\tau)U\}^-\,,\,\{U'\}^-\big\rangle_S\,, \tag{2.19}$$

where $A(\partial_x,\tau)\,U$ is compactly supported and $\{\mathcal{T}(\partial_x,n,\tau)U\}^- \in [H_2^{-1/2}(S)]^4$ is the generalized trace of the stress vector on the boundary surface $S = \partial\Omega^-$. Note that since the operator $A(\partial_x,\tau)$ is strongly elliptic and $A(\partial_x,\tau)\,U$ has a compact support, therefore, actually, U is an analytic vector function of the real variables (x_1,x_2,x_3) in the vicinity of infinity (in the domain $\Omega^-\setminus\mathrm{supp}\,A(\partial_x,\tau)\,U$) and conditions (2.17) can be understood in the usual classical pointwise sense. Hence, the improper integral over Ω^- in formula (2.19) is convergent and well defined.

2.3. **Formulation of the basic transmission problems.** Here we formulate the basic transmission problems in the classical pointwise sense and in the weak sense, when the whole space $\mathbb{R}^3$ is divided into two simply connected regions $\mathbb{R}^3 = \overline{\Omega^{(1)}} \cup \overline{\Omega^{(2)}}$, where $\Omega^{(1)}$ is a bounded domain with a smooth boundary S and $\Omega^{(2)}$ is its unbounded complement, $\Omega^{(2)} = \mathbb{R}^3 \setminus \overline{\Omega^{(1)}}$. We assume that the region $\Omega^{(1)}$ is filled up with a material subject to the termo-electro-magneto-elasticity model, while the region $\Omega^{(2)}$ is filled up with a material subject to the thermo-elasticity model. The thermo-mechanical and electro-magnetic characteristics, material constants, differential and boundary operators associated with the domain $\Omega^{(\beta)}$ for $\beta = 1,2$, we equip with the superscript (β).

In what follows, we assume that all unknown vector functions and the given vector functions depend on the complex parameter τ, however, we will not show explicitly this dependence and drop the argument τ in the case of functions, but we will keep the argument in the differential and stress operators.

Definition 2.2. A vector $U^{(1)} = (U_1^{(1)}, \dots, U_6^{(1)})^\top$ is called regular in the domain $\Omega^{(1)}$ if

$$U^{(1)} \in [C^1(\overline{\Omega^{(1)}})]^6 \cap [C^2(\Omega^{(1)})]^6.$$

Similarly, a vector $U^{(2)} = (U_1^{(2)}, \dots, U_4^{(2)})^\top$ is called regular in the domain $\Omega^{(2)}$ if

$$U^{(2)} \in [C^1(\overline{\Omega^{(2)}})]^4 \cap [C^2(\Omega^{(2)})]^4 \cap \mathbf{Z}_\tau(\Omega^{(2)}) \,.$$

The basic transmission problem $(\mathbf{TD})_\tau$: *Find regular solutions $U^{(1)} = (U_1^{(1)}, \dots, U_6^{(1)})^\top$ and $U^{(2)} = (U_1^{(2)}, \dots, U_4^{(2)})^\top$ to the equations*

$$A^{(1)}(\partial_x, \tau)\, U^{(1)}(x) = \Phi^{(1)}(x)\,, \quad x \in \Omega^{(1)}, \tag{2.20}$$

$$A^{(2)}(\partial_x, \tau)\, U^{(2)}(x) = \Phi^{(2)}(x)\,, \quad x \in \Omega^{(2)}, \tag{2.21}$$

satisfying on the interface S the following transmission conditions:

$$\{U_j^{(1)}(x)\}^+ - \{U_j^{(2)}(x)\}^- = f_j(x), \quad j = 1,2,3, \quad x \in S, \tag{2.22}$$

$$\{U_6^{(1)}(x)\}^+ - \{U_4^{(2)}(x)\}^- = f_6(x), \quad x \in S, \tag{2.23}$$

$$\{\mathcal{T}^{(1)}(\partial_x, n, \tau) U^{(1)}(x)\}_j^+ - \{\mathcal{T}^{(2)}(\partial_x, n, \tau) U^{(2)}(x)\}_j^- = F_j(x), \;\; j = 1,2,3, \;\; x \in S, \tag{2.24}$$

$$\{\mathcal{T}^{(1)}(\partial_x, n, \tau) U^{(1)}(x)\}_6^+ - \{\mathcal{T}^{(2)}(\partial_x, n, \tau) U^{(2)}(x)\}_4^- = F_6(x), \quad x \in S, \tag{2.25}$$

and the Dirichlet type boundary conditions

$$\{U_j^{(1)}(x)\}^+ = f_j(x), \quad j = 4,5, \quad x \in S. \tag{2.26}$$

The basic transmission problem $(\mathbf{TN})_\tau$: *Find regular solutions $U^{(1)}$ and $U^{(2)}$ to equations* (2.20)–(2.21) *satisfying transmission conditions* (2.22)–(2.25) *and the Neumann type boundary conditions*

$$\{\mathcal{T}^{(1)}(\partial_x, n, \tau) U^{(1)}(x)\}_j^+ = F_j(x), \quad j = 4,5, \quad x \in S. \tag{2.27}$$

Here, the differential operators $A^{(\beta)}(\partial_x, \tau)$ and generalized stress operators $\mathcal{T}^{(\beta)}(\partial_x, n, \tau)$, $\beta = 1,2$, are defined by (2.1), (2.8) *and* (2.13), (2.14), *respectively. The data of the problems satisfy the following inclusions:*

$$\Phi^{(1)} = (\Phi_1^{(1)}, \dots, \Phi_6^{(1)})^\top \in [C(\Omega^{(1)})]^6,$$
$$\Phi^{(2)} = (\Phi_1^{(2)}, \dots, \Phi_4^{(2)})^\top \in [C(\Omega^{(2)})]^4, \quad \text{supp}\, \Phi^{(2)} \text{ is compact},$$
$$f_j \in C^1(S), \quad F_j \in C(S), \quad j = 1, 2, \dots, 6.$$

Note that the transmission conditions relate one-sided limits (traces) of similar fields: equations (2.22) relate the components of the displacement vectors $u^{(1)}$ and $u^{(2)}$, equation (2.23) relates temperature functions $U_6^{(1)} = \vartheta^{(1)}$ and $U_4^{(2)} = \vartheta^{(2)}$, equation (2.24) relates components of the mechanical stress vectors, and finally, equation (2.25) relates normal components of the heat flux vectors. Further, equation (2.26) describes the Dirichlet conditions for the electric and magnetic potentials, while equation (2.27) corresponds to the Neumann type boundary conditions for the prescribed normal components of the electric displacement and magnetic induction vectors.

Remark 2.3. Note that Green's formulas can be extended to the general Sobolev $W_p^1(\Omega)$ spaces with arbitrary $p > 1$. For example, if $U^{(1)} \in [W_p^1(\Omega^{(1)})]^6$ with $A^{(1)}(\partial_x, \tau) U^{(1)} \in \big[L_p(\Omega^{(1)})\big]^6$ and $U' \in [W_{p'}^1(\Omega^{(1)})]^6$ with $1/p + 1/p' = 1$, then formula (2.12) holds true due to the inclusion $\{U'\}^+ \in [B_{p',p'}^{1-1/p'}(\partial\Omega^{(1)})]^6 = [B_{p',p'}^{-1/p}(\partial\Omega^{(1)})]^6$ and defines the generalized trace of the stress vector $\{\mathcal{T}^{(1)}(\partial_x, n, \tau) U^{(1)}\}^+ \in [B_{p,p}^{-1/p}(\partial\Omega^{(1)})]^6$ on $\partial\Omega^{(1)}$.

Similarly, if vector functions $U^{(2)} \in [W_p^1(\Omega^{(2)})]^4$ and $U' \in [W_{p'}^1(\Omega^{(2)})]^4$ satisfy the decay conditions at infinity (2.17), and $A^{(2)}(\partial_x, \tau)\, U^{(2)} \in \big[L_{p,\,\text{comp}}(\Omega^{(2)})\big]^4$, then formula (2.19) holds true due to the inclusion $\{U'\}^- \in [B_{p',p'}^{-1/p}(\partial\Omega^{(2)})]^4$ and defines the generalized trace of the stress vector $\{\mathcal{T}^{(2)}(\partial_x, n, \tau) U^{(2)}\}^- \in [B_{p,p}^{-1/p}(\partial\Omega^{(2)})]^4$ on $\partial\Omega^{(2)}$.

In these cases, the symbol $\langle\, \cdot \,, \cdot \,\rangle_S$ denotes the duality pairing of the space $[B_{p,p}^{-1/p}(S)]^m$ with the space $[B_{p',p'}^{1/p}(S)]^m$ for $m = 6$ and $m = 4$, respectively.

These generalized Green's formulas give us possibility to formulate the transmission problems in a week sense.

Weak formulation of the basic transmission problems $(\mathbf{TD})_\tau$ and $(\mathbf{TN})_\tau$:
Find vector functions

$$U^{(1)} \in [W_p^1(\Omega^{(1)})]^6 \quad \textit{and} \quad U^{(2)} \in [W^1_{p,\,\mathrm{loc}}(\Omega^{(2)})]^4 \cap \mathbf{Z}_\tau(\Omega^{(2)}), \quad p>1,$$

satisfying the differential equations (2.20) *and* (2.21) *in the distributional sense, transmission conditions* (2.22)–(2.23) *and the Dirichlet type boundary condition* (2.26) *in the usual trace sense, transmission conditions* (2.24)–(2.25) *and the Neumann type boundary condition* (2.27) *in the generalized functional sense defined by Green's formulas* (2.12) *and* (2.19).
In the case of weak setting, we assume that

$$\Phi^{(1)} \in [L_p(\Omega^{(1)})]^6, \quad \Phi^{(2)} \in [L_{p,\,\mathrm{comp}}(\Omega^{(2)})]^4,$$
$$f_j \in B^{1-\frac{1}{p}}_{p,p}(S), \quad F_j \in B^{-\frac{1}{p}}_{p,p}(S), \quad j=1,2,\dots,6.$$

Recall that for $p=2$ *we have* $B^{\pm\frac{1}{2}}_{2,2}(S) = H_2^{\pm\frac{1}{2}}(S)$.

2.4. **Formulation of the boundary-transmission problems for layered composite structures.** Let us now consider a bounded elastic composite structure $\overline{\Omega^{(1)}} \cup \overline{\Omega^{(2)}}$ with the interface $S^{(1)}$ and the exterior boundary $S^{(2)}$, assuming that in the region $\Omega^{(1)}$, we have again the GTEME model and in the region $\Omega^{(2)}$ the GTE model. Evidently, $\partial\Omega^{(1)} = S^{(1)}$ and $\partial\Omega^{(2)} = S^{(1)} \cup S^{(2)}$. For $x \in S^{(\beta)}$, by $n(x)$ we denote again the outward unit normal vector to the surfaces $S^{(\beta)}$, $\beta = 1,2$.

The boundary-transmission problems in the case under consideration are formulated as follows.

We are looking for regular solutions $U^{(1)} = (U_1^{(1)},\dots,U_6^{(1)})^\top \in [C^1(\overline{\Omega^{(1)}})]^6 \cap [C^2(\Omega^{(1)})]^6$ and $U^{(2)} = (U_1^{(2)},\dots,U_4^{(2)})^\top \in [C^1(\overline{\Omega^{(2)}})]^4 \cap [C^2(\Omega^{(2)})]^4$ to the corresponding differential equations (2.20) and (2.21), respectively, satisfying the boundary-transmission conditions formulated in the problems $(\mathrm{TD})_\tau$ or $(\mathrm{TN})_\tau$ on the interface $S^{(1)}$ and one of the following boundary conditions on the exterior boundary $S^{(2)}$:
(D) Dirichlet boundary condition

$$\{U^{(2)}(x)\}^+ = f^*(x), \quad x \in S^{(2)};$$

(N) Neumann boundary condition

$$\{\mathcal{T}^{(2)}(\partial_x, n, \tau) U^{(2)}(x)\}^+ = F^*(x), \quad x \in S^{(2)};$$

(M) Mixed type boundary conditions

$$\{U^{(2)}(x)\}^+ = f^{(D)}(x), \quad x \in S_D^{(2)},$$
$$\{\mathcal{T}^{(2)}(\partial_x, n, \tau) U^{(2)}(x)\}^+ = F^{(N)}(x), \quad x \in S_N^{(2)},$$

where $S_D^{(2)}$ and $S_N^{(2)}$ are non-overlapping open submanifolds, $S^{(2)} = \overline{S_D^{(2)}} \cup \overline{S_N^{(2)}}$, $S_D^{(2)} \cap S_N^{(2)} = \varnothing$; $f^* = (f_1^*,\dots,f_4^*)^\top$, $F^* = (F_1^*,\dots,F_4^*)^\top$, $f^{(D)} = (f_1^{(D)},\dots,f_4^{(D)})^\top$, and $F^{(N)} = (F_1^{(N)},\dots,F_4^{(N)})^\top$ are the given vector functions from the appropriate continuous function spaces.

In the case of weak setting of the problems we look for solutions $U^{(1)} \in [W_p^1(\Omega^{(1)})]^6$ and $U^{(2)} \in [W_p^1(\Omega^{(2)})]^4$ of the differential equations (2.20) and (2.21) in the distributional sense, and satisfying the above-listed boundary and transmission conditions in the trace sense for the Dirichlet data and in the generalized functional trace sense for the Neumann data. In this case, the data of the problem satisfy the inclusions:

$$\begin{aligned}
&\Phi^{(1)} \in [L_p(\Omega^{(1)})]^6, && \Phi^{(2)} \in [L_p(\Omega^{(2)})]^4,\\
&f_j \in B^{1-\frac{1}{p}}_{p,p}(S^{(1)}), && F_j \in B^{-\frac{1}{p}}_{p,p}(S^{(1)}), \quad j=1,2,\dots,6,\\
&f_j^* \in B^{1-\frac{1}{p}}_{p,p}(S^{(2)}), && F_j^* \in B^{-\frac{1}{p}}_{p,p}(S^{(2)}), \quad j=1,2,\dots,4,\\
&f_j^{(D)} \in B^{1-\frac{1}{p}}_{p,p}(S_D^{(2)}), && F_j^{(N)} \in B^{-\frac{1}{p}}_{p,p}(S_N^{(2)}), \quad j=1,2,\dots,4.
\end{aligned}$$

In what follows,we refer the above problems as $(\mathrm{DTD})_\tau$, $(\mathrm{NTD})_\tau$, $(\mathrm{MTD})_\tau$, $(\mathrm{DTN})_\tau$, $(\mathrm{NTN})_\tau$, and $(\mathrm{MTN})_\tau$, where the first letter indicates the type of boundary conditions on the exterior boundary $S^{(2)}$, while the next two letters indicate the type of boundary-transmission conditions on the interface $S^{(1)}$.

