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A NOTE ON THE BILINEAR FRACTIONAL INTEGRAL OPERATOR ACTING ON MORREY SPACES

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Abstract. The boundedness of the bilinear fractional integral operator is investigated. This bilinear fractional integral operator goes back to Kenig and Stein. The paper is oriented to the boundedness of the operator on products of Morrey spaces. This paper uses some averaging techniques to control the Morrey norm. Compared to the earlier work by He and Yan, one feels that the technique can be applied to other function spaces. Among others, the averaging operator will reduce the matters to the existing results.

1. Introduction

Recently, He and Yan investigated fractional integral operators of Grafakos type acting on Morrey spaces [8]. In this paper, by using some known results, we propose to simplify their proofs. Let $0 < q \le p < \infty$. Define the *Morrey norm* $\|\cdot\|_{\mathcal{M}^p_q}$ by

$$\|f\|_{\mathcal{M}^p_q} \equiv \sup\left\{|Q|^{\frac{1}{p}-\frac{1}{q}}\|f\|_{L^q(Q)} \,:\, Q \text{ is a dyadic cube in } \mathbb{R}^n\right\}$$

for a measurable function f.

The *Morrey space* $\mathcal{M}^p_q(\mathbb{R}^n)$ is the set of all the measurable functions f for which $\|f\|_{\mathcal{M}^p_q}$ is finite. We recall the definition of the dyadic cubes precisely in Section 2. A simple geometric observation shows that

$$\|f\|_{\mathcal{M}^p_q} \sim \sup\left\{|Q|^{\frac{1}{p}-\frac{1}{q}}\|f\|_{L^q(Q)} \,:\, Q \text{ is a cube in } \mathbb{R}^n\right\}$$

for any measurable function f. Here let us content ourselves with the intuitive understanding that p serves as the global integrability, as is hinted by the dilation mapping $f \mapsto f(t\cdot)$, and that q serves as the local integrability. We handle the following bilinear operator defined in [5, 13].

Definition 1.1. The *bilinear fractional integral operator of Grafakos type* $\mathcal{J}_\alpha$, $0 < \alpha < n$ is given by

$$\mathcal{J}_\alpha[f_1, f_2](x) \equiv \int_{\mathbb{R}^n} \frac{f_1(x+y)\, f_2(x-y)}{|y|^{n-\alpha}}\, dy \quad (x \in \mathbb{R}^n),$$

where f_1, f_2 are non-negative integrable functions defined in $\mathbb{R}^n$.

The operator $\mathcal{I}_\alpha[f_1, f_2]$, $0 < \alpha < 2n$, defined by

$$\mathcal{I}_\alpha[f_1, f_2](x) \equiv \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{f_1(y_1)\, f_2(y_2)}{\big(|x-y_1| + |x-y_2|\big)^{2n-\alpha}}\, dy \quad (x \in \mathbb{R}^n)$$

for non-negative integrable functions f_1 and f_2 defined in $\mathbb{R}^n$, is a contrast to $\mathcal{J}_\alpha[f_1, f_2]$. These two operators with $0 < \alpha < n$ pass the fractional integral operator I_α in the bilinear case, where I_α is the fractional integral operator

$$I_\alpha f(x) \equiv \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy \quad (x \in \mathbb{R}^n)$$

for a nonnegative measurable function $f : \mathbb{R}^n \to [0, \infty]$.

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Here and below we assume that the functions are non-negative to ignore the issue of the convergence of the integral definining $\mathcal{J}_\alpha[f_1, f_2](x)$.

The operator $\mathcal{I}_\alpha[f_1, f_2]$ acting on Morrey spaces is investigated by many authors in many settings such as the generalized Morrey spaces [1], the weighted setting [6, 10], the case equipped with the rough kernel [9, 20] and the non-doubling setting [11, 21]. See also [3, 22] for the case of commutators generated by $\mathcal{I}_\alpha$ and other functions. However we do not find so much about the action of the operator $\mathcal{J}_\alpha$ on Morrey spaces. Works [4, 8] considered the boundedness property of $\mathcal{J}_\alpha$. Our aim here is to prove the following estimate

Theorem 1.2. *Let*

$$0 < \alpha < n, \quad 1 < q_1 \le p_1 < \infty, \quad 1 < q_2 \le p_2 < \infty, \quad 1 \le t \le s < \infty.$$

Define p and q by

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Assume that

$$\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}, \quad \frac{q}{p} = \frac{t}{s}, \quad s < \min(q_1, q_2).$$

Then for all $f_1 \in \mathcal{M}^{p_1}_{q_1}(\mathbb{R}^n)$ and $f_2 \in \mathcal{M}^{p_2}_{q_2}(\mathbb{R}^n)$,

$$\|\mathcal{J}_\alpha[f_1, f_2]\|_{\mathcal{M}^s_t} \lesssim \|f_1\|_{\mathcal{M}^{p_1}_{q_1}} \|f_2\|_{\mathcal{M}^{p_2}_{q_2}}.$$

As is pointed out in [8], the assumption $\frac{q}{p} = \frac{t}{s}$ is essential.

