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SOLUTIONS OF SOME DIOPHANTINE EQUATIONS IN TERMS OF HORADAM SEQUENCE

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Abstract. Let a, b, and P be integers such that $(a,b) \neq (0,0)$. In this study, we give all solutions of the equations $x^2 - Pxy - y^2 = \pm\left(b^2 - Pab - a^2\right)$, $x^2 - (P^2+4)y^2 = \pm 4(b^2 - Pab - a^2)$, $x^2 - (P^2+4)y^2 = \pm 4(b^2 - Pab - a^2)^2$, $x^2 - Pxy + y^2 = b^2 - Pab + a^2$, $x^2 - (P^2-4)y^2 = 4(b^2 - Pab + a^2)$, and $x^2 - (P^2-4)y^2 = 4(b^2 - Pab + a^2)^2$ in terms of the second order recurrence sequences when $|b^2 - Pab \pm a^2|$ is odd prime.

1. Introduction

The second order recurrence sequence $\{W_n\} = \{W_n(a,b;P,Q)\}$ is defined by

$$W_0 = a,\ W_1 = b, \quad \text{and} \quad W_n = PW_{n-1} + QW_{n-2} \quad \text{for} \quad n \geq 2,$$

where a, b, P, and Q are integers with $PQ \neq 0$ and $(a,b) \neq (0,0)$. Particular cases of $\{W_n\}$ are the Lucas sequence of the first kind $\{U_n(P,Q)\} = \{W_n(0,1;P,Q)\}$ and the Lucas sequence of the second kind $\{V_n(P,Q)\} = \{W_n(2,P;P,Q)\}$. Now we define the sequence $\{X_n\} = \{X_n(a,b;P,Q)\}$ by

$$X_0 = 2b - aP, \quad X_1 = bP + 2aQ, \quad \text{and} \quad X_n = PX_{n-1} + QX_{n-2} \quad \text{for} \quad n \geq 2.$$

It is convenient to consider $\{X_n\}$ to be the companion sequence of $\{W_n\}$, in the same way that $\{V_n\}$ is the companion sequence of $\{U_n\}$. Let α and β be the roots of the equation $x^2 - Px - Q = 0$. Then $\alpha = (P + \sqrt{P^2+4Q})/2$ and $\beta = (P - \sqrt{P^2+4Q})/2$. Clearly, $\alpha + \beta = P$, $\alpha - \beta = \sqrt{P^2+4Q}$, and $\alpha\beta = -Q$. Assume that $P^2 + 4Q \neq 0$. Then Binet formulas of $\{W_n\}$ and $\{X_n\}$ are given by

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta} \quad \text{and} \quad X_n = A\alpha^n + B\beta^n, \tag{1.1}$$

where $A = b - a\beta$ and $B = b - a\alpha$. It can be seen that $AB = b^2 - abP - a^2Q$. Moreover, it can be easily shown that there are the following relations between the terms of the sequences $\{W_n\}$, $\{X_n\}$, $\{U_n\}$, and $\{V_n\}$ given by

$$X_n = W_{n+1} + QW_{n-1} = PW_n + 2QW_{n-1}, \tag{1.2}$$

$$(P^2+4Q)W_n = X_{n+1} + QX_{n-1}, \tag{1.3}$$

$$W_n = bU_n + aQU_{n-1} \quad \text{and} \quad X_n = bV_n + aQV_{n-1} \tag{1.4}$$

for $n \geq 1$. It is well known that the numbers U_n and V_n for negative subscripts are defined as

$$U_{-n} = \frac{-U_n}{(-Q)^n} \text{ and } V_{-n} = \frac{V_n}{(-Q)^n}$$

for $n \geq 1$. By using (1.1) together with (1.4), it is convenient to define the numbers W_n and X_n for negative subscripts by

$$W_{-n} = \frac{A\alpha^{-n} - B\beta^{-n}}{\alpha - \beta} \quad \text{and} \quad X_{-n} = A\alpha^{-n} + B\beta^{-n}.$$

Then it follows that

$$W_{-n} = \frac{-bU_n + aU_{n+1}}{(-Q)^n} \quad \text{and} \quad X_{-n} = \frac{bV_n - aV_{n+1}}{(-Q)^n} \tag{1.5}$$

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and therefore

$$W_{-n} = bU_{-n} + aQU_{-n-1} \quad \text{and} \quad X_{-n} = bV_{-n} + aQV_{-n-1}.$$

Thus it is seen that identities (1.2), (1.3), and (1.4) hold for all integers n. For more information about the sequence one can consult [2,10,11,13,15].

In the literature, integer solutions of the equations $x^2 - Pxy - y^2 = 1$, $x^2 - Pxy - y^2 = -1$, $x^2 - (P^2+4)y^2 = 4$, $x^2 - (P^2+4)y^2 = -4$, $x^2 - Pxy + y^2 = 1$, and $x^2 - (P^2-4)y^2 = 4$ are given in terms of the sequences $\{U_n(P, \pm 1)\}$ and $\{V_n(P, \pm 1)\}$(see [4–9,12,16]). More clearly, we can state them by

Equations	**Solutions**
$x^2 - Pxy - y^2 = 1$	$(x, y) = \pm(U_n(P, 1), U_{n-1}(P, 1))$ with n odd,
$x^2 - Pxy - y^2 = -1$	$(x, y) = \pm(U_n(P, 1), U_{n-1}(P, 1))$ with n even,
$x^2 - (P^2+4)y^2 = 4$	$(x, y) = \pm(V_n(P, 1), U_n(P, 1))$ with n even,
$x^2 - (P^2+4)y^2 = -4$	$(x, y) = \pm(V_n(P, 1), U_n(P, 1))$ with n odd,
$x^2 - Pxy + y^2 = 1$	$(x, y) = \pm(U_n(P, -1), U_{n-1}(P, -1))$,
$x^2 - (P^2-4)y^2 = 4$	$(x, y) = \pm(V_n(P, -1), U_n(P, -1))$.

Moreover, if $P^2 \pm 4$ is square free, then all integer solutions of the equations $x^2 - Pxy - y^2 = P^2+4, x^2 - Pxy - y^2 = -(P^2+4)$, and $x^2 - Pxy + y^2 = -(P^2-4)$ are given in terms of the sequence $\{V_n(P, \pm 1)\}$(see [7]). When $P^2 \pm 4$ is square free, we get

Equations	**Solutions**
$x^2 - Pxy - y^2 = P^2 + 4$	$(x, y) = \pm(V_n(P, 1), V_{n-1}(P, 1))$ with n even,
$x^2 - Pxy - y^2 = -(P^2+4)$	$(x, y) = \pm(V_n, (P, 1), V_{n-1}(P, 1))$ with n odd,
$x^2 - Pxy + y^2 = -(P^2-4)$	$(x, y) = \pm(V_n(P, -1), V_{n-1}(P, -1))$.

In this paper, we give all integer solutions of the equations

$$\begin{aligned}
x^2 - Pxy - y^2 =& b^2 - Pab - a^2, \; x^2 - Pxy - y^2 = -(b^2 - Pab - a^2) \\
x^2 - (P^2+4)y^2 =& 4(b^2 - Pab - a^2), \; x^2 - (P^2+4)y^2 = -4(b^2 - Pab - a^2), \\
x^2 - (P^2+4)y^2 =& 4(b^2 - Pab - a^2)^2, \; x^2 - (P^2+4)y^2 = -4(b^2 - Pab - a^2)^2, \\
x^2 - Pxy + y^2 =& b^2 - Pab + a^2, \; x^2 - (P^2-4)y^2 = 4(b^2 - Pab + a^2),
\end{aligned}$$

and

$$x^2 - (P^2-4)y^2 = 4(b^2 - Pab + a^2)^2$$

in terms of second order recurrence sequences when $|b^2 - Pab \pm a^2|$ is odd prime. In the second section, we give some identities between the sequence $\{W_n\}$ and its companion sequence $\{X_n\}$. After that, we give our main theorem in the third section.