3. Uniqueness Theorems

Due to the linearity of the above-formulated problems, we consider the corresponding homogeneous problems and prove the uniqueness theorems for weak solutions implying the uniqueness for regular solutions as well. In what follows, we assume that the time relaxation parameters $\nu_0^{(1)}$ and $\nu_0^{(2)}$ involved in the equations of the GTEME and GTE models are the same,

$$\nu_0^{(1)} = \nu_0^{(2)} =: \nu_0.$$

Theorem 3.1. *Let the interface surface S be the Lipschitz one and $\tau = \sigma + i\omega$ with $\sigma > \sigma_0 \geqslant 0$ and $\omega \in \mathbb{R}$. The basic homogeneous transmission problem $(TD)_\tau$ has only the trivial weak solution for $p = 2$, while the general weak solution to the homogeneous transmission problem $(TN)_\tau$ reads as a pair of vectors $U^{(1)} = (0, 0, 0, b_1, b_2, 0)^\top$ and $U^{(2)} = (0, 0, 0, 0)^\top$, where b_1 and b_2 are arbitrary complex constants.*

Proof. Let a pair of vector functions

$$(U^{(1)}, U^{(2)}) \in [W_2^1(\Omega^{(1)})]^6 \times \Big([W_2^1(\Omega^{(2)})]^4 \cap \mathbf{Z}_\tau(\Omega^{(2)})\Big)$$

be a weak solution to one of the homogeneous transmission problems listed in the theorem. For arbitrary vector functions $U' = (u_1', u_2', u_3', \varphi', \psi', \vartheta')^\top \in [W_2^1(\Omega^{(1)})]^6$ and $V' = (v_1', v_2', v_3', \theta')^\top \in [W_2^1(\Omega^{(2)})]^4$, from Green's formulas (2.12) and (2.19), we have

$$\int\limits_{\Omega^{(1)}} \mathcal{E}_\tau^{(1)}(U^{(1)}, \overline{U'})\, dx = \big\langle \{\mathcal{T}^{(1)}(\partial_x, n, \tau) U^{(1)}\}^+ \cdot \{U'\}^+ \big\rangle_S\,, \tag{3.1}$$

$$\int\limits_{\Omega^{(2)}} \mathcal{E}_\tau^{(2)}(U^{(2)}, \overline{V'})\, dx = -\big\langle \{\mathcal{T}^{(2)}(\partial_x, n, \tau) U^{(2)}\}^- \cdot \{V'\}^- \big\rangle_S\,, \tag{3.2}$$

where $\mathcal{E}_\tau^{(1)}(\cdot,\cdot)$ and $\mathcal{E}_\tau^{(2)}(\cdot,\cdot)$ are defined by the relations (2.11) and (2.16) respectively, with the material constants associated with the regions $\Omega^{(1)}$ and $\Omega^{(2)}$,

$$\begin{aligned}
\mathcal{E}_\tau^{(1)}(U^{(1)}, \overline{U'}) &= c_{rjkl}^{(1)}\, \partial_l u_k^{(1)}\, \overline{\partial_j u_r'} + \varrho^{(1)}\, \tau^2\, u_r^{(1)}\, \overline{u_r'} + e_{lrj}^{(1)}\, \big(\partial_l \varphi^{(1)}\, \overline{\partial_j u_r'} - \partial_j u_r^{(1)}\, \overline{\partial_l \varphi'}\,\big)\\
&+ q_{lrj}^{(1)} \big(\partial_l \psi^{(1)} \overline{\partial_j u_r'} - \partial_j u_r^{(1)} \overline{\partial_l \psi'}\big) + \varkappa_{jl}^{(1)}\, \partial_l \varphi^{(1)} \overline{\partial_j \varphi'} + a_{jl}^{(1)}\, \big(\partial_l \varphi^{(1)} \overline{\partial_j \psi'} + \partial_j \psi^{(1)} \overline{\partial_l \varphi'}\,\big)\\
&+ \mu_{jl}^{(1)}\, \partial_l \psi^{(1)}\, \overline{\partial_j \psi'} + \lambda_{kj}^{(1)}\, \big[\tau\, \partial_j u_k^{(1)}\, \overline{\vartheta'} - (1 + \nu_0 \tau)\, \vartheta^{(1)}\, \overline{\partial_j u_k'}\,\big]\\
&- p_l^{(1)} \big[\tau \partial_l \varphi^{(1)} \overline{\vartheta'} + (1 + \nu_0 \tau) \vartheta^{(1)} \overline{\partial_l \varphi'}\,\big] - m_l^{(1)} \big[\tau \partial_l \psi^{(1)} \overline{\vartheta'} + (1 + \nu_0 \tau) \vartheta^{(1)} \overline{\partial_l \psi'}\,\big]\\
&+ \eta_{jl}^{(1)}\, \partial_l \vartheta^{(1)}\, \overline{\partial_j \vartheta'} + \tau\, \big(h_0^{(1)} \tau + d_0^{(1)}\big)\, \vartheta^{(1)}\, \overline{\vartheta'}\,,
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
\mathcal{E}_\tau^{(2)}(U^{(2)}, \overline{V'}) &= c_{rjkl}^{(2)}\, \partial_l u_k^{(2)}\, \overline{\partial_j v_r'} + \varrho^{(2)}\, \tau^2\, u_r^{(2)}\, \overline{v_r'}\\
&+ \lambda_{kj}^{(2)}\, \big[\tau\, \partial_j u_k^{(2)}\, \overline{\theta'} - (1 + \nu_0 \tau)\, \vartheta^{(2)}\, \overline{\partial_j v_k'}\,\big] + \eta_{jl}^{(2)}\, \partial_l \vartheta^{(2)}\, \overline{\partial_j \theta'}\\
&+ \tau\, \big(h_0^{(2)} \tau + d_0^{(2)}\big)\, \vartheta^{(2)}\, \overline{\theta'}\,.
\end{aligned} \tag{3.4}$$

If in Green's formulas (3.1) and (3.2) we substitute successively the vectors

$$\big(u_1^{(1)}, u_2^{(1)}, u_3^{(1)}, 0, 0, 0\big)^\top,\ \ \big(0, 0, 0, \varphi^{(1)}, 0, 0\big)^\top,\ \ \big(0, 0, 0, 0, \psi^{(1)}, 0\big)^\top,$$
$$\Big(0, 0, 0, 0, 0, \frac{1 + \nu_0 \tau}{\overline{\tau}}\vartheta^{(1)}\Big)^\top$$

and

$$\big(u_1^{(2)}, u_2^{(2)}, u_3^{(2)}, 0\big)^\top,\ \ \Big(0, 0, 0, \frac{1 + \nu_0 \tau}{\overline{\tau}}\vartheta^{(2)}\Big)^\top$$

in the place of the vectors U' and V' respectively, we get the following relations:

$$\int\limits_{\Omega^{(1)}}\big[c^{(1)}_{rjkl}\partial_l u^{(1)}_k\overline{\partial_j u^{(1)}_r}+\varrho^{(1)}\tau^2 u^{(1)}_r\overline{u^{(1)}_r}+e^{(1)}_{lrj}\,\partial_l\varphi^{(1)}\,\overline{\partial_j u^{(1)}_r}+q^{(1)}_{lrj}\,\partial_l\psi^{(1)}\,\overline{\partial_j u^{(1)}_r}$$
$$-(1+\nu_0\tau)\lambda^{(1)}_{kj}\vartheta^{(1)}\,\overline{\partial_j u^{(1)}_k}\,\big]dx=\int\limits_S\{[\mathcal{T}^{(1)}(\partial,n,\tau)U^{(1)}]_r\}^+\cdot\{u^{(1)}_r\}^+dS, \tag{3.5}$$

$$\int\limits_{\Omega^{(1)}}\big[-e^{(1)}_{lrj}\,\partial_j u^{(1)}_r\,\overline{\partial_l\varphi^{(1)}}+\varkappa^{(1)}_{jl}\,\partial_l\varphi^{(1)}\,\overline{\partial_j\varphi^{(1)}}+a^{(1)}_{jl}\,\partial_j\psi^{(1)}\,\overline{\partial_l\varphi^{(1)}}$$
$$-(1+\nu_0\tau)p^{(1)}_l\,\vartheta^{(1)}\,\overline{\partial_l\varphi^{(1)}}\,\big]\,dx=\int\limits_S\{[\mathcal{T}^{(1)}(\partial,n,\tau)U^{(1)}]_4\}^+\cdot\{\varphi^{(1)}\}^+dS, \tag{3.6}$$

$$\int\limits_{\Omega^{(1)}}\big[-q^{(1)}_{lrj}\,\partial_j u^{(1)}_r\,\overline{\partial_l\psi^{(1)}}+a^{(1)}_{jl}\,\partial_l\varphi^{(1)}\,\overline{\partial_j\psi^{(1)}}+\mu^{(1)}_{jl}\,\partial_l\psi^{(1)}\,\overline{\partial_j\psi^{(1)}}$$
$$-(1+\nu_0\tau)m^{(1)}_l\,\vartheta^{(1)}\,\overline{\partial_l\psi^{(1)}}\,\big]\,dx=\int\limits_S\{[\mathcal{T}^{(1)}(\partial,n,\tau)U^{(1)}]_5\}^+\cdot\{\psi^{(1)}\}^+dS, \tag{3.7}$$

$$\int\limits_{\Omega^{(1)}}\big\{(1+\nu_0\overline{\tau})\big[\lambda^{(1)}_{kj}\overline{\vartheta^{(1)}}\partial_j u^{(1)}_k-p^{(1)}_l\overline{\vartheta^{(1)}}\partial_l\varphi^{(1)}-m^{(1)}_l\overline{\vartheta^{(1)}}\partial_l\psi^{(1)}+(h^{(1)}_0\tau+d^{(1)}_0)|\vartheta^{(1)}|^2\big]$$
$$+\frac{1+\nu_0\overline{\tau}}{\tau}\eta^{(1)}_{jl}\partial_l\vartheta^{(1)}\overline{\partial_j\vartheta^{(1)}}\big\}dx=\frac{1+\nu_0\overline{\tau}}{\tau}\int\limits_S\{[\mathcal{T}^{(1)}(\partial,n,\tau)U^{(1)}]_6\}^+\cdot\{\vartheta^{(1)}\}^+dS, \tag{3.8}$$

$$\int\limits_{\Omega^{(2)}}\big[c^{(2)}_{rjkl}\partial_l u^{(2)}_k\overline{\partial_j u^{(2)}_r}+\varrho^{(2)}\tau^2 u^{(2)}_r\overline{u^{(2)}_r}-(1+\nu_0\tau)\lambda^{(2)}_{kj}\vartheta^{(2)}\,\overline{\partial_j u^{(2)}_k}\,\big]dx$$
$$=-\int\limits_S\{[\mathcal{T}^{(2)}(\partial,n,\tau)U^{(2)}]_r\}^-\cdot\{u^{(2)}_r\}^-dS, \tag{3.9}$$

$$\int\limits_{\Omega^{(2)}}\big\{(1+\nu_0\,\overline{\tau})\big[\lambda^{(2)}_{kj}\overline{\vartheta^{(2)}}\partial_j u^{(2)}_k+(h^{(2)}_0\tau+d^{(2)}_0)|\vartheta^{(2)}|^2\big]+\frac{1+\nu_0\overline{\tau}}{\tau}\eta^{(2)}_{jl}\partial_l\vartheta^{(2)}\overline{\partial_j\vartheta^{(2)}}\big\}dx$$
$$=-\frac{1+\nu_0\,\overline{\tau}}{\tau}\int\limits_S\{[\mathcal{T}^{(2)}(\partial,n,\tau)U^{(2)}]_4\}^-\cdot\{\vartheta^{(2)}\}^-dS. \tag{3.10}$$

Now, if we add termwise equation (3.5), the complex conjugate of equations (3.6)–(3.8), equation (3.9), and the complex conjugate of equation (3.10), and take into account symmetry properties (2.2) of coefficients for both models and the homogeneous transmission and boundary conditions, we arrive at the relation

$$\int\limits_{\Omega^{(1)}}\Big\{c^{(1)}_{rjkl}\,\partial_l u^{(1)}_k\,\overline{\partial_j u^{(1)}_r}+\varrho^{(1)}\,\tau^2|u^{(1)}|^2+\varkappa^{(1)}_{jl}\,\partial_l\varphi^{(1)}\,\overline{\partial_j\varphi^{(1)}}$$
$$+a^{(1)}_{jl}\big(\partial_l\psi^{(1)}\,\overline{\partial_j\varphi^{(1)}}+\partial_j\varphi^{(1)}\,\overline{\partial_l\psi^{(1)}}\big)+\mu^{(1)}_{jl}\partial_l\psi^{(1)}\,\overline{\partial_j\psi^{(1)}}$$
$$-2\,\mathrm{Re}\big[p^{(1)}_l(1+\nu_0\tau)\vartheta^{(1)}\,\overline{\partial_l\varphi^{(1)}}\,\big]-2\,\mathrm{Re}\big[m^{(1)}_l(1+\nu_0\tau)\vartheta^{(1)}\,\overline{\partial_l\psi^{(1)}}\,\big]$$
$$+(1+\nu_0\tau)(h^{(1)}_0\overline{\tau}+d^{(1)}_0)\,|\vartheta^{(1)}|^2+\frac{1+\nu_0\tau}{\overline{\tau}}\,\eta^{(1)}_{jl}\,\partial_l\vartheta^{(1)}\,\overline{\partial_j\vartheta^{(1)}}\Big\}\,dx$$
$$+\int\limits_{\Omega^{(2)}}\Big\{c^{(2)}_{rjkl}\,\partial_l u^{(2)}_k\,\overline{\partial_j u^{(2)}_r}+\varrho^{(2)}\,\tau^2|u^{(2)}|^2+(1+\nu_0\tau)(h^{(2)}_0\overline{\tau}+d^{(2)}_0)\,|\vartheta^{(2)}|^2$$
$$+\frac{1+\nu_0\tau}{\overline{\tau}}\,\eta^{(2)}_{jl}\,\partial_l\vartheta^{(2)}\,\overline{\partial_j\vartheta^{(2)}}\Big\}\,dx=0. \tag{3.11}$$

Due to the relations (2.5) and the positive definiteness of the matrix $\Lambda^{(1)}$ defined in (2.7), the following relations

$$
\begin{gathered}
c^{(\beta)}_{rjkl}\,\partial_r u^{(\beta)}_j\,\overline{\partial_k u^{(\beta)}_l} \geq \lambda_0\,\varepsilon^{(\beta)}_{kj}\,\varepsilon^{(\beta)}_{kj}\,, \quad \eta^{(\beta)}_{jl}\,\partial_l\vartheta^{(\beta)}\,\overline{\partial_j\vartheta^{(\beta)}} \geq \lambda_0\,|\nabla\vartheta^{(\beta)}|^2, \quad \beta=1,2,\\
[\varkappa^{(1)}_{jl}\,\partial_l\varphi^{(1)}\,\overline{\partial_j\varphi^{(1)}} + a^{(1)}_{jl}\,(\partial_l\psi^{(1)}\,\overline{\partial_j\varphi^{(1)}} + \partial_j\varphi^{(1)}\,\overline{\partial_l\psi^{(1)}}) + \mu^{(1)}_{jl}\,\partial_l\psi^{(1)}\,\overline{\partial_j\psi^{(1)}}]\\
\geq \lambda_0(|\nabla\varphi^{(1)}|^2 + |\nabla\psi^{(1)}|^2),
\end{gathered}
\tag{3.12}
$$

hold true, where λ_0 is a positive constant and $\nabla=(\partial_1,\partial_2,\partial_3)$. Using the equalities

$$
\begin{gathered}
\tau^2=\sigma^2-\omega^2+2i\sigma\omega, \qquad \frac{1+\nu_0\tau}{\overline{\tau}} = \frac{\sigma+\nu_0(\sigma^2-\omega^2)}{|\tau|^2} + i\,\frac{\omega(1+2\sigma\nu_0)}{|\tau|^2},\\
(1+\nu_0\tau)(h^{(\beta)}_0\overline{\tau}+d^{(\beta)}_0) = d^{(\beta)}_0 + \nu_0 h^{(\beta)}_0|\tau|^2 + (h^{(\beta)}_0+\nu_0 d^{(\beta)}_0)\sigma + i\,\omega(\nu_0 d^{(\beta)}_0 - h^{(\beta)}_0)
\end{gathered}
$$

and separating the imaginary part of relation (3.11), we find

$$
\omega\Bigg(\sum_{\beta=1}^{2}\int\limits_{\Omega^{(\beta)}}\Big\{2\,\varrho^{(\beta)}\,\sigma\,|u^{(\beta)}|^2 + (\nu_0 d^{(\beta)}_0 - h^{(\beta)}_0)|\vartheta^{(\beta)}|^2
+ \frac{1+2\sigma\nu_0}{|\tau|^2}\,\eta^{(\beta)}_{jl}\,\partial_l\vartheta^{(\beta)}\,\overline{\partial_j\vartheta^{(\beta)}}\Big\}\,dx\Bigg) = 0.
$$