Theorem 1.2 partially extends the following result by Kenig and Stein [13, Theorem 2]

Proposition 1.3. *Let $0 < \alpha < n$ and $1 < p_1, p_2 < \infty$. Assume that $\dfrac{1}{p_1} + \dfrac{1}{p_2} > \dfrac{\alpha}{n}$, so we can define $s > 0$ by $\dfrac{1}{s} = \dfrac{1}{p_1} + \dfrac{1}{p_2} - \dfrac{\alpha}{n}$. Then for all $f_1 \in L^{p_1}(\mathbb{R}^n)$ and $f_2 \in L^{p_2}(\mathbb{R}^n)$,*

$$\|\mathcal{J}_\alpha[f_1, f_2]\|_{L^s} \lesssim \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}.$$

In the first half of [8], He and Yan proved the boundedness of the operator and used the Hölder inequality under the assumption

$$\frac{q_1}{p_1} = \frac{q_2}{p_2}, \quad \frac{1}{\max\left(q_1', \frac{\alpha}{n} p_1\right)} + \frac{1}{\max\left(q_2', \frac{\alpha}{n} p_2\right)} > 1, \tag{1.1}$$

so

$$\frac{q}{p} = \frac{q_1}{p_1} = \frac{q_2}{p_2}$$

and there exists $u \in (1, \infty)$ such that

$$\frac{\alpha}{n} p_1 < u < \left(\frac{\alpha}{n} p_2\right)', \quad (q_2)' < u < q_1.$$

Define s_1, s_2, t_1 and t_2 by

$$\frac{u}{s_1} = \frac{u}{p_1} - \frac{\alpha}{n}, \quad \frac{u'}{s_2} = \frac{u'}{p_2} - \frac{\alpha}{n}, \quad \frac{t_1}{s_1} = \frac{q_1}{p_1}, \quad \frac{t_2}{s_2} = \frac{q_2}{p_2},$$

so $1 < t_1 \le s_1 < \infty$ and $1 < t_2 \le s_2 < \infty$. Then

$$\frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2}, \quad \frac{1}{t} = \frac{1}{t_1} + \frac{1}{t_2},$$

since

$$\frac{q}{p} = \frac{q_1}{p_1} = \frac{q_2}{p_2}.$$

Meanwhile, by the Hölder inequality we have

$$\mathcal{J}_\alpha[f_1, f_2](x) \le \left(\int_{\mathbb{R}^n} \frac{|f_1(x+y)|^u}{|y|^{n-\alpha}} dy\right)^{\frac{1}{u}} \left(\int_{\mathbb{R}^n} \frac{|f_2(x-y)|^{u'}}{|y|^{n-\alpha}} dy\right)^{\frac{1}{u'}}$$

for any $1 < u < \infty$. Consequently, by the Hölder inequality once again, we obtain

$$\left\|\mathcal{J}_\alpha[f_1, f_2]\right\|_{\mathcal{M}^s_t} \le \left\|I^{(u)}_\alpha f_1\right\|_{\mathcal{M}^{s_1}_{t_1}} \left\|I^{(u')}_\alpha f_2\right\|_{\mathcal{M}^{s_2}_{t_2}},$$

where $I^{(v)}_\alpha f \equiv (I_\alpha[|f|^v])^{\frac{1}{v}}$. If we use the Adams theorem, asserting that $I^{(v)}_\alpha$ maps $\mathcal{M}^P_Q(\mathbb{R}^n)$ boundedly to $\mathcal{M}^S_T(\mathbb{R}^n)$ whenever $v < Q \le P < \infty$, $v < T \le S < \infty$, $\frac{v}{S} = \frac{v}{P} - \frac{\alpha}{n}$ and $\frac{P}{Q} = \frac{S}{T}$, we obtain

$$\left\|\mathcal{J}_\alpha[f_1, f_2]\right\|_{\mathcal{M}^s_t} \lesssim \left\|f_1\right\|_{\mathcal{M}^{p_1}_{q_1}} \left\|f_2\right\|_{\mathcal{M}^{p_2}_{q_2}}.$$

Thus Theorem 1.2 is significant when (1.1) fails. See [4, Theorem 2.2] for the bilinear fractional integral operator of Kenig–Stein type equipped with the rough kernel.