2. Preliminaries

In this section, we give some identities, theorems, and lemmas, which will be used later. The following identities concerning the sequence $\{W_n\}$ and its companion sequence $\{X_n\}$ hold.

$$X_n^2 - (P^2+4Q)W_n^2 = 4(-Q)^n(b^2 - Pab - Qa^2), \tag{2.1}$$

$$W_{n+1}^2 - PW_{n+1}W_n - QW_n^2 = (-Q)^n(b^2 - Pab - Qa^2), \tag{2.2}$$

$$W_n^2 - PW_{n+1}W_{n-1} = (-Q)^{n-1}(b^2 - Pab - Qa^2), \tag{2.3}$$

$$X_{n+1}^2 - PX_{n+1}X_n - QX_n^2 = -(-Q)^n(P^2+4Q)(b^2 - Pab - Qa^2), \tag{2.4}$$

and

$$X_{n+1}X_{n-1} - X_n^2 = (-Q)^{n-1}(P^2+4Q)(b^2 - Pab - Qa^2). \tag{2.5}$$

One can find the above identities in [2] and [15]. Let

$$W_n^* = bW_n + aQW_{n-1} \quad \text{and} \quad X_n^* = bX_n + aQX_{n-1}. \tag{2.6}$$

Then it can be shown that

$$bW_n - aW_{n+1} = (b^2 - Pab - a^2Q)U_n \quad \text{and} \quad bX_n - aX_{n+1} = (b^2 - Pab - a^2Q)V_n \tag{2.7}$$

$$(X_n^*)^2 - (P^2+4Q)(W_n^*)^2 = 4(-Q)^n(b^2 - Pab - Qa^2)^2, \tag{2.8}$$

$$\left(W_{n+1}^*\right)^2 - PW_{n+1}^*W_n^* - Q(W_n^*)^2 = (-Q)^n(b^2 - Pab - Qa^2)^2, \tag{2.9}$$

and

$$\left(X_{n+1}^*\right)^2 - PX_{n+1}^*X_n^* - Q(X_n^*)^2 = -(-Q)^n(P^2+4Q)(b^2 - Pab - Qa^2)^2 \tag{2.10}$$

by (2.1), (2.2), (2.3), (2.4), and (2.5).

From now on, we write W_n, X_n, U_n, and V_n instead of $W_n(a,b;P,1), X_n(a,b;P,1)$, $U_n(P,1)$, and $V_n(P,1)$, respectively. We represent $W_n(a,b;P,-1), X_n(a,b;P,-1)$, $U_n(P,-1)$, and $V_n(P,-1)$ by w_n, x_n, u_n, and v_n, respectively. We write x_n^* and w_n^* instead of $X_n^*\ (a,b;P,-1)$ and $W_n^*(a,b;P,-1)$, respectively. The following three theorems are given in [7].

Theorem 2.1. *Let u and v be integers. Then $u^2-(P^2+4)v^2 = \pm 4$ if and only if $(u,v) = \mp(V_n, U_n)$ for some $n \in \mathbb{Z}$.*

Theorem 2.2. *Let $P > 3$. Then all integer solutions of the equation $u^2 - (P^2-4)v^2 = 4$ are given by $(u,v) = \mp(v_n, u_n)$ with $n \in \mathbb{Z}$.*

Theorem 2.3. *Let $P > 3$. Then the equation $u^2 - (P^2-4)v^2 = -4$ has no integer solutions.*

3. Main Theorems

3.1. **Solutions of some Diophantine equations for $Q = 1$.** In this subsection, we will assume that $Q = 1, P \geq 1$, and $\Delta = b^2 - Pab - a^2$ such that $|\Delta| > 2$ and $|\Delta|$ is prime.

Theorem 3.1. *Let x and y be integers. Then $x^2-(P^2+4)y^2 = \pm 4\Delta$ if and only if $(x,y) = \pm(X_n, W_n)$ or $\pm((-1)^{n-1}X_n, (-1)^nW_n)$ for some $n \in \mathbb{Z}$.*

Proof. If $(x,y) = \pm(X_n, W_n)$ or $\pm((-1)^{n-1}X_n, (-1)^nW_n)$, then it follows that $x^2 - (P^2+4)y^2 = \pm 4\Delta$ by (2.1). Now let $x^2 - (P^2+4)y^2 = \pm 4\Delta$. Assume that $\Delta|y$. Then $\Delta|x$ and this shows that $\Delta^2|x^2 - (P^2+4)y^2$. Then we get $\Delta^2|4\Delta$, but this is impossible, since $|\Delta| > 2$ and $|\Delta|$ is prime. Therefore $\Delta \nmid y$.

It is obvious that $4\Delta = (2b-Pa)^2 - (P^2+4)a^2$. Thus

$$\Delta|[(2b-Pa)^2 - (P^2+4)a^2] \tag{3.1}$$

and

$$\Delta|[x^2 - (P^2+4)y^2]. \tag{3.2}$$

From (3.1) and (3.2), we get

$$\Delta|[a^2\left(x^2 - (P^2+4)y^2\right) - y^2\left((2b-Pa)^2 - (P^2+4)a^2\right)],$$

i.e.,

$$\Delta|[ax + y(2b-Pa)][ax - y(2b-Pa)].$$

Since $|\Delta|$ is prime, it follows that

$$\Delta|[ax + y(2b-Pa)]$$

or

$$\Delta|[ax - y(2b-Pa)].$$

Also, from (3.1) and (3.2), we get

$$\Delta|[a^2(P^2+4)\left(x^2 - (P^2+4)y^2\right) + x^2\left((2b-Pa)^2 - (P^2+4)a^2\right)],$$

i.e.,

$$\Delta|[(2b-Pa)x - ay(P^2+4)][(2b-Pa)x + ay(P^2+4)].$$

This implies that

$$\Delta|[(2b-Pa)x-ay(P^2+4)]$$

or

$$\Delta|[(2b-Pa)x+ay(P^2+4)].$$

Hence, we have

$$\Delta|[ax+y(2b-Pa)] \text{ and } \Delta|[(2b-Pa)x-ay(P^2+4)] \tag{3.3}$$

or

$$\Delta|[ax-y(2b-Pa)] \text{ and } \Delta|[(2b-Pa)x+ay(P^2+4)], \tag{3.4}$$

and

$$\Delta|[ax-y(2b-Pa)] \text{ and } \Delta|[(2b-Pa)x-ay(P^2+4)] \tag{3.5}$$

or

$$\Delta|[ax+y(2b-Pa)] \text{ and } \Delta|[(2b-Pa)x+ay(P^2+4)]. \tag{3.6}$$

Now assume that (3.3) is satisfied. Then we get

$$\Delta|[x\left(ax+y(2b-Pa)\right)-y\left((2b-Pa)x-ay(P^2+4)\right)],$$

i.e.,

$$\Delta \mid a[x^2+(P^2+4)y^2].$$

This implies that $\Delta|a$ or $\Delta|\left(x^2+(P^2+4)y^2\right)$. Assume that $\Delta|a$. Then $\Delta|b$ since $\Delta=b^2-Pab-a^2$. Thus $\Delta^2|\Delta$ and this shows that $\Delta|1$, but this is impossible. Therefore $\Delta|\left(x^2+(P^2+4)y^2\right)$. Then we see that $\Delta|2(P^2+4)y^2$, since $\Delta|\left(x^2-(P^2+4)y^2\right)$. Hence, $\Delta|2(P^2+4)$, since $\Delta\nmid y$. Then it follows that

$$\Delta|\left[(P^2+4)2ay+(2b-Pa)x-ay(P^2+4)\right],$$

i.e.,

$$\Delta|[(2b-Pa)x+ay(P^2+4)].$$

In this case, (3.3) coincides with (3.6). Similarly, it is seen that (3.4) coincides with (3.5).