By inequalities (2.4) and since $\sigma>\sigma_0\geqslant 0$, we conclude $u^{(\beta)}_j=0$ and $\vartheta^{(\beta)}=0$ in $\Omega^{(\beta)}$, $\beta=1,2$, for $\omega\neq 0$. Then from (3.11) we have

$$
\int\limits_{\Omega^{(1)}}\big[\varkappa^{(1)}_{jl}\,\partial_l\varphi^{(1)}\,\overline{\partial_j\varphi^{(1)}} + a^{(1)}_{jl}\,(\partial_l\psi^{(1)}\,\overline{\partial_j\varphi^{(1)}} + \partial_j\varphi^{(1)}\,\overline{\partial_l\psi^{(1)}})
+\mu^{(1)}_{jl}\,\partial_l\psi^{(1)}\,\overline{\partial_j\psi^{(1)}}\,\big]\,dx = 0,
$$

whence, in view of the last inequality in (3.12), we find $\partial_l\varphi^{(1)}=0$, $\partial_l\psi^{(1)}=0$, $l=1,2,3$, in $\Omega^{(1)}$. Thus, if $\omega\neq 0$, then

$$
\begin{gathered}
u^{(1)}=0, \quad \varphi^{(1)}=b_1=\text{const}, \quad \psi^{(1)}=b_2=\text{const}, \quad \vartheta^{(1)}=0 \text{ in } \Omega^{(1)},\\
u^{(2)}=0, \quad \vartheta^{(2)}=0 \text{ in } \Omega^{(2)}.
\end{gathered}
\tag{3.13}
$$

If $\omega=0$, then $\tau=\sigma>0$ and (3.11) can be rewritten in the form

$$
\begin{aligned}
&\int\limits_{\Omega^{(1)}}\Big\{c^{(1)}_{rjkl}\partial_l u^{(1)}_k\,\overline{\partial_j u^{(1)}_r} + \varrho^{(1)}\,\sigma^2|u^{(1)}|^2 + \frac{1+\nu_0\sigma}{\sigma}\,\eta^{(1)}_{jl}\,\partial_l\vartheta^{(1)}\,\overline{\partial_j\vartheta^{(1)}}\Big\}\,dx\\
&+\int\limits_{\Omega^{(1)}}\Big\{\varkappa^{(1)}_{jl}\,\partial_l\varphi^{(1)}\,\overline{\partial_j\varphi^{(1)}} + a^{(1)}_{jl}(\partial_l\psi^{(1)}\,\overline{\partial_j\varphi^{(1)}} + \partial_j\varphi^{(1)}\,\overline{\partial_l\psi^{(1)}}) + \mu^{(1)}_{jl}\partial_l\psi^{(1)}\,\overline{\partial_j\psi^{(1)}}\\
&\quad - 2p^{(1)}_l(1+\nu_0\sigma)\,\mathrm{Re}\big[\vartheta^{(1)}\,\overline{\partial_l\varphi^{(1)}}\,\big] - 2m^{(1)}_l(1+\nu_0\sigma)\,\mathrm{Re}\big[\vartheta^{(1)}\,\overline{\partial_l\psi^{(1)}}\,\big]\\
&\quad + (1+\nu_0\sigma)(h^{(1)}_0\sigma + d^{(1)}_0)\,|\vartheta^{(1)}|^2\Big\}\,dx\\
&+\int\limits_{\Omega^{(2)}}\Big\{c^{(2)}_{rjkl}\partial_l u^{(2)}_k\,\overline{\partial_j u^{(2)}_r} + \varrho^{(2)}\,\sigma^2|u^{(2)}|^2 + \frac{1+\nu_0\sigma}{\sigma}\,\eta^{(2)}_{jl}\,\partial_l\vartheta^{(2)}\,\overline{\partial_j\vartheta^{(2)}}\Big\}\,dx\\
&+\int\limits_{\Omega^{(2)}}\Big\{(1+\nu_0\sigma)(h^{(2)}_0\sigma + d^{(2)}_0)\,|\vartheta^{(2)}|^2\Big\}\,dx = 0.
\end{aligned}
\tag{3.14}
$$

The integrands in the first and third integrals are nonnegative. Let us show that the integrand in the second integral is nonnegative, as well. Introducing the following notation

$$
\zeta_j := \partial_j\varphi^{(1)}, \quad \zeta_{j+3} := \partial_j\psi^{(1)}, \quad \zeta_7 := -\vartheta^{(1)}, \quad \zeta_8 := -\sigma\vartheta^{(1)}, \quad j=1,2,3,
$$

and

$$\Theta := (\zeta_1, \zeta_2, \dots, \zeta_8)^\top,$$

we deduce the relation

$$\begin{aligned}
&\varkappa_{jl}^{(1)}\,\partial_l\varphi^{(1)}\,\overline{\partial_j\varphi^{(1)}} + a_{jl}^{(1)}(\partial_l\psi^{(1)}\,\overline{\partial_j\varphi^{(1)}} + \partial_j\varphi^{(1)}\,\overline{\partial_l\psi^{(1)}})\\
&\quad + \mu_{jl}^{(1)}\partial_l\psi^{(1)}\,\overline{\partial_j\psi^{(1)}} - 2p_l^{(1)}(1+\nu_0\sigma)\,\mathrm{Re}\big[\vartheta^{(1)}\,\overline{\partial_l\varphi^{(1)}}\,\big]\\
&\quad - 2m_l^{(1)}(1+\nu_0\sigma)\,\mathrm{Re}\big[\vartheta^{(1)}\,\overline{\partial_l\psi^{(1)}}\,\big] + (1+\nu_0\sigma)(h_0^{(1)}\sigma + d_0^{(1)})\,|\vartheta^{(1)}|^2\\
&= [\varkappa_{jl}^{(1)}\,\partial_l\varphi^{(1)} + a_{jl}^{(1)}\,\partial_l\psi^{(1)} + p_j^{(1)}(-\vartheta^{(1)}) + \nu_0 p_j^{(1)}(-\sigma\vartheta^{(1)})]\,\overline{\partial_j\varphi^{(1)}}\\
&\quad + [a_{jl}^{(1)}\,\partial_l\varphi^{(1)} + \mu_{jl}^{(1)}\,\partial_l\psi^{(1)} + m_j^{(1)}(-\vartheta^{(1)}) + \nu_0 m_j^{(1)}(-\sigma\vartheta^{(1)})]\,\overline{\partial_j\psi^{(1)}}\\
&\quad + [p_l^{(1)}\partial_l\varphi^{(1)} + m_l^{(1)}\,\partial_l\psi^{(1)} + d_0^{(1)}(-\vartheta^{(1)}) + h_0^{(1)}(-\sigma\vartheta^{(1)})](-\overline{\vartheta^{(1)}})\\
&\quad + [\nu_0 p_l^{(1)}\partial_l\varphi^{(1)} + \nu_0 m_l^{(1)}\,\partial_l\psi^{(1)} + h_0^{(1)}(-\vartheta^{(1)}) + \nu_0 h_0^{(1)}(-\sigma\vartheta^{(1)})](-\sigma\overline{\vartheta^{(1)}})\\
&\quad + \sigma(d_0^{(1)}\nu_0 - h_0^{(1)})\,|\vartheta^{(1)}|^2\\
&= [\varkappa_{jl}^{(1)}\,\zeta_l + a_{jl}^{(1)}\,\zeta_{l+3} + p_j^{(1)}\zeta_7 + \nu_0 p_j^{(1)}\zeta_8)]\,\overline{\zeta}_j\\
&\quad + [a_{jl}^{(1)}\,\zeta_l + \mu_{jl}^{(1)}\,\zeta_{l+3} + m_j^{(1)}\zeta_7 + \nu_0 m_j^{(1)}\zeta_8]\,\overline{\zeta}_{j+3}\\
&\quad + [p_l^{(1)}\zeta_l + m_l^{(1)}\,\zeta_{l+3} + d_0^{(1)}\zeta_7 + h_0^{(1)}\zeta_8]\,\overline{\zeta}_7\\
&\quad + [\nu_0 p_l^{(1)}\zeta_l + \nu_0 m_l^{(1)}\,\zeta_{l+3} + h_0^{(1)}\zeta_7 + \nu_0 h_0^{(1)}\zeta_8]\,\overline{\zeta}_8\\
&\quad + \sigma(d_0^{(1)}\nu_0 - h_0^{(1)})\,|\vartheta^{(1)}|^2\\
&= \sum_{p,q=1}^{8} M_{pq}\zeta_q\,\overline{\zeta}_p + \sigma(d_0^{(1)}\nu_0 - h_0^{(1)})\,|\vartheta^{(1)}|^2\\
&= M\,\Theta\cdot\Theta + \sigma(d_0^{(1)}\nu_0 - h_0^{(1)})\,|\vartheta^{(1)}|^2 \ge C_0\,|\Theta|^2,
\end{aligned}$$

with some positive constant C_0, due to the positive definiteness of the matrix M defined by (2.6) and the inequality $\sigma(d_0^{(1)}\nu_0 - h_0^{(1)}) > 0$. Taking into account inequalities (2.4) and $\sigma > 0$, it can be easily checked that $(1+\nu_0\sigma)(h_0^{(2)}\sigma + d_0^{(2)})$ is positive, hence the integrand in the forth integral in (3.14) is also nonnegative. Therefore, from (3.14) we see that the relations (3.13) hold true for $\omega = 0$, as well.

Thus equalities (3.13) hold for arbitrary $\tau = \sigma + i\omega$ with $\sigma > \sigma_0 \geqslant 0$ and $\omega \in \mathbb{R}$. On the one hand, this implies that the transmission problem $(\mathrm{TD})_\tau$ possesses only the trivial weak solution, since on the interface S due to the homogeneous boundary conditions for $\varphi^{(1)}$ and $\psi^{(1)}$, we have $b_1 = b_2 = 0$. On the other hand, a general weak solution to the transmission problem $(\mathrm{TN})_\tau$ reads as $U^{(1)} = (0,0,0,b_1,b_2,0)^\top$ and $U^{(2)} = (0,0,0,0)^\top$ with arbitrary complex constants b_1 and b_2. This completes the proof. □

Theorem 3.2. *Let the interface surface $S^{(1)}$, the exterior boundary $S^{(2)}$, and the boundary curve $\ell = \partial S_D^{(2)} = \partial S_N^{(2)}$ be Lipschitz continuous. Let $\tau = \sigma + i\omega$ with $\sigma > \sigma_0 \ge 0$ and $\omega \in \mathbb{R}$. Then the homogeneous boundary-transmission problems $(DTD)_\tau$, $(NTD)_\tau$, and $(MTD)_\tau$ have only the trivial weak solutions for p = 2, while the general weak solution to the homogeneous boundary-transmission problems $(DTN)_\tau$, $(NTN)_\tau$, and $(MTN)_\tau$ is a pair of vector functions $U^{(1)} = (0,0,0,b_1,\,b_2,0)^\top$ and $U^{(2)} = (0,0,0,0)^\top$, where b_1 and b_2 are arbitrary complex constants.*

Proof. Let a pair of vectors $(U^{(1)}, U^{(2)})$ be a weak solution to one of the homogeneous transmission problems listed in the theorem. Then for arbitrary vector-functions $U' = (u_1', u_2', u_3', \varphi', \psi', \vartheta')^\top \in [W_2^1(\Omega^{(1)})]^6$ and $V' = (v_1', v_2', v_3', \theta')^\top \in [W_2^1(\Omega^{(2)})]^4$, we have the following Green's formulas:

$$\int\limits_{\Omega^{(1)}} \mathcal{E}_\tau^{(1)}(U^{(1)}, \overline{U'})\,dx = \int\limits_{S^{(1)}} \{\mathcal{T}^{(1)}(\partial_x, n, \tau)U^{(1)}\}^+ \cdot \{U'\}^+ dS\,,$$

$$\int\limits_{\Omega^{(2)}} \mathcal{E}_\tau^{(2)}(U^{(2)},\overline{V'})\,dx = -\int\limits_{S^{(1)}} \{\mathcal{T}^{(2)}(\partial_x,n,\tau)U^{(2)}\}^- \cdot \{V'\}^- dS + \int\limits_{S^{(2)}} \{\mathcal{T}^{(2)}(\partial_x,n,\tau)U^{(2)}\}^+ \cdot \{V'\}^+ dS\,,$$

where $\mathcal{E}_\tau^{(1)}(\cdot\,,\,\cdot)$ and $\mathcal{E}_\tau^{(2)}(\cdot\,,\,\cdot)$ are defined in (3.3) and (3.4).

The arguments used in the proof of Theorem 3.1 extend mutatis mutandis to the present case and we arrive at the relation (3.11), leading to the equalities

$$u^{(1)}=0,\quad \varphi^{(1)}=b_1=\mathrm{const},\quad \psi^{(1)}=b_2=\mathrm{const},\quad \vartheta^{(1)}=0 \text{ in } \Omega^{(1)},$$
$$u^{(2)}=0,\quad \vartheta^{(2)}=0 \text{ in } \Omega^{(2)},$$

where b_1 and b_2 are arbitrary complex constants. Therefore the proof of the theorem follows from the homogeneous boundary and transmission conditions. □

4. Existence and Regularity Results

4.1. **Existence results for the basic transmission problem** $(TD)_\tau$**.** Let us consider the basic transmission problem $(\mathrm{TD})_\tau$ in the weak setting sense for the homogeneous differential equations

$$A^{(1)}(\partial_x,\tau)\,U^{(1)}(x,\tau)=0\,,\quad x\in\Omega^{(1)},\tag{4.1}$$

$$A^{(2)}(\partial_x,\tau)\,U^{(2)}(x,\tau)=0\,,\quad x\in\Omega^{(2)},\tag{4.2}$$

where $A^{(1)}$ and $A^{(2)}$ are defined by (2.1) and (2.13) respectively, and the sought for vectors

$$U^{(1)}=(U_1^{(1)},\dots,U_6^{(1)})^\top\in[W_p^1(\Omega^{(1)})]^6,\quad U^{(2)}=(U_1^{(2)},\dots,U_4^{(2)})^\top\in[W_p^1(\Omega^{(2)})]^4\cap\mathbf{Z}_\tau(\Omega^{(2)}),$$

satisfy on S (see (2.22)–(2.26)) the following transmission and Dirichlet type boundary conditions:

$$\{U_j^{(1)}(x)\}^+-\{U_j^{(2)}(x)\}^-=f_j(x),\quad j=1,2,3,\quad x\in S,\tag{4.3}$$

$$\{U_j^{(1)}(x)\}^+=f_j(x),\quad j=4,5,\quad x\in S,\tag{4.4}$$

$$\{U_6^{(1)}(x)\}^+-\{U_4^{(2)}(x)\}^-=f_6(x),\quad x\in S,\tag{4.5}$$

$$\{\mathcal{T}^{(1)}(\partial_x,n,\tau)U^{(1)}(x)\}_j^+-\{\mathcal{T}^{(2)}(\partial_x,n,\tau)U^{(2)}(x)\}_j^-=F_j(x), j=1,2,3,\ \ x\in S,\tag{4.6}$$

$$\{\mathcal{T}^{(1)}(\partial_x,n,\tau)U^{(1)}(x)\}_6^+-\{\mathcal{T}^{(2)}(\partial_x,n,\tau)U^{(2)}(x)\}_4^-=F_6(x),\quad x\in S,\tag{4.7}$$

with $p>1$ and the data satisfying the inclusions

$$\begin{aligned} &f_j\in B_{p,p}^{1-\frac1p}(S),\quad j=1,2,3,4,5,6,\quad F_1,F_2,F_3,F_6\in B_{p,p}^{-\frac1p}(S),\\ &S=\partial\Omega^{(1)}=\partial\Omega^{(2)}\ \text{ is a sufficiently smooth surface, say }\ S\in C^\infty.\end{aligned}\tag{4.8}$$

We will investigate the problem by the potential method. For the readers convenience, properties of the layer potentials needed in our analysis are briefly presented in Appendix.