The operator $\mathcal{J}_\alpha$ has a lot to do with the bilinear Hilbert transform defined by

$$\mathcal{H}[f_1, f_2](x) = \lim_{\varepsilon \downarrow 0} \int\limits_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \frac{f_1(x+y)\, f_2(x-y)}{y}\, dy \quad (x \in \mathbb{R}),$$

where f_1 and f_2 are the locally integrable functions. One of the important problems in harmonic analysis is to investigate the boundedness property of the bilinear Hilbert transform. A conjecture of Calderón in 1964 concerned possible extensions of $\mathcal{H}$ to a bounded bilinear operator on products of Lebesgue spaces. A remarkable fact is that $\mathcal{H}$ maps $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ to $L^p(\mathbb{R})$ boundedly if $1 < p_1 \le \infty$, $1 < p_2 \le \infty$, $\dfrac{2}{3} < p < \infty$ and $\dfrac{1}{p} = \dfrac{1}{p_1} + \dfrac{1}{p_2}$ [16, 17]. To understand the boundedness property of this operator, we consider its counterpart to fractional integral operators.

After the authors wrote this article, we were aware that Theorem 1.2 is an unweighted version of [8, Theorem 4.6]. However, our proof differs from theirs in that we use an inequality for norms, while He and Yan used a weighted local estimate. It seems that our results can be extended to Orlicz spaces by using [19, Theorem 7.5]. The details are left for the future works.

2. Preliminaries

For a measurable function f defined on $\mathbb{R}^n$, define a function Mf by

$$Mf(x) \equiv \sup_{B \in \mathcal{B}} \frac{\chi_B(x)}{|B|} \int\limits_B \left|f(y)\right| dy \quad (x \in \mathbb{R}^n). \tag{2.1}$$

The mapping $M : f \mapsto Mf$ is called the *Hardy–Littlewood maximal operator*. It is known that the Hardy–Littlewood maximal operator is bounded on $\mathcal{M}^p_q(\mathbb{R}^n)$ if $1 < q \le p < \infty$ [2]. A dyadic cube is a set of the form Q_{jk} for some $j \in \mathbb{Z}, k = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$. The set of all dyadic cubes is denoted by $\mathcal{D}$; $\mathcal{D} \equiv \{Q_{jk} : j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$. For $j \in \mathbb{Z}$, the set of dyadic cubes of the j-th generation is given by

$$\mathcal{D}_j = \mathcal{D}_j(\mathbb{R}^n) \equiv \left\{Q_{jk} : k \in \mathbb{Z}^n\right\} = \left\{Q \in \mathcal{D} : \ell(Q) = 2^{-j}\right\}.$$

The following lemma can be located as a standard estimate to handle this bilinear fractional integral operator.

Lemma 2.1. *Let $f_1, f_2 \ge 0$ be measurable functions. Then we have*

$$\mathcal{J}_\alpha[f_1, f_2](x) \lesssim \sum_{l=-\infty}^{\infty} \sum_{Q \in \mathcal{D}_l} 2^{l(n-\alpha)} \chi_Q(x) \int\limits_{B(2^{-l})} f_1(x+y)\, f_2(x-y)\, dy \quad (x \in \mathbb{R}^n).$$

Proof. We will follow the idea used in [13, Theorem 2]. See also [18, Theorem 3.2] and [14, 15]. We decompose

$$\begin{aligned}
\mathcal{J}_\alpha[f_1,f_2](x) &= \int_{\mathbb{R}^n} \frac{f_1(x+y)\,f_2(x-y)}{|y|^{n-\alpha}}\,dy \\
&= \sum_{l=-\infty}^{\infty} \int_{B(2^{-l})\setminus B(2^{-l-1})} \frac{f_1(x+y)f_2(x-y)}{|y|^{n-\alpha}}\,dy \\
&\sim \sum_{l=-\infty}^{\infty} 2^{l(n-\alpha)} \int_{B(2^{-l})\setminus B(2^{-l-1})} f_1(x+y)f_2(x-y)\,dy \\
&\le \sum_{l=-\infty}^{\infty} 2^{l(n-\alpha)} \int_{B(2^{-l})} f_1(x+y)f_2(x-y)\,dy.
\end{aligned}$$

Observe that for each $l \in \mathbb{N}$, there uniquely exists a dyadic cube $Q \in \mathcal{D}_l$ such that $x \in Q$. Thus, we obtain the desired result. $\square$

We now recall the averaging technique.