Now, let us show that $2|[(2b-Pa)x\pm ay(P^2+4)]$ and $2|[ax\pm y(2b-Pa)]$. It is seen that $x^2\equiv(Py)^2 \pmod 4$ from the equation $x^2-(P^2+4)y^2=\pm4\Delta$. This implies that x and Py have the same parity. Therefore, we see that $2|[(2b-Pa)x\pm ay(P^2+4)]$ and $2|[ax\pm y(2b-Pa)]$.

Consequently, we should examine two cases

$$2\Delta|[(2b-Pa)x-ay(P^2+4)] \quad \text{and} \quad 2\Delta|[ax-y(2b-Pa)] \tag{3.7}$$

and

$$2\Delta|[(2b-Pa)x+ay(P^2+4)] \quad \text{and} \quad 2\Delta|[ax+y(2b-Pa)]. \tag{3.8}$$

Assume that (3.7) is satisfied. Let

$$u=\frac{(2b-Pa)x-ay(P^2+4)}{2\Delta} \quad \text{and} \quad v=\frac{[(2b-Pa)y-ax]}{2\Delta}.$$

Then it follows that

$$\begin{bmatrix} u \\ v \end{bmatrix}=\frac{1}{2\Delta}\begin{bmatrix} 2b-Pa & -a(P^2+4) \\ -a & 2b-Pa \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

and so a simple computation shows that

$$\begin{bmatrix} 2b-Pa & a(P^2+4) \\ a & 2b-Pa \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}=\begin{bmatrix} 2x \\ 2y \end{bmatrix}. \tag{3.9}$$

By using the identities

$$x^2-(P^2+4)y^2=\pm4\Delta \quad \text{and} \quad 4\Delta=(2b-Pa)^2-(P^2+4)a^2,$$

it is seen that

$$u^2-(P^2+4)v^2=\pm4.$$

Thus we have $(u,v) = \mp(V_n, U_n)$ for some $n \in \mathbb{Z}$ by Theorem 2.1. Then $2x = \pm((2b-Pa)\,V_n + a(P^2+4)U_n)$ and $2y = \pm(aV_n + (2b-Pa)\,U_n)$ by (3.9). By using (1.2), (1.3), and (1.4), we get

$$\begin{aligned} x &= \pm\left((2b-Pa)\,V_n + a(P^2+4)U_n\right)/2 = \pm\left(2bV_n - PaV_n + aV_{n+1} + aV_{n-1}\right)/2 \\ &= \pm\left(bV_n + aV_{n-1}\right) = \pm X_n \end{aligned}$$

and

$$\begin{aligned} y &= \pm\left(aV_n + (2b-Pa)\,U_n\right)/2 = \pm\left(aU_{n+1} + aU_{n-1} + 2bU_n - PaU_n\right)/2 \\ &= \pm\left(bU_n + aU_{n-1}\right) = \pm W_n. \end{aligned}$$

Now assume that (3.8) is satisfied. Let

$$u = \frac{(2b-Pa)x + ay(P^2+4)}{2\Delta} \quad \text{and} \quad v = \frac{[(2b-Pa)y + ax]}{2\Delta}.$$

Then we can see by a simple computation that

$$\begin{bmatrix} 2b-Pa & -a(P^2+4) \\ -a & 2b-Pa \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

and

$$u^2 - (P^2+4)v^2 = \pm 4.$$

Thus we have $(u,v) = \mp(V_m, U_m)$ for some $m \in \mathbb{Z}$ by Theorem 2.1. Similarly, it can be shown that $(x,y) = \pm((-1)^m X_{-m}, (-1)^{m+1} W_{-m})$. Taking $n = -m$, it is seen that

$$\begin{aligned} (x,y) &= \pm\left((-1)^{-n} X_n, (-1)^{-n+1} W_n\right) = \pm((-1)^{-n-1} X_n, (-1)^{-n} W_n) \\ &= \pm\left((-1)^{n-1} X_n, (-1)^n W_n\right). \end{aligned}$$ □

From the above theorem and (2.1), the following corollaries can be given.

Corollary 1. *All integer solutions of the equation* $x^2 - (P^2+4)y^2 = 4\Delta$ *are given by* $(x,y) = \pm(X_{2n}, W_{2n})$ *or* $\pm(-X_{2n}, W_{2n})$ *with* $n \in \mathbb{Z}$.

Corollary 2. *All integer solutions of the equation* $x^2 - (P^2+4)y^2 = -4\Delta$ *are given by* $(x,y) = \pm(X_{2n-1}, W_{2n-1})$ *or* $\pm(X_{2n-1}, -W_{2n-1})$ *with* $n \in \mathbb{Z}$.

Theorem 3.2. *Let* x *and* y *be integers. Then* $x^2 - Pxy - y^2 = \pm\Delta$ *if and only if* $(x,y) = \pm(W_{n+1}, W_n)$ *or* $\pm\left((-1)^n W_n, (-1)^{n+1} W_{n+1}\right)$ *for some* $n \in \mathbb{Z}$.

Proof. If $(x,y) = \pm(W_{n+1}, W_n)$ or $\pm\left((-1)^n W_n, (-1)^{n+1} W_{n+1}\right)$, then it follows that $x^2 - Pxy - y^2 = \pm\Delta$ by (2.2). Assume that $x^2 - Pxy - y^2 = \pm\Delta$. Completing the square gives $(2x-Py)^2 - (P^2+4)y^2 = \pm 4\Delta$.This implies that $(2x - Py, y) = \pm(X_n, W_n)$ or $\pm((-1)^{n-1} X_n, (-1)^n W_n)$ for some $n \in \mathbb{Z}$ by Theorem 3.1. If $(2x - Py, y) = \pm(X_n, W_n)$, then we get $(x,y) = \pm(W_{n+1}, W_n)$. If $(2x - Py, y) = \pm((-1)^{n-1} X_n, (-1)^n W_n)$, then $(x,y) = \pm\left((-1)^{n-1} W_{n-1}, (-1)^n W_n\right)$. □

From the above theorem and (2.2), the following corollaries can be given.

Corollary 3. *All integer solutions of the equation* $x^2 - Pxy - y^2 = \Delta$ *are given by* $(x,y) = \pm(W_{2n+1}, W_{2n})$ *or* $\pm(-W_{2n+1}, W_{2n+2})$ *with* $n \in \mathbb{Z}$.

Corollary 4. *All integer solutions of the equation* $x^2 - Pxy - y^2 = -\Delta$ *are given by* $(x,y) = \pm(W_{2n}, W_{2n-1})$ *or* $\pm(W_{2n}, -W_{2n+1})$ *with* $n \in \mathbb{Z}$.

Since $b^2 - 3ab + a^2 = (b-a)^2 - (b-a)a - a^2$, we can give the following corollaries.

Corollary 5. *Let* $|b^2 - 3ab + a^2|$ *be a prime number. Then all integer solutions of the equation* $x^2 - xy - y^2 = b^2 - 3ab + a^2$ *are given by* $(x,y) = \pm(W_{2n+1}, W_{2n})$ *or* $\pm(-W_{2n+1}, W_{2n+2})$ *with* $n \in \mathbb{Z}$, *where* $W_n = W_n(a, b-a, 1, 1)$.