We look for the vectors $U^{(1)}$ and $U^{(2)}$ in the form of single layer potentials associated with the operators $A^{(1)}(\partial_x,\tau)$ and $A^{(2)}(\partial_x,\tau)$ (see Appendix, formulas (5.2) and Corollary 5.4)

$$U^{(1)}(x)=V_S^{(1)}\varphi(x),\quad x\in\Omega^{(1)},\quad \varphi=(\varphi_1,\dots,\varphi_6)^\top\in\big[B_{p,p}^{-\frac1p}(S)\big]^6,$$
$$U^{(2)}(x)=V_S^{(2)}\psi(x),\quad x\in\Omega^{(2)},\quad \psi=(\psi_1,\dots,\psi_4)^\top\in\big[B_{p,p}^{-\frac1p}(S)\big]^4.$$

The transmission and boundary conditions (4.3)–(4.7) and properties of single layer potentials, presented in Appendix (see Theorems 5.1–5.3) lead then for φ and ψ on S to the following system of integral equations:

$$\sum_{l=1}^{6}[\mathcal{H}_S^{(1)}]_{jl}\,\varphi_l-\sum_{p=1}^{4}[\mathcal{H}_S^{(2)}]_{jp}\,\psi_p=f_j \text{ on } S,\ \ j=1,2,3,\tag{4.9}$$

$$\sum_{l=1}^{6}[\mathcal{H}_S^{(1)}]_{4l}\,\varphi_l = f_4 \text{ on } S, \tag{4.10}$$

$$\sum_{l=1}^{6}[\mathcal{H}_S^{(1)}]_{5l}\,\varphi_l = f_5 \text{ on } S, \tag{4.11}$$

$$\sum_{l=1}^{6}[\mathcal{H}_S^{(1)}]_{6l}\,\varphi_l - \sum_{p=1}^{4}[\mathcal{H}_S^{(2)}]_{4p}\,\psi_p = f_6 \text{ on } S, \tag{4.12}$$

$$\sum_{l=1}^{6}\Big[-\frac{1}{2}\,I_6 + \mathcal{K}_S^{(1)}\Big]_{jl}\,\varphi_l - \sum_{p=1}^{4}\Big[\frac{1}{2}\,I_4 + \mathcal{K}_S^{(2)}\Big]_{jp}\,\psi_p = F_j \text{ on } S, \ \ j=1,2,3, \tag{4.13}$$

$$\sum_{l=1}^{6}\Big[-\frac{1}{2}\,I_6 + \mathcal{K}_S^{(1)}\Big]_{6l}\,\varphi_l - \sum_{p=1}^{4}\Big[\frac{1}{2}\,I_4 + \mathcal{K}_S^{(2)}\Big]_{4p}\,\psi_p = F_6 \text{ on } S, \tag{4.14}$$

where the integral operators $\mathcal{H}_S^{(l)}$ and $\mathcal{K}_S^{(l)}$ are associated with the single layer potentials and are defined by (5.7) and (5.8), respectively.

To prove the unique solvability of the above system, we proceed as follows. Due to the invertibility of the operators (see Theorem 5.3)

$$\mathcal{H}_S^{(1)} : [B_{p,p}^{-\frac{1}{p}}(S)]^6 \to [B_{p,p}^{1-\frac{1}{p}}(S)]^6, \qquad \mathcal{H}_S^{(2)} : [B_{p,p}^{-\frac{1}{p}}(S)]^4 \to [B_{p,p}^{1-\frac{1}{p}}(S)]^4, \tag{4.15}$$

we can introduce new unknown vector functions

$$h = (h_1,\dots,h_6)^\top \in [B_{p,p}^{1-\frac{1}{p}}(S)]^6, \qquad g = (g_1,\dots,g_4)^\top \in [B_{p,p}^{1-\frac{1}{p}}(S)]^4,$$

by the relations $h := \big[\mathcal{H}_S^{(1)}\big]\varphi$ and $g := \big[\mathcal{H}_S^{(2)}\big]\psi$ implying

$$\varphi = \big[\mathcal{H}_S^{(1)}\big]^{-1}h, \quad \psi = \big[\mathcal{H}_S^{(2)}\big]^{-1}g.$$

Evidently, then we have

$$U^{(1)}(x) = V_S^{(1)}\big([\mathcal{H}_S^{(1)}]^{-1}\,h\big)(x) \ \text{ in } \ \Omega^{(1)}, \tag{4.16}$$

$$U^{(2)}(x) = V_S^{(2)}\big([\mathcal{H}_S^{(2)}]^{-1}\,g\big)(x) \ \text{ in } \ \Omega^{(2)}, \tag{4.17}$$

where $[\mathcal{H}_S^{(j)}]^{-1}$ is the inverse to the operator $\mathcal{H}_S^{(j)}$, $j=1,2$, in (4.15) (see Theorem 5.3). Note that these unknown densities are, actually, the traces on S of the sought for vectors

$$h = \big\{U^{(1)}\big\}^+, \quad g = \big\{U^{(2)}\big\}^-. \tag{4.18}$$

By Theorem 5.3, we have

$$\big\{\mathcal{T}^{(1)}V_S^{(1)}\big([\mathcal{H}_S^{(1)}]^{-1}\,h\big)\big\}^+ = \Big(-\frac{1}{2}I_6 + \mathcal{K}_S^{(1)}\Big)[\mathcal{H}_S^{(1)}]^{-1}\,h \equiv \mathcal{A}_S^{(1)+}h,$$

$$\big\{\mathcal{T}^{(2)}V_S^{(2)}\big([\mathcal{H}_S^{(2)}]^{-1}\,g\big)\big\}^- = \Big(\frac{1}{2}I_4 + \mathcal{K}_S^{(2)}\Big)[\mathcal{H}_S^{(2)}]^{-1}\,g \equiv \mathcal{A}_S^{(2)-}g,$$

with Steklov-Poincaré type operators (see Appendix, formulas (5.12)–(5.13))

$$\mathcal{A}_S^{(1)+} := \Big(-\frac{1}{2}I_6 + \mathcal{K}_S^{(1)}\Big)[\mathcal{H}_S^{(1)}]^{-1}, \qquad \mathcal{A}_S^{(2)-} := \Big(\frac{1}{2}I_4 + \mathcal{K}_S^{(2)}\Big)[\mathcal{H}_S^{(2)}]^{-1}.$$

System (4.9)–(4.14) can be rewritten then for new unknown vectors h and g as follows

$$h_j - g_j = f_j \quad j=1,2,3, \tag{4.19}$$

$$h_4 = f_4, \tag{4.20}$$

$$h_5 = f_5, \tag{4.21}$$

$$h_6 - g_4 = f_6, \tag{4.22}$$

$$\big[\mathcal{A}_S^{(1)+}h\big]_j - \big[\mathcal{A}_S^{(2)-}g\big]_j = F_j, \quad j=1,2,3, \tag{4.23}$$

$$\big[\mathcal{A}_S^{(1)+}h\big]_6 - \big[\mathcal{A}_S^{(2)-}g\big]_4 = F_6. \tag{4.24}$$

As we see, the unknowns h_4 and h_5 are uniquely defined by equations (4.20) and (4.21) and the above system can be rewritten as

$$h_j - g_j = f_j, \quad j = 1,2,3, \tag{4.25}$$

$$h_6 - g_4 = f_6, \tag{4.26}$$

$$\sum_{l=1,2,3,6} [\mathcal{A}_S^{(1)+}]_{jl}\, h_l - \sum_{l=1}^{4} [\mathcal{A}_S^{(2)-}]_{jl}\, g_l = F_j - [\mathcal{A}_S^{(1)+}]_{j4} f_4 - [\mathcal{A}_S^{(1)+}]_{j5} f_5, \quad j = 1,2,3, \tag{4.27}$$

$$\sum_{l=1,2,3,6} [\mathcal{A}_S^{(1)+}]_{6l}\, h_l - \sum_{l=1}^{4} [\mathcal{A}_S^{(2)-}]_{4l}\, g_l = F_6 - [\mathcal{A}_S^{(1)+}]_{64} f_4 - [\mathcal{A}_S^{(1)+}]_{65}\, f_5, \tag{4.28}$$

$$h_4 = f_4, \tag{4.29}$$

$$h_5 = f_5. \tag{4.30}$$

Further, let

$$\widetilde{h} := (h_1, h_2, h_3, h_6)^\top,$$

$$\widetilde{f} = (f_1, f_2, f_3, f_6)^\top \in [B_{p,p}^{1-\frac{1}{p}}(S)]^4, \quad \widetilde{F} = (\widetilde{F}_1, \widetilde{F}_2, \widetilde{F}_3, \widetilde{F}_6)^\top \in [B_{p,p}^{-\frac{1}{p}}(S)]^4, \tag{4.31}$$

$$\widetilde{F}_j = F_j - [\mathcal{A}_S^{(1)+}]_{j4} f_4 - [\mathcal{A}_S^{(1)+}]_{j5} f_5, \quad j = 1,2,3,6. \tag{4.32}$$

From equations (4.25)–(4.30), we then get

$$\widetilde{h} - g = \widetilde{f}, \tag{4.33}$$

$$\widetilde{\mathcal{A}}_S^{(1)+}\, \widetilde{h} - \mathcal{A}_S^{(2)-}\, g = \widetilde{F}, \tag{4.34}$$

$$h_4 = f_4, \tag{4.35}$$

$$h_5 = f_5, \tag{4.36}$$

where

$$\widetilde{\mathcal{A}}_S^{(1)+} := \begin{bmatrix} [\mathcal{A}_S^{(1)+}]_{11} & [\mathcal{A}_S^{(1)+}]_{12} & [\mathcal{A}_S^{(1)+}]_{13} & [\mathcal{A}_S^{(1)+}]_{16} \\ [\mathcal{A}_S^{(1)+}]_{21} & [\mathcal{A}_S^{(1)+}]_{22} & [\mathcal{A}_S^{(1)+}]_{23} & [\mathcal{A}_S^{(1)+}]_{26} \\ [\mathcal{A}_S^{(1)+}]_{31} & [\mathcal{A}_S^{(1)+}]_{32} & [\mathcal{A}_S^{(1)+}]_{33} & [\mathcal{A}^{(1)+}]_{36} \\ [\mathcal{A}_S^{(1)+}]_{61} & [\mathcal{A}_S^{(1)+}]_{62} & [\mathcal{A}_S^{(1)+}]_{63} & [\mathcal{A}_S^{(1)+}]_{66} \end{bmatrix}_{4\times 4}.$$

Finally, system (4.33)–(4.36) can be equivalently rewritten in more convenient form

$$\widetilde{h} = g + \widetilde{f}, \tag{4.37}$$

$$h_4 = f_4, \tag{4.38}$$

$$h_5 = f_5, \tag{4.39}$$

$$\big[\widetilde{\mathcal{A}}_S^{(1)+} - \mathcal{A}_S^{(2)-}\big]\, g = \widetilde{F}^*, \tag{4.40}$$

where $\widetilde{F}^* = \widetilde{F} - \widetilde{\mathcal{A}}_S^{(1)+}\widetilde{f} \in [B_{p,p}^{-\frac{1}{p}}(S)]^4$.

Thus, the solvability of system (4.19)–(4.24) is equivalently reduced to the solvability of the matrix pseudodifferential equation (4.40).

Next, we prove the following lemma.

Lemma 4.1. *The operator*

$$\widetilde{\mathcal{A}}_S^{(1)+} - \mathcal{A}_S^{(2)-} \;:\; [B_{p,p}^{1-\frac{1}{p}}]^4 \to [B_{p,p}^{-\frac{1}{p}}]^4, \quad p > 1, \tag{4.41}$$

is invertible.

Proof. Note that the integral operators $\mathcal{H}_S^{(j)}$ and $\pm\frac{1}{2}I^{(j)} + \mathcal{K}_S^{(j)}$ are elliptic pseudodifferential operators of order -1 and 0, respectively. This implies that $\mathcal{A}_S^{(j)\pm}$, $j = 1,2$, are also elliptic pseudodifferential operators of order $+1$ (see [7]). More detailed analysis shows that the principal homogeneous symbol matrix of the operators $\mathcal{A}_S^{(1)+}$ and $-\mathcal{A}_S^{(2)-}$ are strongly elliptic (see [7,21]). Therefore, the operator $\widetilde{\mathcal{A}}_S^{(1)+} - \mathcal{A}_S^{(2)-}$ is a strongly elliptic pseudodifferential operator of order $+1$. Therefore, due to the general theory of pseudodifferential equations on manifolds without boundary, if we show invertibility of operator (4.41) for $p = 2$, then it will imply that of operator (4.41) for all $p > 1$.

Keeping in mind that $B_{2,2}^{\pm\frac{1}{2}}(S) = H_2^{\pm\frac{1}{2}}(S)$, we have to show the invertibility of the operator

$$\widetilde{\mathcal{A}}_S^{(1)+} - \mathcal{A}_S^{(2)-} : [H_2^{\frac{1}{2}}(S)]^4 \to [H_2^{-\frac{1}{2}}(S)]^4. \tag{4.42}$$

Using Theorem 5.5 we easily deduce the coercivity inequalities for arbitrary vector function $\widetilde{g} = (\widetilde{g}_1, \widetilde{g}_2, \widetilde{g}_3, \widetilde{g}_4)^\top \in \big[H_2^{\frac{1}{2}}(S)\big]^4$,

$$\operatorname{Re} \big\langle \big(\widetilde{\mathcal{A}}_S^{(1)+} + \widetilde{\mathcal{C}}^{(1)}\big)\widetilde{g}, \widetilde{g}\big\rangle_S \geq C_1 \|\widetilde{g}\|^2_{[H_2^{\frac{1}{2}}(S)]^4},$$
$$\operatorname{Re} \big\langle \big(-\mathcal{A}_S^{(2)-} + \mathcal{C}^{(2)}\big)\widetilde{g}, \widetilde{g}\big\rangle_S \geq C_2 \|\widetilde{g}\|^2_{[H_2^{\frac{1}{2}}(S)]^4},$$

where $\widetilde{\mathcal{C}}^{(1)} : \big[H_2^{\frac{1}{2}}(S)\big]^4 \to \big[H_2^{-\frac{1}{2}}(S)\big]^4$ and $\mathcal{C}^{(2)} : \big[H_2^{\frac{1}{2}}(S)\big]^4 \to \big[H_2^{-\frac{1}{2}}(S)\big]^4$ are compact operators. Consequently, the following coercivity inequality

$$\operatorname{Re} \big\langle \big(\widetilde{\mathcal{A}}_S^{(1)+} - \mathcal{A}_S^{(2)-} + \widetilde{\mathcal{C}}^{(1)} + \mathcal{C}^{(2)}\big)\widetilde{g}, \widetilde{g}\big\rangle_S \geq C \|\widetilde{g}\|^2_{[H_2^{\frac{1}{2}}(S)]^4}, \quad C = \text{const} > 0, \tag{4.43}$$

holds implying that operator (4.42) is Fredholm one with zero index (see, e.g., [38, Ch. 2]).

Further, we show that the null space of operator (4.42) is trivial.