Lemma 2.2. *Suppose that the parameters p,q,s,t satisfy*

$$1 < q \le p < \infty, \quad 1 < t \le s < \infty, \quad q < t, \quad p < s$$

or

$$1 = q \le p < \infty, \quad 1 = t \le s < \infty, \quad p < s.$$

Assume that $\{Q_j\}_{j=1}^\infty \subset \mathcal{D}(\mathbb{R}^n)$, $\{a_j\}_{j=1}^\infty \subset \mathcal{M}^s_t(\mathbb{R}^n)$ and $\{\lambda_j\}_{j=1}^\infty \subset [0,\infty)$ fulfill

$$\mathrm{supp}(a_j) \subset Q_j, \quad \left\| \sum_{j=1}^\infty \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}^p_q} < \infty. \tag{2.2}$$

Then $f = \sum_{j=1}^\infty \lambda_j a_j$ converges in $\mathcal{S}'(\mathbb{R}^n) \cap L^q_{\mathrm{loc}}(\mathbb{R}^n)$ and satisfies

$$\|f\|_{\mathcal{M}^p_q} \lesssim_{p,q,s,t} \left\| \sum_{j=1}^\infty \lambda_j \frac{\|a_j\|_{\mathcal{M}^s_t}}{|Q_j|^{\frac{1}{s}}} \chi_{Q_j} \right\|_{\mathcal{M}^p_q}. \tag{2.3}$$

Proof. This estimate is essentially obtained in [12] if $q > 1$ and in [7] if $q = 1$. Although we distinguished these cases in these papers, we can combine them, since the case of $q = 1$ can almost be emerged into the case of $q > 1$.

Let us suppose $q > 1$ for the time being. Let $0 < \eta < \infty$. We will use the *powered Hardy–Littlewood maximal operator $M^{(\eta)}$* defined by

$$M^{(\eta)} f(x) \equiv \sup_{R>0} \left(\frac{1}{|B(x,R)|} \int_{B(x,R)} |f(y)|^\eta dy \right)^{\frac{1}{\eta}}$$

for a measurable function $f : \mathbb{R}^n \to \mathbb{C}$. If $\eta = 1$, then we write M instead of $M^{(\eta)}$. To prove this, we resort to the duality. For the time being, we assume that there exists $N \in \mathbb{N}$ such that $\lambda_j = 0$ whenever $j \ge N$. Let us assume in addition that the a_j's are non-negative. Fix a non-negative function g that is supported on a cube Q such that $\|g\|_{L^{q'}} \le |Q|^{\frac{1}{q'}-\frac{1}{p'}}$. By duality, we will show

$$\int_{\mathbb{R}^n} f(x)\,g(x)\,dx \lesssim_{p,q,s,t} \left\| \sum_{j=1}^\infty \lambda_j \frac{\|a_j\|_{\mathcal{M}^s_t}}{|Q_j|^{\frac{1}{s}}} \chi_{Q_j} \right\|_{\mathcal{M}^p_q}, \tag{2.4}$$

to obtain

$$\|f\|_{\mathcal{M}^p_q} \lesssim_{p,q,s,t} \left\| \sum_{j=1}^\infty \lambda_j \frac{\|a_j\|_{\mathcal{M}^s_t}}{|Q_j|^{\frac{1}{s}}} \chi_{Q_j} \right\|_{\mathcal{M}^p_q}.$$

Assume first that each Q_j contains Q as a proper subset. If we group the j's such that Q_j are identical, we can assume that each Q_j is a j-th parent of Q for each $j \in \mathbb{N}$. Then we have

$$\int_{\mathbb{R}^n} f(x)\, g(x)\, dx = \sum_{j=1}^\infty \lambda_j \int_Q a_j(x)\, g(x)\, dx \le \sum_{j=1}^\infty \lambda_j \|a_j\|_{L^q(Q)} \|g\|_{L^{q'}(Q)}$$

from $f = \sum_{j=1}^\infty \lambda_j a_j$. By the size condition of a_j and g, we obtain

$$\int_{\mathbb{R}^n} f(x)\, g(x)\, dx \le \sum_{j=1}^\infty \lambda_j \|a_j\|_{\mathcal{M}^s_t} |Q|^{\frac{1}{q}-\frac{1}{s}} |Q|^{\frac{1}{q'}-\frac{1}{p'}} = \sum_{j=1}^\infty \lambda_j \|a_j\|_{\mathcal{M}^s_t} |Q|^{\frac{1}{p}-\frac{1}{s}}.$$