Corollary 6. *Let* $|b^2 - 3ab + a^2|$ *be a prime number. Then all integer solutions of the equation* $x^2 - xy - y^2 = -(b^2 - 3ab + a^2)$ *are given by* $(x,y) = \pm(W_{2n}, W_{2n-1})$ *or* $\pm(W_{2n}, -W_{2n+1})$ *with* $n \in \mathbb{Z}$, *where* $W_n = W_n(a, b-a, 1, 1)$.

Theorem 3.3. *Let P^2+4 be square free. Then $x^2-Pxy-y^2=\pm(P^2+4)\Delta$ for some integers x and y if and only if $(x,y)=\pm(X_{n+1},X_n)$ or $\pm((-1)^nX_{n-1},(-1)^{n-1}X_n)$ for some $n\in\mathbb{Z}$.*

Proof. If $(x,y)=\pm(X_{n+1},X_n)$ or $\pm((-1)^nX_{n-1},(-1)^{n-1}X_n)$, then it follows that $x^2-Pxy-y^2=\pm(P^2+4)\Delta$ by (2.4). Now assume that P^2+4 is square free and $x^2-Pxy-y^2=\pm(P^2+4)\Delta$ for some integers x and y. Then $(2x-Py)^2-(P^2+4)y^2=\pm4(P^2+4)\Delta$. Since P^2+4 is square free, it is seen that $(P^2+4)|(2x-Py)$. Therefore, if we take

$$u=\frac{2x-Py}{P^2+4}\quad\text{and}\quad v=y,$$

then we get $v^2-\left(P^2+4\right)u^2=\pm4\Delta$. This implies that $(v,u)=\mp(X_n,W_n)$ or $\pm((-1)^{n-1}X_n,(-1)^nW_n)$ for some $n\in\mathbb{Z}$ by Theorem 3.1. If $(v,u)=\mp(X_n,W_n)$, then it follows that $y=v=\pm X_n$ and

$$\begin{aligned}x&=\left(\left(P^2+4\right)u+Pv\right)/2=\pm\left(\left(P^2+4\right)W_n+PX_n\right)/2\\&=\pm\left(X_{n+1}+X_{n-1}+PX_n\right)/2\\&=\pm X_{n+1}\end{aligned}$$

by (1.3). Similarly, it can be seen that $(x,y)=\pm((-1)^nX_{n-1},(-1)^{n-1}X_n)$ if $(v,u)=\pm((-1)^{n-1}X_n,(-1)^nW_n)$. □

We can give the following corollaries from the above theorem and (2.4).

Corollary 7. *Let P^2+4 be square free. Then all integer solutions of the equation $x^2-Pxy-y^2=(P^2+4)\Delta$ are given by $(x,y)=\pm(X_{2n+2},X_{2n+1})$ or $\pm(-X_{2n},X_{2n+1})$ with $n\in\mathbb{Z}$.*

Corollary 8. *Let P^2+4 be square free. Then all integer solutions of the equation $x^2-Pxy-y^2=-(P^2+4)\Delta$ are given by $(x,y)=\pm(X_{2n+1},X_{2n})$ or $\pm(X_{2n-1},-X_{2n})$ with $n\in\mathbb{Z}$.*

Theorem 3.4. *Let x and y be integers. Then $x^2-(P^2+4)y^2=\pm4\Delta^2$ if and only if $(x,y)=\pm(X_n^*,W_n^*)$, $\pm((-1)^{n-1}X_n^*,(-1)^nW_n^*)$, or $\pm(\Delta V_n,\Delta U_n)$ for some $n\in\mathbb{Z}$.*

Proof. If $(x,y)=\pm(X_n^*,W_n^*),\pm((-1)^{n-1}X_n^*,(-1)^nW_n^*)$, or $\pm(\Delta V_n,\Delta U_n)$, then it follows that $x^2-(P^2+4)y^2=\pm4\Delta^2$ by (2.1) and (2.8). Let $x^2-(P^2+4)y^2=\pm4\Delta^2$. Now we divide the proof into two cases:

Case I: Assume that $\Delta\nmid y$.

It is obvious that $4\Delta=(2b-Pa)^2-(P^2+4)a^2$. Thus

$$\Delta|[(2b-Pa)^2-(P^2+4)a^2] \tag{3.10}$$

and

$$\Delta|[x^2-(P^2+4)y^2]. \tag{3.11}$$

From (3.10) and (3.11), we get

$$\Delta|[a^2\left(x^2-(P^2+4)y^2\right)-y^2\left((2b-Pa)^2-(P^2+4)a^2\right)],$$

i.e.,

$$\Delta|[ax+y(2b-Pa)][ax-y(2b-Pa)].$$

Since $|\Delta|$ is a prime number, it follows that

$$\Delta|[ax+y(2b-Pa)]$$

or

$$\Delta|[ax-y(2b-Pa)].$$

Also, from (3.10) and (3.11), we get

$$\Delta|[a^2(P^2+4)\left(x^2-(P^2+4)y^2\right)+x^2\left((2b-Pa)^2-(P^2+4)a^2\right)],$$

i.e.,

$$\Delta|[(2b-Pa)x-ay(P^2+4)][(2b-Pa)x+ay(P^2+4)].$$

This implies that

$$\Delta|[(2b-Pa)x-ay(P^2+4)]$$

or

$$\Delta|[(2b-Pa)x+ay(P^2+4)].$$

Hence, we have

$$\Delta|[ax+y(2b-Pa)] \quad \text{and} \quad \Delta|[(2b-Pa)x-ay(P^2+4)] \tag{3.12}$$

or

$$\Delta|[ax-y(2b-Pa)] \text{ and } \Delta|[(2b-Pa)x+ay(P^2+4)] \tag{3.13}$$

and

$$\Delta|[ax-y(2b-Pa)] \quad \text{and} \quad \Delta|[(2b-Pa)x-ay(P^2+4)] \tag{3.14}$$

or

$$\Delta|[ax+y(2b-Pa)] \quad \text{and} \quad \Delta|[(2b-Pa)x+ay(P^2+4)]. \tag{3.15}$$

Now assume that (3.12) is satisfied. Then we get

$$\Delta|[x\,(ax+y(2b-Pa))-y\left((2b-Pa)x-ay(P^2+4)\right)],$$

i.e.,

$$\Delta \mid a[x^2+(P^2+4)y^2].$$

This implies that $\Delta|a$ or $\Delta|\left(x^2+(P^2+4)y^2\right)$. Assume that $\Delta|a$. Then $\Delta|b$, since $\Delta=b^2-Pab-a^2$. Thus $\Delta^2|\Delta$ and this shows that $\Delta|1$, which is impossible. Therefore $\Delta|\left(x^2+(P^2+4)y^2\right)$. Then we see that $\Delta|2(P^2+4)y^2$ since $\Delta|\left(x^2-(P^2+4)y^2\right)$. Hence $\Delta|2(P^2+4)$ since $\Delta\nmid y$. Then it follows that

$$\Delta|\left[(P^2+4)2ay+(2b-Pa)x-ay(P^2+4)\right],$$

i.e.,

$$\Delta|[(2b-Pa)x+ay(P^2+4)].$$

In this case, (3.12) coincides with (3.15). Similarly, it is seen that (3.13) coincides with (3.14).

It can be seen that $2|[(2b-Pa)x\pm ay(P^2+4)]$ and $2|[ax\pm y(2b-Pa)]$.