Let $\widetilde{\varphi} = \big(\widetilde{\varphi}_1, \widetilde{\varphi}_2, \widetilde{\varphi}_3, \widetilde{\varphi}_4\big)^\top \in \big[H_2^{\frac{1}{2}}(S)\big]^4$ be a solution to the homogeneous equation

$$\big[\widetilde{\mathcal{A}}_S^{(1)+} - \mathcal{A}_S^{(2)-}\big]\widetilde{\varphi} = 0 \quad \text{on } S \tag{4.44}$$

and construct the vectors

$$U^{(1)}(x) = V_S^{(1)}\big([\mathcal{H}_S^{(1)}]^{-1}\varphi\big)(x) \ \text{ in } \ \Omega^{(1)}, \tag{4.45}$$
$$U^{(2)}(x) = V_S^{(2)}\big(\big[\mathcal{H}_S^{(2)}\big]^{-1}\widetilde{\varphi}\big)(x) \ \text{ in } \ \Omega^{(2)}, \tag{4.46}$$

where $\varphi = \big(\widetilde{\varphi}_1, \widetilde{\varphi}_2, \widetilde{\varphi}_3, 0, 0, \widetilde{\varphi}_4\big)^\top \in \big[H_2^{\frac{1}{2}}(S)\big]^6$.

By Theorems 5.1–5.3, evidently, $[\mathcal{H}_S^{(1)}]^{-1}\varphi \in \big[H_2^{-\frac{1}{2}}(S)\big]^6$, $[\mathcal{H}_S^{(2)}]^{-1}\widetilde{\varphi} \in \big[H_2^{-\frac{1}{2}}(S)\big]^4$ and using the mapping properties and the jump relations of the single layer potentials, we deduce:

$$U^{(1)} \in \big[W_2^1(\Omega^{(1)})\big]^6, \quad U^{(2)} \in \big[W_{2,\mathrm{loc}}^1(\Omega^{(2)})\big]^4 \cap \mathbf{Z}_\tau(\Omega^{(2)}), \tag{4.47}$$
$$\big\{U^{(1)}\big\}^+ = \big\{V_S^{(1)}\big[\mathcal{H}_S^{(1)}\big]^{-1}\varphi\big\}^+ = \mathcal{H}_S^{(1)}\big(\big[\mathcal{H}_S^{(1)}\big]^{-1}\varphi\big) = \varphi, \tag{4.48}$$
$$\big\{U^{(2)}\big\}^- = \big\{V_S^{(2)}\big[\mathcal{H}_S^{(2)}\big]^{-1}\widetilde{\varphi}\big\}^- = \mathcal{H}_S^{(2)}\big(\big[\mathcal{H}_S^{(2)}\big]^{-1}\widetilde{\varphi}\big) = \widetilde{\varphi}, \tag{4.49}$$
$$\big\{\mathcal{T}^{(1)}U^{(1)}\big\}^+ = \big\{\mathcal{T}^{(1)}V_S^{(1)}\big(\big[\mathcal{H}_S^{(1)}\big]^{-1}\varphi\big)\big\}^+ = \big[-\tfrac{1}{2}I_6 + \mathcal{K}_S^{(1)}\big]\big[\mathcal{H}_S^{(1)}\big]^{-1}\varphi = \mathcal{A}_S^{(1)+}\varphi, \tag{4.50}$$
$$\big\{\mathcal{T}^{(2)}U^{(2)}\big\}^- = \big\{\mathcal{T}^{(2)}V_S^{(2)}\big(\big[\mathcal{H}_S^{(2)}\big]^{-1}\widetilde{\varphi}\big)\big\}^- = \big[\tfrac{1}{2}I_4 + \mathcal{K}_S^{(2)}\big]\big[\mathcal{H}_S^{(2)}\big]^{-1}\widetilde{\varphi} = \mathcal{A}_S^{(2)-}\widetilde{\varphi}. \tag{4.51}$$

Taking into account that $\big[\mathcal{A}_S^{(1)+}\varphi\big]_j = \big[\widetilde{\mathcal{A}}_S^{(1)+}\widetilde{\varphi}\big]_j$ for $j = 1,2,3$, and $\big[\mathcal{A}_S^{(1)+}\varphi\big]_6 = \big[\widetilde{\mathcal{A}}_S^{(1)+}\widetilde{\varphi}\big]_4$, from relations (4.44)–(4.51), we conclude that the vectors defined by (4.45)–(4.46) solve the homogeneous differential equations

$$A^{(\beta)}(\partial_x, \tau)\,U^{(\beta)} = 0 \ \text{ in } \ \Omega^{(\beta)}, \quad \beta = 1,2,$$

and satisfy on S the homogeneous boundary-transmission conditions

$$\{U_j^{(1)}\}^+ - \{U_j^{(2)}\}^- = 0, \quad j = 1,2,3,$$
$$\{U_j^{(1)}\}^+ = 0, \quad j = 4,5,$$
$$\{U_6^{(1)}\}^+ - \{U_4^{(2)}\}^- = 0,$$
$$\{\mathcal{T}^{(1)}U^{(1)}\}_j^+ - \{\mathcal{T}^{(2)}U^{(2)}\}_j^- = 0, \quad j = 1,2,3,$$

$$\{\mathcal{T}^{(1)}U^{(1)}\}_6^+ - \{\mathcal{T}^{(2)}U^{(2)}\}_4^- = 0,$$

i.e., the pair $(U^{(1)}, U^{(2)})$ solves the homogeneous basic boundary-transmission problem $(\mathrm{TD})_\tau$. Therefore $U^{(1)} = 0$ in $\Omega^{(1)}$ and $U^{(2)} = 0$ in $\Omega^{(2)}$ by the uniqueness Theorem 3.1. Consequently, $\widetilde{\varphi}_j = 0$, $j = 1, 2, 3, 4$, in view of (4.49) implying that the null space of the operator (4.42) is trivial which completes the proof of the invertibility of the operators (4.42) and (4.41). □

Note that the operator $\mathfrak{M}$ generated by the left hand side expressions of system (4.19)–(4.24) reads as

$$\mathfrak{M} = \|\mathfrak{M}_{kj}\|_{10\times 10} =$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ [\mathcal{A}_S^{(1)+}]_{11} & \cdot & \cdot & \cdot & \cdot & [\mathcal{A}_S^{(1)+}]_{16} & -[\mathcal{A}_S^{(2)-}]_{11} & \cdot & \cdot & -[\mathcal{A}_S^{(2)-}]_{14} \\ [\mathcal{A}_S^{(1)+}]_{21} & \cdot & \cdot & \cdot & \cdot & [\mathcal{A}_S^{(1)+}]_{26} & -[\mathcal{A}_S^{(2)-}]_{21} & \cdot & \cdot & -[\mathcal{A}_S^{(2)-}]_{24} \\ [\mathcal{A}_S^{(1)+}]_{31} & \cdot & \cdot & \cdot & \cdot & [\mathcal{A}_S^{(1)+}]_{36} & -[\mathcal{A}_S^{(2)-}]_{31} & \cdot & \cdot & -[\mathcal{A}_S^{(2)-}]_{34} \\ [\mathcal{A}_S^{(1)+}]_{61} & \cdot & \cdot & \cdot & \cdot & [\mathcal{A}_S^{(1)+}]_{66} & -[\mathcal{A}_S^{(2)-}]_{41} & \cdot & \cdot & -[\mathcal{A}_S^{(2)-}]_{44} \end{bmatrix}. \tag{4.52}$$

Therefore this system can be rewritten in matrix form as follows

$$\mathfrak{M}\,\Phi = \Psi,$$

where $\mathfrak{M}$ is given by (4.52), $\Phi := (h, g)^\top$ is an unknown vector function, and Ψ is a known vector function, $\Psi := (f_1, \dots, f_6, F_1, F_2, F_3, F_4)^\top$.

The above results imply the following assertions.

Lemma 4.2. *Systems of integral equations* (4.9)–(4.14) *and* (4.19)–(4.24) *are uniquely solvable in the spaces* $\big[B_{p,p}^{-\frac{1}{p}}(S)\big]^6 \times \big[B_{p,p}^{-\frac{1}{p}}(S)\big]^4$ *and* $\big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^6 \times \big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^4$*, respectively, for arbitrary right-hand side functions satisfying conditions* (4.8).

Proof. Follows from Lemma 4.1 and equivalence of systems (4.9)–(4.14), (4.19)–(4.24), and (4.37)–(4.40). □

Lemma 4.3. *The operator*

$$\mathfrak{M} : \big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^{10} \to \big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^6 \times \big[B_{p,p}^{-\frac{1}{p}}(S)\big]^4$$

is invertible.

Proof. Follows from Lemmas 4.1 and 4.2 and the structure of the operator (4.52). □

From (4.37)–(4.40), for the solution vectors $h \in \big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^6$ and $g \in \big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^4$, we derive the following relations:

$$\widetilde{h} = (h_1, h_2, h_3, h_6)^\top = \big[\widetilde{\mathcal{A}}_S^{(1)+} - \mathcal{A}_S^{(2)-}\big]^{-1}\widetilde{F} - \big[\widetilde{\mathcal{A}}_S^{(1)+} - \mathcal{A}_S^{(2)-}\big]^{-1}\mathcal{A}_S^{(2)-}\widetilde{f}, \tag{4.53}$$

$$h_4 = f_4, \tag{4.54}$$

$$h_5 = f_5, \tag{4.55}$$

$$g = (g_1, g_2, g_3, g_4)^\top = \big[\widetilde{\mathcal{A}}_S^{(1)+} - \mathcal{A}_S^{(2)-}\big]^{-1}\widetilde{F} - \big[\widetilde{\mathcal{A}}_S^{(1)+} - \mathcal{A}_S^{(2)-}\big]^{-1}\widetilde{\mathcal{A}}_S^{(1)+}\widetilde{f}. \tag{4.56}$$

Introduce the notation

$$\widetilde{\mathcal{Q}} = \big[\widetilde{\mathcal{Q}}_{kj}\big]_{4\times 4} := \big[\widetilde{\mathcal{A}}_S^{(1)+} - \mathcal{A}_S^{(2)-}\big]^{-1},$$

$$\widetilde{\mathcal{R}} = \big[\widetilde{\mathcal{R}}_{kj}\big]_{4\times 4} := \big[\widetilde{\mathcal{A}}_S^{(1)+} - \mathcal{A}_S^{(2)-}\big]^{-1}\mathcal{A}_S^{(2)-},$$

$$\widetilde{\mathcal{M}} = \big[\widetilde{\mathcal{M}}_{kj}\big]_{4\times 4} := \big[\widetilde{\mathcal{A}}_S^{(1)+} - \mathcal{A}_S^{(2)-}\big]^{-1}\widetilde{\mathcal{A}}_S^{(1)+},$$

and construct the following matrix operators

$$\mathcal{Q} := \begin{bmatrix} \widetilde{\mathcal{Q}}_{11} & \widetilde{\mathcal{Q}}_{12} & \widetilde{\mathcal{Q}}_{13} & 0 & 0 & \widetilde{\mathcal{Q}}_{14} \\ \widetilde{\mathcal{Q}}_{21} & \widetilde{\mathcal{Q}}_{22} & \widetilde{\mathcal{Q}}_{23} & 0 & 0 & \widetilde{\mathcal{Q}}_{24} \\ \widetilde{\mathcal{Q}}_{31} & \widetilde{\mathcal{Q}}_{32} & \widetilde{\mathcal{Q}}_{33} & 0 & 0 & \widetilde{\mathcal{Q}}_{34} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \widetilde{\mathcal{Q}}_{41} & \widetilde{\mathcal{Q}}_{42} & \widetilde{\mathcal{Q}}_{43} & 0 & 0 & \widetilde{\mathcal{Q}}_{44} \end{bmatrix}_{6\times 6},$$

$$\mathcal{R} := \begin{bmatrix} \widetilde{\mathcal{R}}_{11} & \widetilde{\mathcal{R}}_{12} & \widetilde{\mathcal{R}}_{13} & 0 & 0 & \widetilde{\mathcal{R}}_{14} \\ \widetilde{\mathcal{R}}_{21} & \widetilde{\mathcal{R}}_{22} & \widetilde{\mathcal{R}}_{23} & 0 & 0 & \widetilde{\mathcal{R}}_{24} \\ \widetilde{\mathcal{R}}_{31} & \widetilde{\mathcal{R}}_{32} & \widetilde{\mathcal{R}}_{33} & 0 & 0 & \widetilde{\mathcal{R}}_{34} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \widetilde{\mathcal{R}}_{41} & \widetilde{\mathcal{R}}_{42} & \widetilde{\mathcal{R}}_{43} & 0 & 0 & \widetilde{\mathcal{R}}_{44} \end{bmatrix}_{6\times 6}.$$

The relations (4.53)–(4.56) can be rewritten then in the form

$$h = \mathcal{Q}F - \mathcal{R}f \in \big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^6, \quad g = \widetilde{\mathcal{Q}}\widetilde{F} - \widetilde{\mathcal{M}}\widetilde{f} \in \big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^4,$$

where $F = \big(\widetilde{F}_1, \widetilde{F}_2, \widetilde{F}_3, 0, 0, \widetilde{F}_6\big)^\top$, $f = (f_1, f_2, f_3, f_4, f_5, f_6)^\top$, the vectors $\widetilde{F}$ and $\widetilde{f}$ are defined by (4.31)–(4.32). Consequently, from (4.16) and (4.17), we get the following representation of the solution vectors,

$$U^{(1)} = V_S^{(1)}\big(\big[\mathcal{H}_S^{(1)}\big]^{-1} h\big) = V_S^{(1)}\big(\big[\mathcal{H}_S^{(1)}\big]^{-1}\big(\mathcal{Q}F - \mathcal{R}f\big)\big) \ \text{ in } \ \Omega^{(1)}, \tag{4.57}$$

$$U^{(2)} = V_S^{(2)}\big(\big[\mathcal{H}_S^{(2)}\big]^{-1} g\big) = V_S^{(2)}\big(\big[\mathcal{H}_S^{(2)}\big]^{-1}\big(\widetilde{\mathcal{Q}}\widetilde{F} - \widetilde{\mathcal{M}}\widetilde{f}\big)\big) \ \text{ in } \ \Omega^{(2)}. \tag{4.58}$$

Finally, let us formulate the following existence result.

Theorem 4.4. *Let conditions* (4.8) *be fulfilled. Then the basic transmission problem* $(TD)_\tau$*,* (4.1)–(4.7)*, is uniquely solvable in the space* $\big[W_p^1(\Omega^{(1)})\big]^6 \times \Big(\big[W_{p,\mathrm{loc}}^1(\Omega^{(2)})\big]^4 \cap \mathbf{Z}_\tau(\Omega^{(2)})\Big)$ *for* $p > 1$ *and the solution pair of vectors* $(U^{(1)}, U^{(2)})$ *is representable in the form of single layer potentials* (4.57)–(4.58).