Note that

$$\left\| \sum_{j=1}^\infty \lambda_j \frac{\|a_j\|_{\mathcal{M}^s_t}}{|Q_j|^{\frac{1}{s}}} \chi_{Q_j} \right\|_{\mathcal{M}^p_q} \ge \frac{\|a_{j_0}\|_{\mathcal{M}^s_t}}{|Q_{j_0}|^{\frac{1}{s}}} \|\lambda_{j_0} \chi_{Q_{j_0}}\|_{\mathcal{M}^p_q} = \|a_{j_0}\|_{\mathcal{M}^s_t} |Q_{j_0}|^{\frac{1}{p}-\frac{1}{s}} \lambda_{j_0}$$

for each $j_0 \in \mathbb{N}$. Consequently, it follows from the condition $p < s$ that

$$\int_{\mathbb{R}^n} f(x)\, g(x)\, dx \le \sum_{j=1}^\infty |Q|^{\frac{1}{p}-\frac{1}{s}} |Q_j|^{\frac{1}{s}-\frac{1}{p}} \cdot \left\| \sum_{k=1}^\infty \lambda_k \frac{\|a_k\|_{\mathcal{M}^s_t}}{|Q_k|^{\frac{1}{s}}} \chi_{Q_k} \right\|_{\mathcal{M}^p_q} \lesssim \left\| \sum_{j=1}^\infty \lambda_j \frac{\|a_j\|_{\mathcal{M}^s_t}}{|Q_j|^{\frac{1}{s}}} \chi_{Q_j} \right\|_{\mathcal{M}^p_q}.$$

Conversely, assume that Q contains each Q_j. Then we have

$$\int_{\mathbb{R}^n} f(x)\, g(x)\, dx = \sum_{j=1}^\infty \lambda_j \int_{Q_j} a_j(x)\, g(x)\, dx \le \sum_{j=1}^\infty \lambda_j \|a_j\|_{L^t(Q_j)} \|g\|_{L^{t'}(Q_j)}.$$

By the condition of a_j, we obtain

$$\int_{\mathbb{R}^n} f(x)\, g(x)\, dx = \sum_{j=1}^\infty \lambda_j \int_{Q_j} a_j(x)\, g(x)\, dx \le \sum_{j=1}^\infty \lambda_j \|a_j\|_{\mathcal{M}^s_t} |Q_j|^{\frac{1}{t}-\frac{1}{s}} \|g\|_{L^{t'}(Q_j)}.$$

Thus, in terms of the Hardy–Littlewood maximal operator M, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} f(x)\, g(x)\, dx &\le \sum_{j=1}^\infty \lambda_j \frac{\|a_j\|_{\mathcal{M}^s_t}}{|Q_j|^{\frac{1}{s}}} |Q_j| \times \inf_{y \in Q_j} M^{(t')} g(y) \\
&\le \int_{\mathbb{R}^n} \left(\sum_{j=1}^\infty \lambda_j \frac{\|a_j\|_{\mathcal{M}^s_t}}{|Q_j|^{\frac{1}{s}}} \chi_{Q_j}(y) \right) M^{(t')} g(y)\, dy \\
&\le \int_{\mathbb{R}^n} \left(\sum_{j=1}^\infty \lambda_j \frac{\|a_j\|_{\mathcal{M}^s_t}}{|Q_j|^{\frac{1}{s}}} \chi_{Q_j}(y) \right) \chi_Q(y) M^{(t')} g(y) dy.
\end{aligned}$$

Hence, we obtain (2.4) by the Hölder inequality, since $\|\chi_Q M^{(t')} g\|_{L^{q'}} \lesssim |Q|^{\frac{1}{p}-\frac{1}{q}}$. Thus the proof for the case of $q > 1$ is complete.