Consequently, we should examine two cases

$$2\Delta|[(2b-Pa)x-ay(P^2+4)] \quad \text{and} \quad 2\Delta|[ax-y(2b-Pa)] \tag{3.16}$$

and

$$2\Delta|[(2b-Pa)x+ay(P^2+4)] \quad \text{and} \quad 2\Delta|[ax+y(2b-Pa)]. \tag{3.17}$$

Assume that (3.16) is satisfied. Let

$$u=\frac{(2b-Pa)x-ay(P^2+4)}{2\Delta} \quad \text{and} \quad v=\frac{[(2b-Pa)y-ax]}{2\Delta}.$$

Then it follows that

$$\begin{bmatrix} u \\ v \end{bmatrix}=\frac{1}{2\Delta}\begin{bmatrix} 2b-Pa & -a(P^2+4) \\ -a & 2b-Pa \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

and so a simple computation shows that

$$\begin{bmatrix} 2b-Pa & a(P^2+4) \\ a & 2b-Pa \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}=\begin{bmatrix} 2x \\ 2y \end{bmatrix}. \tag{3.18}$$

By using the equalties

$$x^2-(P^2+4)y^2=\pm 4\Delta^2 \quad \text{and} \quad 4\Delta=(2b-Pa)^2-(P^2+4)a^2,$$

it is seen that

$$u^2-(P^2+4)v^2=\pm 4\Delta.$$

Thus we have $(u,v)=\pm(X_n,W_n)$ or $\pm((-1)^{n-1}X_n,(-1)^nW_n)$ for some $n\in\mathbb{Z}$ by Theorem 3.1. If $(u,v)=\pm(X_n,W_n)$, then $2x=\pm((2b-Pa)\,X_n+a(P^2+4)W_n)$ and $2y=\pm(aX_n+(2b-Pa)\,W_n)$ by (3.18). By using (1.2), (1.3), and (2.6), we get

$$\begin{aligned} x &= \pm\left((2b-Pa)\,X_n+a(P^2+4)W_n\right)/2=\pm\left(2bX_n-PaX_n+aX_{n+1}+aX_{n-1}\right)/2 \\ &= \pm\left(bX_n+aX_{n-1}\right)=\pm X_n^* \end{aligned}$$

and

$$y = \pm\left(aX_n + (2b - Pa)\,W_n\right)/2 = \pm\left(aW_{n+1} + aW_{n-1} + 2bW_n - PaW_n\right)/2$$
$$= \pm\left(bW_n + aW_{n-1}\right) = \pm W_n^*.$$

Assume that $(u, v) = \pm\left((-1)^{n-1}X_n, (-1)^n W_n\right)$. Then from (3.18) and (2.7), we get

$$y = \pm\left(a(-1)^{n-1}X_n + (2b - Pa)\,(-1)^n W_n\right)/2$$
$$= \pm\,(-1)^n\left(-aW_{n+1} - aW_{n-1} + 2bW_n - PaW_n\right)/2$$
$$= \pm\,(-1)^n\left(bW_n - aW_{n+1}\right) = \pm(-1)^n \Delta U_n.$$

However, this is impossible since $\Delta \nmid y$.

Now assume that (3.17) is satisfied. Let

$$u = \frac{(2b - Pa)x + ay(P^2 + 4)}{2\Delta} \quad \text{and} \quad v = \frac{[(2b - Pa)y + ax]}{2\Delta}.$$

Then we can see by a simple computation that

$$\begin{bmatrix} 2b - Pa & -a(P^2+4) \\ -a & 2b - Pa \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

and

$$u^2 - (P^2 + 4)v^2 = \pm 4\Delta.$$

Thus we have $(u, v) = \pm(X_n, W_n)$ or $\pm((-1)^{n-1}X_n, (-1)^n W_n)$ for some $n \in \mathbb{Z}$ by Theorem 3.1. Similarly, it can be shown that $(x, y) = \pm((-1)^{n-1}X_n^*, (-1)^n W_n^*)$.

Case II. Assume that $\Delta | y$. Then $\Delta | x$ and therefore

$$(x/\Delta)^2 - (P^2 + 4)(y/\Delta)^2 = \pm 4.$$

Thus we get $(x, y) = \pm(\Delta V_n, \Delta U_n)$ for some integer n by Theorem 2.1. □

Now, we can give the following results by using (2.8) and Theorem 3.4.

Corollary 9. *All integer solutions of the equation $x^2 - (P^2 + 4)y^2 = 4\Delta^2$ are given by $(x, y) = \pm(X_{2n}^*, W_{2n}^*)$, $\pm(-X_{2n}^*, W_{2n}^*)$, or $\pm(\Delta V_{2n}, \Delta U_{2n})$ with $n \in \mathbb{Z}$.*

Corollary 10. *All integer solutions of the equation $x^2 - (P^2 + 4)y^2 = -4\Delta^2$ are given by $(x, y) = \pm(X_{2n+1}^*, W_{2n+1}^*)$, $\pm(X_{2n+1}^*, -W_{2n+1}^*)$, or $\pm(\Delta V_{2n+1}, \Delta U_{2n+1})$ with $n \in \mathbb{Z}$.*

Theorem 3.5. *Let x and y be integers. Then $x^2 - Pxy - y^2 = \pm\Delta^2$ if and only if $(x, y) = \pm(W_{n+1}^*, W_n^*), \pm((-1)^{n-1}W_{n-1}^*, (-1)^n W_n^*)$, or $\pm(\Delta U_{n+1}, \Delta U_n)$ for some $n \in \mathbb{Z}$.*

Proof. If $(x, y) = \pm(W_{n+1}^*, W_n^*), \pm((-1)^{n-1}W_{n-1}^*, (-1)^n W_n^*)$, or $\pm(\Delta U_{n+1}, \Delta U_n)$, then it follows that $x^2 - Pxy - y^2 = \pm\Delta^2$ by (2.2) and (2.9). Assume that $x^2 - Pxy - y^2 = \pm\Delta^2$ for some integers x and y. Then

$$(2x - Py)^2 - (P^2 + 4)y^2 = \pm 4\Delta^2.$$

Taking

$$u = 2x - Py \quad \text{and} \quad v = y, \tag{3.19}$$

we get

$$u^2 - (P^2 + 4)v^2 = \pm 4\Delta^2.$$

Hence, $(u, v) = \pm\,(X_n^*, W_n^*)$, $\pm((-1)^{n-1}X_n^*, (-1)^n W_n^*)$, or $\pm(\Delta V_n, \Delta U_n)$ for some $n \in \mathbb{Z}$ by Theorem 3.4. If $(u, v) = \pm\,(X_n^*, W_n^*)$, then we get $(x, y) = \pm(W_{n+1}^*, W_n^*)$ by (3.19) and (1.2). If $(u, v) = \pm((-1)^{n-1}X_n^*, (-1)^n W_n^*)$, then it is seen that $(x, y) = \pm((-1)^{n-1}W_{n-1}^*, (-1)^n W_n^*)$. If $(u, v) = \pm(\Delta V_n, \Delta U_n)$, it can be shown that $(x, y) = \pm(\Delta U_{n+1}, \Delta U_n)$. □

From (2.9) and Theorem 3.5, we have the following immediate corollaries.

Corollary 11. *All integer solutions of the equation $x^2 - Pxy - y^2 = \Delta^2$ are given by $(x, y) = \pm(W_{2n+1}^*, W_{2n}^*), \pm(-W_{2n-1}^*, W_{2n}^*)$, or $\pm(\Delta U_{2n+1}, \Delta U_{2n})$ with $n \in \mathbb{Z}$.*

Corollary 12. *All integer solutions of the equation* $x^2 - Pxy - y^2 = -\Delta^2$ *are given by* $(x, y) = \pm(W^*_{2n+2}, W^*_{2n+1}), \pm(W^*_{2n}, -W^*_{2n+1})$, *or* $\pm(\Delta U_{2n+2}, \Delta U_{2n+1})$ *with* $n \in \mathbb{Z}$.