Proof. The existence of a solution in the space $\big[W_p^1(\Omega^{(1)})\big]^6 \times \Big(\big[W_{p,\mathrm{loc}}^1(\Omega^{(2)})\big]^4 \cap \mathbf{Z}_\tau(\Omega^{(2)})\Big)$ for $p > 1$ follows from the representation (4.57)–(4.58) and Lemmas 4.2 and 4.3. For $p = 2$, the solution is unique due to Theorem 3.1. To show the uniqueness for $p \neq 2$, we proceed as follows. Let a pair

$$(U^{(1)}, U^{(2)}) \in \big[W_p^1(\Omega^{(1)})\big]^6 \times \Big(\big[W_{p,\mathrm{loc}}^1(\Omega^{(2)})\big]^4 \cap \mathbf{Z}_\tau(\Omega^{(2)})\Big) \ \text{ for } \ p \neq 2$$

be a solution to the homogeneous basic transmission problem $(\mathrm{TD})_\tau$. Due to Corollary 5.4, $U^{(1)}$ and $U^{(2)}$ are uniquely representable in the form of single layer potentials (4.16) and (4.17), respectively, where the densities h and g are the traces on S of the vectors $U^{(1)}$ and $U^{(2)}$ (see (4.18)). Therefore, in view of the homogenous boundary-transmission conditions on S, with the help of the above-employed arguments we arrive at the homogeneous system of equations on S (cf. (4.37)–(4.40)):

$$\begin{aligned} &\widetilde{h} - g = 0, \\ &h_4 = 0, \\ &h_5 = 0, \\ &\big[\widetilde{\mathcal{A}}_S^{(1)+} - \mathcal{A}_S^{(2)-}\big]\, g = 0. \end{aligned}$$

Due to the invertibility of the operator (4.41) (see Lemma 4.1), we deduce that $g = 0$ and $h = 0$ on S implying $U^{(1)} = 0$ in $\Omega^{(1)}$ and $U^{(2)} = 0$ in $\Omega^{(2)}$. This completes the proof. □

Corollary 4.5. *Let S be the Lipschitz one and $p = 2$. Then the basic transmission problem (TD)$_\tau$,* (4.1)–(4.8) *is uniquely solvable in the space* $\big[W_2^1(\Omega^{(1)})\big]^6 \times \Big(\big[W_{2,\mathrm{loc}}^1(\Omega^{(2)})\big]^4 \cap \mathbf{Z}_\tau(\Omega^{(2)})\Big)$ *and the solution pair of vectors* $(U^{(1)}, U^{(2)})$ *is representable again in the form* (4.57)–(4.58).

Proof. We seek for a solution again in the form of single layer potentials (4.16) and (4.17). Using the properties of the single layer potentials presented in Theorem 5.2, the problem is then again reduced to system (4.37)–(4.40) and the coercivity inequality (4.43) leads to the invertibility of the operator (4.42) which completes the proof. □

We have the following regularity result.

Corollary 4.6. *Let $S \in C^{m,\alpha'}$ with $0 < \alpha < \alpha' \leqslant 1$ and $m \geqslant 2$ being an integer. Further, let*

$$f_j \in C^{k,\alpha}(S), \ \ j = 1, \dots, 6, \ \ F_1, F_2, F_3, F_6 \in C^{k-1,\alpha}(S), \ \ 1 \leqslant k \leqslant m-1.$$

Then the basic transmission problem (TD)$_\tau$, (4.1)–(4.7), *is uniquely solvable in the space* $\big[C^{k,\alpha}(\overline{\Omega^{(1)}})\big]^6 \times \Big(\big[C^{k,\alpha}(\overline{\Omega^{(2)}})\big]^4 \cap \mathbf{Z}_\tau(\Omega^{(2)})\Big)$ *and the solution pair of vectors* $(U^{(1)}, U^{(2)})$ *is representable again in the form* (4.57)–(4.58).

Proof. The existence of a unique weak solution representable in the form (4.57)–(4.58) follows from Theorem 4.4. On the other hand, Lemma 4.1 implies that the strongly elliptic pseudodifferential operator

$$\widetilde{\mathcal{A}}_S^{(1)+} - \mathcal{A}_S^{(2)-} \ : \ [C^{k,\alpha}(S)]^4 \to [C^{k-1,\alpha}(S)]^4$$

is invertible. This implies that solution vectors to system (4.37)–(4.40) satisfy the inclusions $h \in [C^{k,\alpha}(S)]^6$ and $g \in [C^{k,\alpha}(S)]^4$. The regularity result then follows from the representation (4.16)–(4.17) (and from (4.57)–(4.58) as well) and the mapping properties of the single layer potentials and the corresponding boundary operators described in Theorem 5.1. □

4.2. **Existence results for the boundary-transmission problem (DTD)$_\tau$.** In this subsection we consider a bounded composite structure $\Omega^{(1)} \cup \Omega^{(2)}$ introduced in Subsection 2.4. Recall that $S^{(1)}$ is the interface between the interior domain $\Omega^{(1)}$ and the exterior domain $\Omega^{(2)}$ and $S^{(2)}$ is the exterior boundary of the composite body. In the region $\Omega^{(1)}$ we have the GTEME model and in the region $\Omega^{(2)}$ the GTE model. Evidently, $\partial\Omega^{(1)} = S^{(1)}$ and $\partial\Omega^{(2)} = S^{(1)} \cup S^{(2)}$, $\overline{\Omega^{(1)}} = \Omega^{(1)} \cup S^{(1)}$, $\overline{\Omega^{(2)}} = \Omega^{(2)} \cup S^{(1)} \cup S^{(2)}$. For simplicity, let us assume that $S^{(1)}, S^{(2)} \in C^\infty$.

We will investigate the boundary-transmission problem (DTD)$_\tau$ in the weak setting sense for the homogeneous differential equations

$$A^{(j)}(\partial_x, \tau)\, U^{(j)}(x, \tau) = 0\,, \quad x \in \Omega^{(j)}, \qquad j = 1, 2,$$

where the differential operators $A^{(1)}(\partial_x, \tau)$ and $A^{(2)}(\partial_x, \tau)$ are defined by (2.1) and (2.13) respectively, and the sought for vectors

$$U^{(1)} = (U_1^{(1)}, \dots, U_6^{(1)})^\top \in [W_p^1(\Omega^{(1)})]^6, \quad U^{(2)} = (U_1^{(2)}, \dots, U_4^{(2)})^\top \in [W_p^1(\Omega^{(2)})]^4,$$

satisfy on the interface $S^{(1)}$ (see (2.22)–(2.26)) the following transmission conditions:

$$\{U_j^{(1)}(x)\}^+ - \{U_j^{(2)}(x)\}^- = f_j(x), \quad j = 1, 2, 3, \quad x \in S^{(1)}, \tag{4.59}$$

$$\{U_6^{(1)}(x)\}^+ - \{U_4^{(2)}(x)\}^- = f_6(x), \quad x \in S^{(1)}, \tag{4.60}$$

$$\{\mathcal{T}^{(1)}(\partial_x, n, \tau)U^{(1)}(x)\}_j^+ - \{\mathcal{T}^{(2)}(\partial_x, n, \tau)U^{(2)}(x)\}_j^- = F_j(x), \quad j = 1, 2, 3, \ \ x \in S^{(1)}, \tag{4.61}$$

$$\{\mathcal{T}^{(1)}(\partial_x, n, \tau)U^{(1)}(x)\}_6^+ - \{\mathcal{T}^{(2)}(\partial_x, n, \tau)U^{(2)}(x)\}_4^- = F_6(x), \quad x \in S^{(1)}, \tag{4.62}$$

$$\{U_j^{(1)}(x)\}^+ = f_j(x), \quad j = 4, 5, \quad x \in S^{(1)}, \tag{4.63}$$

and on the exterior boundary $S^{(2)}$ the Dirichlet boundary conditions:

$$\{U_j^{(2)}(x)\}^+ = f_j^*(x), \quad j = 1, 2, 3, 4, \quad x \in S^{(2)}. \tag{4.64}$$

The data of the problem satisfy the inclusions

$$
\begin{aligned}
&f_j \in B_{p,p}^{1-\frac{1}{p}}(S^{(1)}), \quad j=1,2,3,4,5,6, \quad F_1,F_2,F_3,F_6 \in B_{p,p}^{-\frac{1}{p}}(S^{(1)}),\\
&f_j^* \in B_{p,p}^{1-\frac{1}{p}}(S^{(2)}), \quad j=1,2,3,4.
\end{aligned} \tag{4.65}
$$

We look for solutions $U^{(1)}$ and $U^{(2)}$ in the form of a linear combination of single layer potentials associated with the operators $A^{(1)}$ and $A^{(2)}$ and constructed by the corresponding fundamental matrices $\Gamma^{(1)}$ and $\Gamma^{(2)}$ defined by (5.1), respectively:

$$
U^{(1)}(x) = V_{S^{(1)}}^{(1)}\varphi^{(1)}(x), \quad x \in \Omega^{(1)}, \tag{4.66}
$$

$$
U^{(2)}(x) = V_{S^{(1)}}^{(2)}\psi^{(1)}(x) + V_{S^{(2)}}^{(2)}\psi^{(2)}(x), \quad x \in \Omega^{(2)}, \tag{4.67}
$$

where

$$
\begin{aligned}
\varphi^{(1)} &= (\varphi_1^{(1)},\varphi_2^{(1)},\varphi_3^{(1)},\varphi_4^{(1)},\varphi_5^{(1)},\varphi_6^{(1)})^\top \in [B_{p,p}^{-\frac{1}{p}}(S^{(1)})]^6,\\
\psi^{(1)} &= (\psi_1^{(1)},\psi_2^{(1)},\psi_3^{(1)},\psi_4^{(1)})^\top \in [B_{p,p}^{-\frac{1}{p}}(S^{(1)})]^4,\\
\psi^{(2)} &= (\psi_1^{(2)},\psi_2^{(2)},\psi_3^{(2)},\psi_4^{(2)})^\top \in [B_{p,p}^{-\frac{1}{p}}(S^{(2)})]^4,
\end{aligned}
$$

are unknown density vector functions.

The properties of single layer potentials and the boundary-transmission conditions (4.59)–(4.64) lead to the following system of pseudodifferential equations for $\varphi^{(1)}$, $\psi^{(1)}$, and $\psi^{(2)}$:

$$
\sum_{l=1}^{6}[\mathcal{H}_{S^{(1)}}^{(1)}]_{jl}\,\varphi_l^{(1)} - \sum_{p=1}^{4}[\mathcal{H}_{S^{(1)}}^{(2)}]_{jp}\,\psi_p^{(1)} - [\gamma_{S^{(1)}}^{-}\{V_{S^{(2)}}^{(2)}\psi^{(2)}\}]_j = f_j, \ j=1,2,3, \text{ on } S^{(1)}, \tag{4.68}
$$

$$
\sum_{l=1}^{6}[\mathcal{H}_{S^{(1)}}^{(1)}]_{4l}\,\varphi_l^{(1)} = f_4 \text{ on } S^{(1)}, \tag{4.69}
$$

$$
\sum_{l=1}^{6}[\mathcal{H}_{S^{(1)}}^{(1)}]_{5l}\,\varphi_l^{(1)} = f_5 \text{ on } S^{(1)}, \tag{4.70}
$$

$$
\sum_{l=1}^{6}[\mathcal{H}_{S^{(1)}}^{(1)}]_{6l}\,\varphi_l^{(1)} - \sum_{p=1}^{4}[\mathcal{H}_{S^{(1)}}^{(2)}]_{4p}\,\psi_p^{(1)} - [\gamma_{S^{(1)}}^{-}\{V_{S^{(2)}}^{(2)}\psi^{(2)}\}]_4 = f_6 \text{ on } S^{(1)}, \tag{4.71}
$$

$$
\begin{aligned}
\sum_{l=1}^{6}\Big[-\frac{1}{2}\,I_6 + \mathcal{K}_{S^{(1)}}^{(1)}\Big]_{jl}\,\varphi_l^{(1)} &- \sum_{p=1}^{4}\Big[\frac{1}{2}\,I_4 + \mathcal{K}_{S^{(1)}}^{(2)}\Big]_{jp}\,\psi_p^{(1)}\\
&- [\gamma_{S^{(1)}}^{-}\{\mathcal{T}^{(2)}V_{S^{(2)}}^{(2)}\psi^{(2)}\}]_j = F_j, \quad j=1,2,3, \text{ on } S^{(1)},
\end{aligned} \tag{4.72}
$$

$$
\begin{aligned}
\sum_{l=1}^{6}\Big[-\frac{1}{2}\,I_6 + \mathcal{K}_{S^{(1)}}^{(1)}\Big]_{6l}\,\varphi_l^{(1)} &- \sum_{p=1}^{4}\Big[\frac{1}{2}\,I_4 + \mathcal{K}_{S^{(1)}}^{(2)}\Big]_{4p}\,\psi_p^{(1)}\\
&- [\gamma_{S^{(1)}}^{-}\{\mathcal{T}^{(2)}V_{S^{(2)}}^{(2)}\psi^{(2)}\}]_4 = F_6 \text{ on } S^{(1)},
\end{aligned} \tag{4.73}
$$

$$
\sum_{p=1}^{4}[\mathcal{H}_{S^{(2)}}^{(2)}]_{jp}\,\psi_p^{(2)} + [\gamma_{S^{(2)}}^{+}\{V_{S^{(1)}}^{(2)}\psi^{(1)}\}]_j = f_j^*, \quad j=1,2,3,4, \text{ on } S^{(2)}, \tag{4.74}
$$

where $\gamma_{S^{(j)}}^{\pm}$ denote one-sided traces on $S^{(j)}$, $j=1,2$. The integral operators $\mathcal{H}_{S^{(j)}}^{(l)}$ and $\mathcal{K}_{S^{(j)}}^{(l)}$ are associated with the single layer potentials and are defined by (5.7) and (5.8), respectively.

To prove the unique solvability of the above system, we proceed as follows. Due to the invertibility of the operators

$$
\begin{aligned}
&\mathcal{H}_{S^{(1)}}^{(1)} : [B_{p,p}^{-\frac{1}{p}}(S^{(1)})]^6 \to [B_{p,p}^{1-\frac{1}{p}}(S^{(1)})]^6,\\
&\mathcal{H}_{S^{(j)}}^{(2)} : [B_{p,p}^{-\frac{1}{p}}(S^{(j)})]^4 \to [B_{p,p}^{1-\frac{1}{p}}(S^{(j)})]^4, \ \ j=1,2,
\end{aligned}
$$

we can introduce new unknown vector functions

$$h^{(1)}=(h^{(1)}_1,\dots,h^{(1)}_6)^\top\in[B^{1-\frac1p}_{p,p}(S^{(1)})]^6,$$
$$g^{(1)}=(g^{(1)}_1,\dots,g^{(1)}_4)^\top\in[B^{1-\frac1p}_{p,p}(S^{(1)})]^4,$$
$$g^{(2)}=(g^{(2)}_1,\dots,g^{(2)}_4)^\top\in[B^{1-\frac1p}_{p,p}(S^{(2)})]^4,$$

by the relations

$$\varphi^{(1)}=\big[\mathcal{H}^{(1)}_{S^{(1)}}\big]^{-1}h^{(1)},\quad \psi^{(1)}=\big[\mathcal{H}^{(2)}_{S^{(1)}}\big]^{-1}g^{(1)},\quad \psi^{(2)}=\big[\mathcal{H}^{(2)}_{S^{(2)}}\big]^{-1}g^{(2)}.$$