Theorem 3.6. *Let* $P^2 + 4$ *be square free. Then* $x^2 - Pxy - y^2 = \pm(P^2 + 4)\Delta^2$ *for some integers* x *and* y *if and only if* $(x, y) = \pm(X^*_{n+1}, X^*_n)$, $\pm((-1)^n X^*_{n-1}, (-1)^{n-1} X^*_n)$, *or* $\pm(\Delta V_{n+1}, \Delta V_n)$ *for some* $n \in \mathbb{Z}$.

Proof. If $(x, y) = \pm(X^*_{n+1}, X^*_n), \pm((-1)^n X^*_{n-1}, (-1)^{n-1} X^*_n)$, or $\pm(\Delta V_{n+1}, \Delta V_n)$, then it follows that $x^2 - Pxy - y^2 = \pm(P^2+4)\Delta^2$ by (2.4) and (2.10). Assume that P^2+4 is square free, and $x^2-Pxy-y^2 = \pm(P^2 + 4)\Delta^2$ for some integers x and y. Then

$$(2x - Py)^2 - (P^2 + 4)y^2 = \pm 4(P^2 + 4)\Delta^2.$$

Since $P^2 + 4$ is square free, we get $(P^2 + 4)|(2x - Py)$. Let

$$u = \frac{2x - Py}{P^2 + 4} \quad \text{and} \quad v = y. \tag{3.20}$$

Then it can be seen that

$$v^2 - \left(P^2 + 4\right) u^2 = \pm 4\Delta^2.$$

This implies that $(v, u) = \pm\ (X^*_n, W^*_n)$, $\pm((-1)^{n-1} X^*_n, (-1)^n W^*_n)$, or $\pm(\Delta V_n, \Delta U_n)$ for some $n \in \mathbb{Z}$ by Theorem 3.4. The result follows from (1.3). □

We can give the following results from (2.5) and the above theorem.

Corollary 13. *Let* $P^2 + 4$ *be square free. Then all integer solutions of the equation* $x^2 - Pxy - y^2 = (P^2 + 4)\Delta^2$ *are given by* $(x, y) = \pm(X^*_{2n}, X^*_{2n-1})$, $\pm(-X^*_{2n}, X^*_{2n+1})$, *or* $\pm(\Delta V_{2n}, \Delta V_{2n-1})$ *with* $n \in \mathbb{Z}$.

Corollary 14. *Let* $P^2 + 4$ *be square free. Then all integer solutions of the equation* $x^2 - Pxy - y^2 = -(P^2+4)\Delta^2$ *are given by* $(x, y) = \pm(X^*_{2n+1}, X^*_{2n})$, $\pm(X^*_{2n-1}, -X^*_{2n})$, *or* $\pm(\Delta V_{2n+1}, \Delta V_{2n})$ *with* $n \in \mathbb{Z}$.

3.2. **Solutions of some Diophantine equations for** $Q = -1$**.** In this subsection, we will assume that $P > 3$, $Q = -1$, and $\Delta = b^2 - Pab + a^2$ such that $|\Delta| > 2$ and $|\Delta|$ is prime.

Theorem 3.7. *All integer solutions of the equation* $x^2 - (P^2 - 4)y^2 = 4\Delta$ *are given by* $(x, y) = \pm(x_n, w_n)$ *or* $\pm(-x_n, w_n)$ *with* $n \in \mathbb{Z}$.

Proof. If $(x, y) = \pm(x_n, w_n)$ or $\pm(-x_n, w_n)$, it follows that $x^2 - (P^2 - 4)y^2 = 4\Delta$ by (2.1). Now let $x^2 - (P^2 - 4)y^2 = 4\Delta$ for some integers x and y. It can be shown that $\Delta \nmid y$.

It is obvious that $4\Delta = (2b - Pa)^2 - (P^2 - 4)a^2$. Thus

$$\Delta|[(2b - Pa)^2 - (P^2 - 4)a^2] \tag{3.21}$$

and

$$\Delta|[x^2 - (P^2 - 4)y^2]. \tag{3.22}$$

From (3.21) and (3.22), we get

$$\Delta|[a^2 \left(x^2 - (P^2 - 4)y^2\right) - y^2 \left((2b - Pa)^2 - (P^2 - 4)a^2\right)],$$

i.e.,

$$\Delta|[ax + y(2b - Pa)][ax - y(2b - Pa)].$$

Since $|\Delta|$ is prime, it follows that

$$\Delta|[ax + y(2b - Pa)]$$

or

$$\Delta|[ax - y(2b - Pa)].$$

Also, from (3.21) and (3.22), we get

$$\Delta|[a^2(P^2 - 4) \left(x^2 - (P^2 - 4)y^2\right) + x^2 \left((2b - Pa)^2 - (P^2 - 4)a^2\right)],$$

i.e.,

$$\Delta|[(2b - Pa)x - ay(P^2 - 4)][(2b - Pa)x + ay(P^2 - 4)].$$

This implies that

$$\Delta|[(2b - Pa)x - ay(P^2 - 4)]$$

or

$$\Delta|[(2b-Pa)x+ay(P^2-4)].$$

Hence, we have

$$\Delta|[ax+y(2b-Pa)] \text{ and } \Delta|[(2b-Pa)x-ay(P^2-4)] \tag{3.23}$$

or

$$\Delta|[ax-y(2b-Pa)] \text{ and } \Delta|[(2b-Pa)x+ay(P^2-4)] \tag{3.24}$$

and

$$\Delta|[ax-y(2b-Pa)] \text{ and } \Delta|[(2b-Pa)x-ay(P^2-4)] \tag{3.25}$$

or

$$\Delta|[ax+y(2b-Pa)] \text{ and } \Delta|[(2b-Pa)x+ay(P^2-4)]. \tag{3.26}$$

Now assume that (3.23) is satisfied. Then we get

$$\Delta|[x\,(ax+y(2b-Pa))-y\left((2b-Pa)x-ay(P^2-4)\right)],$$

i.e.,

$$\Delta \mid a[x^2+(P^2-4)y^2].$$

This implies that $\Delta|a$ or $\Delta|\left(x^2+(P^2-4)y^2\right)$. Assume that $\Delta|a$. Then $\Delta|b$, since $\Delta=b^2-Pab+a^2$. Thus $\Delta^2|\Delta$ and this shows that $\Delta|1$, which is impossible. Therefore $\Delta|\left(x^2+(P^2-4)y^2\right)$. Then we see that $\Delta|2(P^2-4)y^2$ since $\Delta|\left(x^2-(P^2-4)y^2\right)$. Hence, $\Delta|2(P^2-4)$, since $\Delta\nmid y$. Then it follows that

$$\Delta|\left[(P^2-4)2ay+(2b-Pa)x-ay(P^2-4)\right],$$

i.e.,

$$\Delta|[(2b-Pa)x+ay(P^2-4)].$$

In this case, (3.23) coincides with (3.26). Similarly, it is seen that (3.24) coincides with (3.25).

Now, let us show that $2|[(2b-Pa)x\pm ay(P^2-4)]$ and $2|[ax\pm y(2b-Pa)]$. It is seen that $x^2\equiv(Py)^2$ (mod4) from the equation $x^2-(P^2-4)y^2=4\Delta$. This implies that x and Py have the same parity. Therefore, we see that $2|[(2b-Pa)x\pm ay(P^2-4)]$ and $2|[ax\pm y(2b-Pa)]$.