Then

$$U^{(1)}(x)=V^{(1)}_{S^{(1)}}\big(\big[\mathcal{H}^{(1)}_{S^{(1)}}\big]^{-1}h^{(1)}\big)(x),\quad x\in\Omega^{(1)},$$
$$U^{(2)}(x)=V^{(2)}_{S^{(1)}}\big(\big[\mathcal{H}^{(2)}_{S^{(1)}}\big]^{-1}g^{(1)}\big)(x)+V^{(2)}_{S^{(2)}}\big(\big[\mathcal{H}^{(2)}_{S^{(2)}}\big]^{-1}g^{(2)}\big)(x),\quad x\in\Omega^{(2)},$$

and system (4.68)–(4.74) can be rewritten as follows

$$h^{(1)}_j-g^{(1)}_j-\big[\gamma^-_{S^{(1)}}\big\{V^{(2)}_{S^{(2)}}\big(\big[\mathcal{H}^{(2)}_{S^{(2)}}\big]^{-1}g^{(2)}\big)\big\}\big]_j=f_j,\ j=1,2,3,\ \text{on } S^{(1)}, \tag{4.75}$$

$$h^{(1)}_4=f_4 \text{ on } S^{(1)}, \tag{4.76}$$

$$h^{(1)}_5=f_5 \text{ on } S^{(1)}, \tag{4.77}$$

$$h^{(1)}_6-g^{(1)}_4-\big[\gamma^-_{S^{(1)}}\big\{V^{(2)}_{S^{(2)}}\big(\big[\mathcal{H}^{(2)}_{S^{(2)}}\big]^{-1}g^{(2)}\big)\big\}\big]_4=f_6 \text{ on } S^{(1)}, \tag{4.78}$$

$$\sum_{l=1}^{6}\big[\mathcal{A}^{(1)+}_{S^{(1)}}\big]_{jl}h^{(1)}_l-\sum_{p=1}^{4}\big[\mathcal{A}^{(2)-}_{S^{(1)}}\big]_{jp}g^{(1)}_p-\big[\gamma^-_{S^{(1)}}\big\{\mathcal{T}^{(2)}V^{(2)}_{S^{(2)}}\big(\big[\mathcal{H}^{(2)}_{S^{(2)}}\big]^{-1}g^{(2)}\big)\big\}\big]_j=F_j,\qquad j=1,2,3,\ \text{on } S^{(1)}, \tag{4.79}$$

$$\sum_{l=1}^{6}\big[\mathcal{A}^{(1)+}_{S^{(1)}}\big]_{6l}h^{(1)}_l-\sum_{p=1}^{4}\big[\mathcal{A}^{(2)-}_{S^{(1)}}\big]_{4p}g^{(1)}_p-\big[\gamma^-_{S^{(1)}}\big\{\mathcal{T}^{(2)}V^{(2)}_{S^{(2)}}\big(\big[\mathcal{H}^{(2)}_{S^{(2)}}\big]^{-1}g^{(2)}\big)\big\}\big]_4=F_6 \text{ on } S^{(1)}, \tag{4.80}$$

$$g^{(2)}_j+\big[\gamma^+_{S^{(2)}}\big\{V^{(2)}_{S^{(1)}}\big(\big[\mathcal{H}^{(2)}_{S^{(1)}}\big]^{-1}g^{(1)}\big)\big\}\big]_j=f^*_j,\quad j=1,2,3,4,\ \text{on } S^{(2)}, \tag{4.81}$$

where $\mathcal{A}^{(1)+}_{S^{(1)}}$ and $\mathcal{A}^{(2)-}_{S^{(1)}}$ are the Steklov-Poincaré type operators associated with the interface manifold $S^{(1)}$ (see Appendix, formulas (5.12)–(5.13))

$$\mathcal{A}^{(1)+}_{S^{(1)}}:=\Big(-\frac12 I_6+\mathcal{K}^{(1)}_{S^{(1)}}\Big)[\mathcal{H}^{(1)}_{S^{(1)}}]^{-1},\qquad \mathcal{A}^{(2)-}_{S^{(1)}}:=\Big(\frac12 I_4+\mathcal{K}^{(2)}_{S^{(1)}}\Big)[\mathcal{H}^{(2)}_{S^{(1)}}]^{-1}.$$

Note that the traces of the potentials

$$\gamma^-_{S^{(1)}}\big\{V^{(2)}_{S^{(2)}}\big(\big[\mathcal{H}^{(2)}_{S^{(2)}}\big]^{-1}g^{(2)}\big)\big\},\ \gamma^-_{S^{(1)}}\big\{\mathcal{T}^{(2)}V^{(2)}_{S^{(2)}}\big(\big[\mathcal{H}^{(2)}_{S^{(2)}}\big]^{-1}g^{(2)}\big)\big\},\ \gamma^+_{S^{(2)}}\big\{V^{(2)}_{S^{(1)}}\big(\big[\mathcal{H}^{(2)}_{S^{(1)}}\big]^{-1}g^{(1)}\big)\big\}, \tag{4.82}$$

are smoothing operators, since $S^{(1)}$ and $S^{(2)}$ are disjoint surfaces, $S^{(1)}\cap S^{(2)}=\varnothing$.

Therefore the operator $\mathfrak{D}$ generated by the left-hand side expressions in system (4.75)–(4.81) can be written as follows:

$$\mathfrak{D}=\|\mathfrak{D}_{kj}\|_{14\times14}=\mathfrak{N}+\mathfrak{L}, \tag{4.83}$$

with

$$\mathfrak{N}=\|\mathfrak{N}_{kj}\|_{14\times14}=\begin{bmatrix}\mathfrak{M} & [0]_{10\times4}\\ [0]_{4\times10} & I_4\end{bmatrix}_{14\times14},$$

where the operator $\mathfrak{M}=[\mathfrak{M}_{kj}]_{10\times10}$ is given by (4.52) with $S^{(1)}$ for S, $I_4=[\delta_{kj}]_{4\times4}$ is the unit matrix, and $\mathfrak{L}=[\mathfrak{L}_{kj}]_{14\times14}$ is infinitely smoothing operator generated by the summands of system (4.75)–(4.81) involving operators (4.82).

For the 14-dimensional unknown vector function Φ and for the known vector function Ψ constructed by the transmission and boundary data, we introduce the following notation:

$$\Phi:=\big(h^{(1)},\,g^{(1)},\,g^{(2)}\big)^\top\in\mathbb{X}_p,$$

$$\Psi := \big(f_1,\, f_2,\, f_3,\, f_4,\, f_5,\, f_6,\; F_1,\, F_2,\, F_3,\, F_6,\; f_1^*,\, f_2^*,\, f_3^*,\, f_4^*\big)^\top \in \mathbb{Y}_p,$$

where

$$\mathbb{X}_p := [B_{p,p}^{1-\frac{1}{p}}(S^{(1)})]^{10} \times [B_{p,p}^{1-\frac{1}{p}}(S^{(2)})]^4,$$
$$\mathbb{Y}_p := [B_{p,p}^{1-\frac{1}{p}}(S^{(1)})]^{6} \times [B_{p,p}^{-\frac{1}{p}}(S^{(1)})]^4 \times [B_{p,p}^{1-\frac{1}{p}}(S^{(2)})]^4.$$

The system of equations (4.75)–(4.81) can be then rewritten in the matrix form

$$\mathfrak{D}\Phi = \Psi, \;\text{ i.e., }\; \big(\mathfrak{N} + \mathfrak{L}\big)\Phi = \Psi.$$

Evidently, we have the following mapping properties

$$\mathfrak{N} : \mathbb{X}_p \to \mathbb{Y}_p, \tag{4.84}$$
$$\mathfrak{L} : \mathbb{X}_p \to \mathbb{Y}_p. \tag{4.85}$$

Now we prove the following

Lemma 4.7. *The operator*

$$\mathfrak{N} + \mathfrak{L} : \mathbb{X}_p \to \mathbb{Y}_p \tag{4.86}$$

is invertible.

Proof. Note that operator (4.85) is compact due to the above mentioned smoothing property of the operator $\mathfrak{L}$. On the other hand, in view of Lemma 4.3 and relation (4.83), we conclude that the operator (4.84) is invertible. Therefore operator (4.86) is the Fredholm one with zero index. Let us show that the null-space of the operator (4.86) is trivial which will complete the proof. To this end, let us assume that $\widetilde{\Phi} = \big(\widetilde{h}^{(1)},\, \widetilde{g}^{(1)},\, \widetilde{g}^{(2)}\big)^\top \in \mathbb{X}_p$ is a solution to the homogeneous equation $\big(\mathfrak{N}+\mathfrak{L}\big)\Phi = 0$. Then in accordance with relations (4.75)–(4.81) the pair of vector functions

$$U^{(1)}(x) = V_{S^{(1)}}^{(1)}\big(\big[\mathcal{H}_{S^{(1)}}^{(1)}\big]^{-1}\widetilde{h}^{(1)}\big)(x), \quad x \in \Omega^{(1)},$$
$$U^{(2)}(x) = V_{S^{(1)}}^{(2)}\big(\big[\mathcal{H}_{S^{(1)}}^{(2)}\big]^{-1}\widetilde{g}^{(1)}\big)(x) + V_{S^{(2)}}^{(2)}\big(\big[\mathcal{H}_{S^{(2)}}^{(2)}\big]^{-1}\widetilde{g}^{(2)}\big)(x), \quad x \in \Omega^{(2)},$$

solve the homogeneous boundary-transmission problem $(\mathrm{DTD})_\tau$. Therefore $U^{(1)} = 0$ in $\Omega^{(1)}$ and $U^{(2)} = 0$ in $\Omega^{(2)}$ by the uniqueness Theorem 3.2. Using the continuity property of the single layer potentials across the integration surface and the uniqueness theorems for the interior and exterior Dirichlet problems for the operators $A^{(j)}(\partial_x, \tau)$, $j = 1, 2$, we deduce that $V_{S^{(1)}}^{(1)}\big(\big[\mathcal{H}_{S^{(1)}}^{(1)}\big]^{-1}\widetilde{h}^{(1)}\big)(x) = 0$ and $V_{S^{(1)}}^{(2)}\big(\big[\mathcal{H}_{S^{(1)}}^{(2)}\big]^{-1}\widetilde{g}^{(1)}\big)(x) + V_{S^{(2)}}^{(2)}\big(\big[\mathcal{H}_{S^{(2)}}^{(2)}\big]^{-1}\widetilde{g}^{(2)}\big)(x) = 0$ in the whole space $\mathbb{R}^3$ (see, [7, Theorems 2.25 and 2.26]). By the jump relations presented in Theorem 5.3, we finally conclude that $\widetilde{h}^{(1)} = 0$ on $S^{(1)}$, $\widetilde{g}^{(1)} = 0$ on $S^{(1)}$, and $\widetilde{g}^{(2)} = 0$ on $S^{(2)}$, which completes the proof. □

This lemma implies directly the following assertion.

Lemma 4.8. *Let conditions* (4.65) *be satisfied with* $p > 1$. *The systems of pseudodifferential equations* (4.68)–(4.74) *and* (4.75)–(4.81) *be uniquely solvable in appropriate function spaces for arbitrary right-hand side functions.*

We now can prove the existence and regularity theorems of solutions to the problem $(DTD)_\tau$.

Theorem 4.9. *Let conditions* (4.65) *be satisfied with* $p > 1$. *Then the boundary-transmission problem* $(DTD)_\tau$ *is uniquely solvable in the spaces* $[W_p^1(\Omega^{(1)})]^6 \times [W_p^1(\Omega^{(2)})]^4$ *and the solution vectors* $U^{(j)}$, $j = 1, 2$, *are representable in the form of a linear combination of single layer potentials* (4.66)–(4.67), *where the density vectors* $\varphi^{(1)} \in [B_{p,p}^{-\frac{1}{p}}(S^{(1)})]^6$, $\psi^{(1)} \in [B_{p,p}^{-\frac{1}{p}}(S^{(1)})]^4$ *and* $\psi^{(2)} \in [B_{p,p}^{-\frac{1}{p}}(S^{(2)})]^4$ *are defined from the uniquely solvable system of pseudodifferential equations* (4.68)–(4.74).

Proof. Is word for word similar to that of Theorem 4.4. □

Corollary 4.10. *Let $S^{(1)}, S^{(2)} \in C^{m,\alpha'}$ with $0<\alpha<\alpha'\leqslant 1$ and $m\geqslant 2$ being an integer. Further, let*

$$f_j \in C^{k,\alpha}(S^{(1)}),\ \ j=1,\dots,6,\ \ F_1,F_2,F_3,F_6 \in C^{k-1,\alpha}(S^{(1)}),$$
$$f_j^* \in C^{k,\alpha}(S^{(2)}),\ \ j=1,\dots,4,\ \ 1\leqslant k\leqslant m-1.$$

Then the transmission problem $(DTD)_\tau$ is uniquely solvable in the space $\big[C^{k,\alpha}(\overline{\Omega^{(1)}})\big]^6\times\big[C^{k,\alpha}(\overline{\Omega^{(2)}})\big]^4$ and the solution pair of vectors $(U^{(1)},U^{(2)})$ is representable in the form (4.66)–(4.67).

Proof. Is word for word similar to that of Corollary 4.6. □

5. Appendix

Here we collect some results from references [7, 21], and [22] which are employed in the main text of the present paper.

Fundamental matrices $\Gamma^{(j)}(x,\tau)$ of the operators $A^{(j)}(\partial_x,\tau)$, $j=1,2$, can be constructed with the help of the Fourier transform

$$\Gamma^{(j)}(x,\tau)=\mathcal{F}^{-1}_{\xi\to x}\big[\big(A^{(j)}(-i\,\xi,\tau)\big)^{-1}\big]\,, \tag{5.1}$$

where $\big(A^{(j)}(-i\,\xi,\tau)\big)^{-1}$ is the matrix inverse to $A(-i\,\xi,\tau)$, and $\mathcal{F}_{x\to\xi}$ and $\mathcal{F}^{-1}_{\xi\to x}$ denote the direct and inverse distributional Fourier transforms in the space of tempered distributions which for regular summable functions f and g read as follows

$$\mathcal{F}_{x\to\xi}[f]=\int\limits_{\mathbb{R}^3} f(x)\,e^{i\,x\cdot\xi}dx,\qquad \mathcal{F}^{-1}_{\xi\to x}[g]=\frac{1}{(2\pi)^3}\int\limits_{\mathbb{R}^3} g(\xi)\,e^{-i\,x\cdot\xi}d\xi,$$

where $x=(x_1,x_2,x_3)$ and $\xi=(\xi_1,\xi_2,\xi_3)$.

These fundamental matrices solve the following distributional equations

$$A^{(j)}(\partial_x,\tau)\Gamma^{(j)}(x,\tau)=I^{(j)}\,\delta(x),$$

where $I^{(1)}=I_6$ and $I^{(2)}=I_4$ are 6×6 and 4×4 unit matrices and $\delta(x)$ is Dirac's distribution.

The entries of the matrices $\Gamma^{(1)}(x,\tau)$ and $\Gamma^{(2)}(x,\tau)$ in the vicinity of the origin have the property

$$\Gamma^{(1)}(x,\tau)=\begin{bmatrix}[\mathcal{O}(|x|^{-1})]_{5\times5} & [\mathcal{O}(1)]_{5\times1}\\ [\mathcal{O}(1)]_{1\times5} & \mathcal{O}(|x|^{-1})\end{bmatrix}_{6\times6},$$

$$\Gamma^{(2)}(x,\tau)=\begin{bmatrix}[\mathcal{O}(|x|^{-1})]_{3\times3} & [\mathcal{O}(1)]_{3\times1}\\ [\mathcal{O}(1)]_{1\times3} & \mathcal{O}(|x|^{-1})\end{bmatrix}_{4\times4},$$

while at infinity they have the following asymptotic behaviour

$$\Gamma^{(1)}(x,\tau)=\begin{bmatrix}
\mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-5}) & \mathcal{O}(|x|^{-5}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-4})\\
\mathcal{O}(|x|^{-5}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-5}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-4})\\
\mathcal{O}(|x|^{-5}) & \mathcal{O}(|x|^{-5}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-4})\\
\mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-1}) & \mathcal{O}(|x|^{-1}) & \mathcal{O}(|x|^{-2})\\
\mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-1}) & \mathcal{O}(|x|^{-1}) & \mathcal{O}(|x|^{-2})\\
\mathcal{O}(|x|^{-4}) & \mathcal{O}(|x|^{-4}) & \mathcal{O}(|x|^{-4}) & \mathcal{O}(|x|^{-2}) & \mathcal{O}(|x|^{-2}) & \mathcal{O}(|x|^{-3})
\end{bmatrix}_{6\times6},$$

$$\Gamma^{(2)}(x,\tau)=\begin{bmatrix}
\mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-5}) & \mathcal{O}(|x|^{-5}) & \mathcal{O}(|x|^{-4})\\
\mathcal{O}(|x|^{-5}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-5}) & \mathcal{O}(|x|^{-4})\\
\mathcal{O}(|x|^{-5}) & \mathcal{O}(|x|^{-5}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-4})\\
\mathcal{O}(|x|^{-4}) & \mathcal{O}(|x|^{-4}) & \mathcal{O}(|x|^{-4}) & \mathcal{O}(|x|^{-3})
\end{bmatrix}_{4\times4}.$$

Let $\Omega=\Omega^+$ be a bounded domain with a simply connected boundary $S=\partial\Omega^+$ and $\Omega^-=\mathbb{R}^3\setminus\overline{\Omega}$. Introduce the generalized single layer potentials

$$V_S^{(j)}(g^{(j)})(x)=\int\limits_S \Gamma^{(j)}(x-y,\tau)\,g^{(j)}(y)\,dS_y,\quad j=1,2,\quad x\in\mathbb{R}^3\setminus S, \tag{5.2}$$

$g^{(1)} = (g_1^{(1)}, \dots, g_6^{(1)})^\top$ and $g^{(2)} = (g_1^{(2)}, \dots, g_4^{(2)})^\top$ are the density vector functions defined on S.