Consequently, we should examine two cases

$$2\Delta|[(2b-Pa)x-ay(P^2-4)] \quad \text{and} \quad 2\Delta|[ax-y(2b-Pa)] \tag{3.27}$$

and

$$2\Delta|[(2b-Pa)x+ay(P^2-4)] \quad \text{and} \quad 2\Delta|[ax+y(2b-Pa)]. \tag{3.28}$$

Assume that (3.27) is satisfied. Let

$$u=\frac{(2b-Pa)x-ay(P^2-4)}{2\Delta} \quad \text{and} \quad v=\frac{[(2b-Pa)y-ax]}{2\Delta}.$$

Then it follows that

$$\begin{bmatrix} u \\ v \end{bmatrix}=\frac{1}{2\Delta}\begin{bmatrix} 2b-Pa & -a(P^2-4) \\ -a & 2b-Pa \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

and so a simple computation shows that

$$\begin{bmatrix} 2b-Pa & a(P^2-4) \\ a & 2b-Pa \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}=\begin{bmatrix} 2x \\ 2y \end{bmatrix}. \tag{3.29}$$

Since $x^2-(P^2-4)y^2=4\Delta$, using the equalty

$$4\Delta=(2b-Pa)^2-(P^2-4)a^2,$$

it is seen that

$$u^2-(P^2-4)v^2=4.$$

Thus we have $(u,v)=\mp(v_n,u_n)$ for some $n\in\mathbb{Z}$ by Theorem 2.2. Then $2x=\pm((2b-Pa)\,v_n+a(P^2-4)u_n)$ and $2y=\pm(av_n+(2b-Pa)\,u_n)$ by (3.29). By using (1.2), (1.3), and (1.4), we get

$$\begin{aligned} x&=\pm\left((2b-Pa)\,v_n+a(P^2-4)u_n\right)/2=\pm\left(2bv_n-Pav_n+av_{n+1}-av_{n-1}\right)/2 \\ &=\pm\left(bv_n-av_{n-1}\right)=\pm x_n \end{aligned}$$

and

$$\begin{aligned} y &= \pm\left(av_n + (2b - Pa)\,u_n\right)/2 = \pm\left(au_{n+1} - au_{n-1} + 2bu_n - Pau_n\right)/2 \\ &= \pm\left(bu_n - au_{n-1}\right) = \pm w_n. \end{aligned}$$

Now assume that (3.28) is satisfied. Let

$$u = \frac{(2b - Pa)x + ay(P^2 - 4)}{2\Delta} \quad \text{and} \quad v = \frac{[(2b - Pa)y + ax]}{2\Delta}.$$

Then we can see by a simple computation that

$$\begin{bmatrix} 2b - Pa & -a(P^2 - 4) \\ -a & 2b - Pa \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

and

$$u^2 - (P^2 - 4)v^2 = 4,$$

since $x^2 - (P^2 - 4)y^2 = 4\Delta$. Thus we have $(u, v) = \mp(v_n, u_n)$ for some $n \in \mathbb{Z}$ by Theorem 2.2. Similarly, it can be shown that $(x, y) = \pm(x_{-n}, -w_{-n})$. □

Since the equation $x^2 - (P^2 - 4)y^2 = -4$ has no integer solutions by Theorem 2.3, using the same argument in the proof of the above theorem, we can give the following theorem.

Theorem 3.8. *The equation $x^2 - (P^2 - 4)y^2 = -4\Delta$ has no integer solutions.*

Corollary 15. *The equation $x^2 - Pxy + y^2 = -\Delta$ has no integer solutions.*

Proof. Assume that $x^2 - Pxy + y^2 = -\Delta$ for some integers x and y. Completing the square gives $(2x - Py)^2 - (P^2 - 4)y^2 = -4\Delta$, which is impossible by Theorem 3.8. □

We can give the following corollaries from Theorem 3.8.

Corollary 16. *The equation $x^2 - (P^2 - 4)y^2 = -\Delta$ has no integer solutions.*

Corollary 17. *Let P be odd. Then the equation $x^2 - (P^2 - 4)y^2 = -16\Delta$ has no integer solutions.*

Proof. Assume that P is odd and $x^2 - (P^2 - 4)y^2 = -16\Delta$ for some integers x and y. Then it is seen that x and y are even and this implies that $(x/2)^2 - (P^2 - 4)(y/2)^2 = -4\Delta$, which is impossible by Theorem 3.8. □

Corollary 18. *Let P be odd. Then the equation $x^2 - Pxy + y^2 = -4\Delta$ has no integer solutions.*

Proof. Since $x^2 - Pxy + y^2 = -4\Delta$ if and only if $(2x - Py)^2 - (P^2 - 4)y^2 = -16\Delta$, the proof follows. □

Theorem 3.9. *All integer solutions of the equation $x^2 - Pxy + y^2 = \Delta$ are given by $(x, y) = \pm(w_{n+1}, w_n)$ with $n \in \mathbb{Z}$.*

Proof. If $(x, y) = \pm(w_{n+1}, w_n)$, then it follows that $x^2 - Pxy + y^2 = \Delta$ by (2.2). Assume that $x^2 - Pxy + y^2 = \Delta$. Completing the square gives $(2x - Py)^2 - (P^2 - 4)y^2 = 4\Delta$. This implies that $(2x - Py, y) = \pm(x_n, w_n)$ or $\pm(x_n, -w_n)$ for some $n \in \mathbb{Z}$ by Theorem 3.7. Hence, $(x, y) = \pm(w_{n+1}, w_n)$ or $\pm(w_{n-1}, w_n)$. Since the role of x and y is symmetric, the proof follows. □

Theorem 3.10. *Let $P^2 - 4$ be square free. Then all integer solutions of the equation $x^2 - Pxy + y^2 = -(P^2 - 4)\Delta$ are given by $(x, y) = \pm(x_{n+1}, x_n)$ with $n \in \mathbb{Z}$.*

Proof. If $(x, y) = \pm(x_{n+1}, x_n)$, then it follows that $x^2 - Pxy + y^2 = -(P^2 - 4)\Delta$ by (2.4). Now assume that $P^2 - 4$ is square free and $x^2 - Pxy + y^2 = -(P^2 - 4)\Delta$ for some integers x and y. Then $(2x - Py)^2 - (P^2 - 4)y^2 = -4(P^2 - 4)\Delta$. Since $P^2 - 4$ is square free, it is seen that $(P^2 - 4)|(2x - Py)$. Therefore, taking

$$u = \frac{2x - Py}{P^2 - 4} \quad \text{and} \quad v = y,$$

we get $v^2 - \left(P^2 - 4\right)u^2 = 4\Delta$. This implies that $(v, u) = \pm(x_n, w_n)$ or $\pm(-x_n, w_n)$ for some $n \in \mathbb{Z}$ by Theorem 3.7. Hence, the proof follows from (1.2) and (1.3). □

From Theorem 3.8, we can give the following corollary.

Corollary 19. *Let P^2-4 be square free. Then the equation $x^2-Pxy+y^2=(P^2-4)\Delta$ has no integer solutions.*

Since the proof of the following theorem is similar to that of Theorem 3.4, we omit it.