Theorem 5.1. *Let $S \in C^{m,\alpha'}$, $0 < \alpha < \alpha' \leqslant 1$, and let $m \geqslant 1$ and $k \leqslant m-1$ be nonnegative integers. Then the operators*

$$
\begin{aligned}
V_S^{(1)} &: [C^{k,\alpha}(S)]^6 \to [C^{k+1,\alpha}(\overline{\Omega^\pm})]^6,\\
V_S^{(2)} &: [C^{k,\alpha}(S)]^4 \to [C^{k+1,\alpha}(\overline{\Omega^\pm})]^4,
\end{aligned}
$$

are continuous.

For any $g^{(1)} \in [C^{0,\alpha}(S)]^6$ and $g^{(2)} \in [C^{0,\alpha}(S)]^4$, and for any $x \in S$, the following jump relations

$$\{V_S^{(1)}(g^{(1)})(x)\}^\pm = \mathcal{H}_S^{(1)}\, g^{(1)}(x), \tag{5.3}$$

$$\{\mathcal{T}^{(1)}(\partial_x, n(x), \tau)\, V_S^{(1)}(g^{(1)})(x)\}^\pm = [\,\mp 2^{-1} I_6 + \mathcal{K}_S^{(1)}\,]\, g^{(1)}(x), \tag{5.4}$$

$$\{V_S^{(2)}(g^{(2)})(x)\}^\pm = \mathcal{H}_S^{(2)}\, g^{(2)}(x), \tag{5.5}$$

$$\{\mathcal{T}^{(2)}(\partial_x, n(x), \tau)\, V_S^{(2)}(g^{(2)})(x)\}^\pm = [\,\mp 2^{-1} I_4 + \mathcal{K}_S^{(2)}\,]\, g^{(2)}(x) \tag{5.6}$$

hold, where

$$\mathcal{H}_S^{(j)}\, g^{(j)}(x) := \int\limits_S \Gamma^{(j)}(x-y,\tau)\, g^{(j)}(y)\, dS_y\,, \quad x \in S, \quad j = 1,2, \tag{5.7}$$

$$\mathcal{K}_S^{(j)}\, g^{(j)}(x) := \int\limits_S [\,\mathcal{T}^{(j)}(\partial_x, n(x), \tau)\, \Gamma^{(j)}(x-y,\tau)\,]\, g^{(j)}(y)\, dS_y\,, \quad x \in S, \quad j = 1,2. \tag{5.8}$$

The following operators

$$
\begin{aligned}
\mathcal{H}_S^{(1)} &: [C^{k,\alpha}(S)]^6 \to [C^{k+1,\alpha}(S)]^6\,, \qquad & \mathcal{H}_S^{(2)} &: [C^{k,\alpha}(S)]^4 \to [C^{k+1,\alpha}(S)]^4\,,\\
\mathcal{K}_S^{(1)} &: [C^{k,\alpha}(S)]^6 \to [C^{k,\alpha}(S)]^6\,, & \mathcal{K}_S^{(2)} &: [C^{k,\alpha}(S)]^4 \to [C^{k,\alpha}(S)]^4
\end{aligned} \tag{5.9}
$$

are continuous. Moreover, the operators (5.9) *are invertible.*

Theorem 5.2. *Let S be a Lipschitz surface. The operators $V_S^{(j)}$, $\mathcal{H}_S^{(j)}$, and $\mathcal{K}_S^{(j)}$, $j = 1,2$, defined by* (5.2)*,* (5.7)*, and* (5.8)*, can be extended to the continuous mappings*

$$
\begin{aligned}
V_S^{(1)} &: \big[H_2^{-\frac12}(S)\big]^6 \to \big[H_2^1(\Omega^+)\big]^6, \qquad & V_S^{(1)} &: \big[H_2^{-\frac12}(S)\big]^6 \to \big[H_{2,\,\mathrm{loc}}^1(\Omega^-)\big]^6,\\
V_S^{(2)} &: \big[H_2^{-\frac12}(S)\big]^4 \to \big[H_2^1(\Omega^+)\big]^4, & V_S^{(2)} &: \big[H_2^{-\frac12}(S)\big]^4 \to \big[H_{2,\,\mathrm{loc}}^1(\Omega^-)\big]^4 \cap \mathbf{Z}_\tau(\Omega^-).\\
\mathcal{H}_S^{(1)} &: [H_2^{-\frac12}(S)]^6 \to [H_2^{\frac12}(S)]^6\,, & \mathcal{K}_S^{(1)} &: [H_2^{-\frac12}(S)]^6 \to [H_2^{-\frac12}(S)]^6\,,\\
\mathcal{H}_S^{(2)} &: [H_2^{-\frac12}(S)]^4 \to [H_2^{\frac12}(S)]^4\,, & \mathcal{K}_S^{(2)} &: [H_2^{-\frac12}(S)]^4 \to [H_2^{-\frac12}(S)]^4\,.
\end{aligned}
$$

Moreover, the operators

$$\mathcal{H}_S^{(1)} : [H_2^{-\frac12}(S)]^6 \to [H_2^{\frac12}(S)]^6, \qquad \mathcal{H}_S^{(2)} : [H_2^{-\frac12}(S)]^4 \to [H_2^{\frac12}(S)]^4\,,$$

are invertible.

Theorem 5.3. *Let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$, $S \in C^\infty$. The operators $V_S^{(j)}$, $\mathcal{H}_S^{(j)}$, and $\mathcal{K}_S^{(j)}$, $j = 1, 2$, can be extended to the following continuous operators*

$$\begin{aligned}
V_S^{(1)} &: [B^s_{p,p}(S)]^6 \to [H_p^{s+1+\frac{1}{p}}(\Omega^+)]^6 \quad \Big[[B^s_{p,p}(S)]^6 \to [H_{p,\,\mathrm{loc}}^{s+1+\frac{1}{p}}(\Omega^-)]^6 \Big],\\
&: [B^s_{p,q}(S)]^6 \to [B_{p,q}^{s+1+\frac{1}{p}}(\Omega^+)]^6 \quad \Big[[B^s_{p,q}(S)]^6 \to [B_{p,q,\,\mathrm{loc}}^{s+1+\frac{1}{p}}(\Omega^-)]^6 \Big],\\
V_S^{(2)} &: [B^s_{p,p}(S)]^4 \to [H_p^{s+1+\frac{1}{p}}(\Omega^+)]^4 \quad \Big[[B^s_{p,p}(S)]^4 \to [H_{p,\,\mathrm{loc}}^{s+1+\frac{1}{p}}(\Omega^-)]^4 \cap \mathbf{Z}_\tau(\Omega^-) \Big],\\
&: [B^s_{p,q}(S)]^4 \to [B_{p,q}^{s+1+\frac{1}{p}}(\Omega^+)]^4 \quad \Big[[B^s_{p,q}(S)]^4 \to [B_{p,q,\,\mathrm{loc}}^{s+1+\frac{1}{p}}(\Omega^-)]^4 \cap \mathbf{Z}_\tau(\Omega^-) \Big],\\
\mathcal{H}_S^{(1)} &: [H^s_p(S)]^6 \to [H^{s+1}_p(S)]^6 \quad \Big[[B^s_{p,q}(S)]^6 \to [B^{s+1}_{p,q}(S)]^6 \Big],\\
\mathcal{K}_S^{(1)} &: [H^s_p(S)]^6 \to [H^{s}_p(S)]^6 \quad \Big[[B^s_{p,q}(S)]^6 \to [B^{s}_{p,q}(S)]^6 \Big],\\
\mathcal{H}_S^{(2)} &: [H^s_p(S)]^4 \to [H^{s+1}_p(S)]^4 \quad \Big[[B^s_{p,q}(S)]^4 \to [B^{s+1}_{p,q}(S)]^4 \Big],\\
\mathcal{K}_S^{(2)} &: [H^s_p(S)]^4 \to [H^{s}_p(S)]^4 \quad \Big[[B^s_{p,q}(S)]^4 \to [B^{s}_{p,q}(S)]^4 \Big].
\end{aligned}$$

For $s > -1$ the jump relations (5.3)–(5.6) *remain valid in appropriate function spaces.*
The operators

$$\begin{aligned}
\mathcal{H}_S^{(1)} &: [B^s_{p,q}(S)]^6 \to [B^{s+1}_{p,q}(S)]^6, \qquad s \in \mathbb{R},\ \ p > 1,\ \ q \ge 1,\\
\mathcal{H}_S^{(2)} &: [B^s_{p,q}(S)]^4 \to [B^{s+1}_{p,q}(S)]^4, \qquad s \in \mathbb{R},\ \ p > 1,\ \ q \ge 1,
\end{aligned}$$

are invertible.

Corollary 5.4. *Let $s \in \mathbb{R}$, $1 < p < \infty$, $S \in C^\infty$. Arbitrary solutions to the homogeneous equations*

$$A^{(1)}(\partial,\tau)U^{(1)} = 0 \ \ in \ \ \Omega, \quad U^{(1)} \in [W^1_p(\Omega)]^6, \ \ p > 1,$$

and

$$A^{(2)}(\partial,\tau)U^{(2)} = 0 \ \ in \ \ \Omega, \quad U^{(2)} \in [W^1_p(\Omega)]^4, \ \ p > 1,$$

are uniquely representable in the form

$$U^{(1)} = V_S^{(1)}\big(\big[\mathcal{H}_S^{(1)}\big]^{-1} g^{(1)}\big) \ \ with \ \ g^{(1)} = \{U^{(1)}\}^+ \in [B_{p,p}^{1-\frac{1}{p}}(\partial\Omega)]^6\,, \tag{5.10}$$

$$U^{(2)} = V_S^{(2)}\big(\big[\mathcal{H}_S^{(2)}\big]^{-1} g^{(2)}\big) \ \ with \ \ g^{(2)} = \{U^{(2)}\}^+ \in [B_{p,p}^{1-\frac{1}{p}}(\partial\Omega)]^4. \tag{5.11}$$

The representations (5.10) *and* (5.11) *hold true for Lipschitz domain Ω and $p = 2$.*

Further, let us introduce the Steklov-Poincaré type operators

$$\begin{aligned}
\mathcal{A}_S^{(1)\pm} &:= \big(\mp 2^{-1}\, I_6 + \mathcal{K}_S^{(1)} \big) \big[\mathcal{H}_S^{(1)}\big]^{-1},\\
\mathcal{A}_S^{(2)\pm} &:= \big(\mp 2^{-1}\, I_4 + \mathcal{K}_S^{(2)} \big) \big[\mathcal{H}_S^{(2)}\big]^{-1},
\end{aligned}$$

which are related to the single layer potentials by the relations

$$\mathcal{A}_S^{(1)\pm} g^{(1)} = \Big\{ \mathcal{T}^{(1)}(\partial_x, n(x), \tau)\, V_S^{(1)}\big(\big[\mathcal{H}_S^{(1)}\big]^{-1} g^{(1)}\big) \Big\}^{\pm}\,, \tag{5.12}$$

$$\mathcal{A}_S^{(2)\pm} g^{(2)} = \Big\{ \mathcal{T}^{(2)}(\partial_x, n(x), \tau)\, V_S^{(2)}\big(\big[\mathcal{H}_S^{(2)}\big]^{-1} g^{(2)}\big) \Big\}^{\pm}\,. \tag{5.13}$$

Theorem 5.5. *Let S be a Lipschitz surface and $\tau = \sigma + i\omega$ with $\sigma > 0$ and $\omega \in \mathbb{R}$. Then for all $g^{(1)} \in [H_2^{\frac{1}{2}}(S)]^6$ and $g^{(2)} \in [H_2^{\frac{1}{2}}(S)]^4$, the coercivity inequalities*

$$\mathrm{Re}\,\langle (\pm\,\mathcal{A}_S^{(1)\pm} + \mathcal{C}^{(1)}) g^{(1)}\,,\, g^{(1)} \rangle_S \ge C_1\, \|g^{(1)}\|^2_{[H_2^{\frac{1}{2}}(S)]^6}\,,$$

$$\mathrm{Re}\,\langle (\pm\,\mathcal{A}_S^{(2)\pm} + \mathcal{C}^{(2)}) g^{(2)}\,,\, g^{(2)} \rangle_S \ge C_2\, \|g^{(2)}\|^2_{[H_2^{\frac{1}{2}}(S)]^4}$$

hold, where

$$\mathcal{C}^{(1)} \ : \ [H_2^{\frac{1}{2}}(S)]^6 \to [H_2^{-\frac{1}{2}}(S)]^6 \ \ and \ \ \mathcal{C}^{(2)} \ : \ [H_2^{\frac{1}{2}}(S)]^4 \to [H_2^{-\frac{1}{2}}(S)]^4$$

are compact operators and C_j, $j=1,2$, *are positive constants.*
The operators

$$\mathcal{A}_S^{(1)-} \ : \ [H_2^{\frac{1}{2}}(S)]^6 \to [H_2^{-\frac{1}{2}}(S)]^6\,,$$
$$\mathcal{A}_S^{(2)\pm} \ : \ [H_2^{\frac{1}{2}}(S)]^4 \to [H_2^{-\frac{1}{2}}(S)]^4\,,$$

are invertible, while

$$\mathcal{A}_S^{(1)+} \ : \ [H_2^{\frac{1}{2}}(S)]^6 \to [H_2^{-\frac{1}{2}}(S)]^6$$

is the Fredholm one of index zero with the null space spanned over the vectors

$$\Psi^{(1)} = (0,0,0,1,0,0)^\top, \quad \Psi^{(2)} = (0,0,0,0,1,0)^\top. \tag{5.14}$$

Theorem 5.6. *Let* $s\in\mathbb{R}$, $1<p<\infty$, $1\le q\le\infty$, $S\in C^\infty$. *The operators*

$$\mathcal{A}_S^{(1)-} \ : \ [B_{p,q}^{s+1}(S)]^6 \to [B_{p,q}^{s}(S)]^6, \qquad s\in\mathbb{R}, \ \ p>1, \ \ q\ge 1,$$
$$\mathcal{A}_S^{(2)\pm} \ : \ [B_{p,q}^{s+1}(S)]^4 \to [B_{p,q}^{s}(S)]^4, \qquad s\in\mathbb{R}, \ \ p>1, \ \ q\ge 1,$$

are invertible, while the operator

$$\mathcal{A}_S^{(1)+} \ : \ [B_{p,q}^{s+1}(S)]^6 \to [B_{p,q}^{s}(S)]^6, \qquad s\in\mathbb{R}, \ \ p>1, \ \ q\ge 1,$$

is Fredholm of zero index with a two-dimensional null space spanned over the vectors (5.14).

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