Theorem 3.11. *All integer solutions of the equation $x^2-(P^2-4)y^2=4\Delta^2$ are given by $(x,y)=\pm(x_n^*,w_n^*),\pm(-x_n^*,w_n^*)$, or $\pm(\Delta v_n,\Delta u_n)$ with $n\in\mathbb{Z}$.*

Corollary 20. *All integer solutions of the equation $x^2-Pxy+y^2=\Delta^2$ are given by $(x,y)=\pm(w_{n+1}^*,w_n^*)$ or $\pm(\Delta u_{n+1},\Delta u_n)$ with $n\in\mathbb{Z}$.*

Theorem 3.12. *The equation $x^2-(P^2-4)y^2=-4\Delta^2$ has no integer solutions.*

Proof. If we follow the way as in the proof of Theorem 3.4, then we have the equation $u^2-(P^2-4)v^2=-4\Delta$, where $u=\left[(2b-Pa)x+ay(P^2-4)\right]/2\Delta$ and $v=\left[(2b-Pa)y+ax\right]/2\Delta$ or $u=\left[(2b-Pa)x-ay(P^2-4)\right]/2\Delta$ and $v=\left[(2b-Pa)y-ax\right]/2\Delta$. Since the equation $u^2-(P^2-4)v^2=-4\Delta$ is impossible by Theorem 3.8, the equation $x^2-(P^2-4)y^2=-4\Delta^2$ has no integer solutions. □

Corollary 21. *Let P^2-4 be square free. Then all integer solutions of the equation $x^2-Pxy+y^2=-(P^2-4)\Delta^2$ are given by $(x,y)=\pm(x_{n+1}^*,x_n^*)$ or $\pm(\Delta v_{n+1},\Delta v_n)$ with $n\in\mathbb{Z}$.*

Corollary 22. *The equation $x^2-Pxy+y^2=-\Delta^2$ has no integer solutions.*

Corollary 23. *The equation $x^2-(P^2-4)y^2=-\Delta^2$ has no integer solutions.*

Corollary 24. *Let P be odd. Then the equation $x^2-(P^2-4)y^2=-16\Delta^2$ has no integer solutions.*

Corollary 25. *Let P be odd. Then the equation $x^2-Pxy+y=-4\Delta^2$ has no integer solutions.*

Corollary 26. *Suppose that $|b^2-ba-a^2|$ is prime. Then all integer solutions of the equation $x^2-3xy+y^2=b^2-ba-a^2$ are given by $(x,y)=\pm(W_{2n+2},W_{2n})$ with $n\in\mathbb{Z}$, where $W_n=W_n(a,b;1,1)$.*

Proof. Suppose that $(x,y)=\pm(W_{2n+2},W_{2n})$. Then it is easy to see that

$$\begin{aligned}W_{2n+2}^2-3W_{2n+2}W_{2n}+W_{2n}^2 =&(W_{2n+2}-W_{2n})^2-(W_{2n+2}-W_{2n})W_{2n}-W_{2n}^2\\ =&W_{2n+1}^2-W_{2n+1}W_{2n}-W_{2n}^2=b^2-ba-a^2\end{aligned}$$

by (2.2). Now suppose that $x^2-3xy+y^2=b^2-ba-a^2$ for some integers x and y. Then $(x-y)^2-y(x-y)-y^2=b^2-ba-a^2$ and therefore $(x-y,y)=\pm(W_{2n+1},W_{2n})$ or $\pm(-W_{2n+1},W_{2n+2})$ for some $n\in\mathbb{Z}$ by Corollary 3, where $W_n=W_n(a,b;1,1)$. If $(x-y,y)=\pm(W_{2n+1},W_{2n})$, then $(x,y)=\pm(W_{2n+2},W_{2n})$. If $(x-y,y)=\pm(-W_{2n+1},W_{2n+2})$, then $(x,y)=\pm(W_{2n+2},W_{2n})$. Since the role of x and y is symmetric, the proof follows. □

The following corollary can be proved in a similar way.

Corollary 27. *Suppose that $|b^2-ba-a^2|$ is prime. Then all integer solutions of the equation $x^2-3xy+y^2=-(b^2-ba-a^2)$ are given by $(x,y)=\pm(W_{2n+1},W_{2n-1})$ with $n\in\mathbb{Z}$, where $W_n=W_n(a,b;1,1)$.*

Corollary 28. *Suppose that $|b^2-3ba+a^2|$ is prime. Then all integer solutions of the equation $x^2-3xy+y^2=b^2-3ba+a^2$ are given by $(x,y)=\pm(W_{2n+2},W_{2n})$ with $n\in\mathbb{Z}$, where $W_n=W_n(a,b-a;1,1)$.*

Proof. Suppose that $(x,y)=\pm(W_{2n+2},W_{2n})$ with $W_n=W_n(a,b-a;1,1)$. Then it can be seen that

$$\begin{aligned}W_{2n+2}^2-3W_{2n+2}W_{2n}+W_{2n}^2 =&(W_{2n+2}-W_{2n})^2-(W_{2n+2}-W_{2n})W_{2n}-W_{2n}^2\\ =&W_{2n+1}^2-W_{2n+1}W_{2n}-W_{2n}^2\\ =&(b-a)^2-(b-a)a-a^2\\ =&b^2-3ba+a^2\end{aligned}$$

by (2.2). Now suppose that $x^2 - 3xy + y^2 = b^2 - 3ba + a^2$ for some integers x and y. Then $(x-y)^2 - y(x-y) - y^2 = (b-a)^2 - a(b-a) - a^2$ and therefore $(x-y, y) = \pm(W_{2n+1}, W_{2n})$ or $(x-y, y) = \pm(-W_{2n+1}, W_{2n+2})$ for some $n \in \mathbb{Z}$ by Corollary 3, where $W_n = W_n(a, b-a; 1, 1)$. Let $(x-y, y) = \pm(W_{2n+1}, W_{2n})$. Then $y = \pm W_{2n}$ and $x - y = \pm W_{2n+1}$, which implies that $x = \pm(W_{2n+1} + W_{2n}) = \pm W_{2n+2}$. Let $(x-y, y) = \pm(-W_{2n+1}, W_{2n+2})$. Then $y = \pm W_{2n+2}$ and $x - y = \pm(-W_{2n+1})$. Thus $x = \pm(-W_{2n+1} + W_{2n+2}) = \pm W_{2n}$. This completes the proof. □

Corollary 29. *Suppose that* $|b^2 - 3ba + a^2|$ *is prime. Then all integer solutions of the equation* $x^2 - 3xy + y^2 = -(b^2 - 3ba + a^2)$ *are given by* $(x, y) = \pm(W_{2n+1}, W_{2n-1})$ *with* $n \in \mathbb{Z}$, *where* $W_n = W_n(a, b-a; 1, 1)$.

By using Corollaries 1 and 2, we can give the following corollaries.

Corollary 30. *Suppose that* $|b^2 - 3ba + a^2|$ *is prime. Then all integer solutions of the equation* $x^2 - 5y^2 = 4(b^2 - 3ba + a^2)$ *are given by* $(x, y) = \pm(X_{2n}, W_{2n})$ *or* $\pm(-X_{2n}, W_{2n})$ *with* $n \in \mathbb{Z}$, *where* $W_n = W_n(a, b-a; 1, 1)$ *and* $X_n = X_n(a, b-a; 1, 1)$.

Corollary 31. *Suppose that* $|b^2 - 3ba + a^2|$ *is prime. Then all integer solutions of the equation* $x^2 - 5y^2 = -4(b^2 - 3ba + a^2)$ *are given by* $(x, y) = \pm(X_{2n-1}, W_{2n-1})$ *or* $\pm(X_{2n-1}, -W_{2n-1})$ *with* $n \in \mathbb{Z}$, *where* $W_n = W_n(a, b-a; 1, 1)$ *and* $X_n = X_n(a, b-a; 1, 1)$.

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