The case of $q = 1$ is a minor modification of the above proof. First, if each Q_j contains Q as a proper subset, the same argument as above works. If each Q contains Q_j, then we can take $g = |Q|^{\frac{1}{p}-1} \chi_Q$ to obtain

$$\int_Q f(x)\, g(x)\, dx \lesssim_{p,s} \left\| \sum_{j=1}^\infty \lambda_j \frac{\|a_j\|_{\mathcal{M}^s_1}}{|Q_j|^{\frac{1}{s}}} \chi_{Q_j} \right\|_{\mathcal{M}^p_1}.$$

We go through the same argument as before, where we will replace $M^{(t')}g$ by 1. Since $\|\chi_Q 1\|_{L^\infty}=1$, we do not have to resort to the boundedness of the maximal operator $M^{(t')}$ as we did in the estimate $\|\chi_Q M^{(t')}g\|_{L^{q'}} \lesssim |Q|^{\frac{1}{p}-\frac{1}{q}}$. So, the proof is complete in this case. $\square$

Lemma 2.3. *Let*

$$0<\alpha<2n, \quad 1<q_j \le p_j<\infty, \quad 0<q\le p<\infty, \quad 0<t\le s<\infty$$

for $j=1,2$. *Assume*

$$\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}, \quad \frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}, \quad \frac{1}{s}=\frac{1}{p}-\frac{\alpha}{n}, \quad \frac{q}{p}=\frac{t}{s}.$$

Then

$$|R|^{\frac{1}{s}-\frac{1}{t}}\left\|\sum_{Q\in\mathcal{D}}\frac{\chi_Q}{\ell(Q)^{2n-\alpha}}\int_{(3Q)^2} f_1(y_1)\,f_2(y_2)\,dy_1\,dy_2\right\|_{L^t(R)} \lesssim \prod_{j=1}^{2}\|f_j\|_{\mathcal{M}^{p_j}_{q_j}} \tag{2.5}$$

for any cube R and for all non-negative measurable functions f_1, f_2.

See the proof of [4, Theorem 2.2] for a similar approach.

Proof. Let $L=L(x)$ be a positive number that is specified shortly. We decompose

$$\begin{aligned}
&\sum_{Q\in\mathcal{D}}\frac{\chi_Q(x)}{\ell(Q)^{2n-\alpha}}\int_{(3Q)^2} f_1(y_1)\,f_2(y_2)\,dy_1\,dy_2\\
&\quad=\sum_{Q\in\mathcal{D},\ell(Q)\le L}\frac{\chi_Q(x)}{\ell(Q)^{2n-\alpha}}\int_{(3Q)^2} f_1(y_1)\,f_2(y_2)\,dy_1\,dy_2\\
&\qquad+\sum_{Q\in\mathcal{D},\ell(Q)> L}\frac{\chi_Q(x)}{\ell(Q)^{2n-\alpha}}\int_{(3Q)^2} f_1(y_1)\,f_2(y_2)\,dy_1\,dy_2 =: S_1+S_2.
\end{aligned}$$

First, we estimate the quantity S_1:

$$S_1 \lesssim \sum_{Q\in\mathcal{D},\ell(Q)\le L}\chi_Q(x)\,\ell(Q)^\alpha Mf_1(x)\,Mf_2(x)\sim L^\alpha Mf_1(x)\,Mf_2(x).$$

Next, we estimate the quantity S_2. By Hölder's inequality,

$$\begin{aligned}
S_2 &\lesssim \sum_{Q\in\mathcal{D},\ell(Q)>L}\frac{\chi_Q(x)}{\ell(Q)^{2n-\alpha}}|Q|^{\frac{1}{q_1'}}\|f_1\|_{L^{q_1}(3Q)}\cdot|Q|^{\frac{1}{q_2'}}\|f_1\|_{L^{q_2}(3Q)}\\
&\lesssim \sum_{Q\in\mathcal{D},\ell(Q)>L}\chi_Q(x)|Q|^{-\frac{1}{s}}\|f_1\|_{\mathcal{M}^{p_1}_{q_1}}\|f_2\|_{\mathcal{M}^{p_2}_{q_2}}\sim L^{-\frac{n}{s}}\|f_1\|_{\mathcal{M}^{p_1}_{q_1}}\|f_2\|_{\mathcal{M}^{p_2}_{q_2}}.
\end{aligned}$$

Hence we obtain

$$\sum_{Q\in\mathcal{D}}\frac{\chi_Q(x)}{\ell(Q)^{2n-\alpha}}\int_{(3Q)^2} f_1(y_1)\,f_2(y_2)\,dy_1\,dy_2 \lesssim L^\alpha Mf_1(x)\,Mf_2(x)+L^{-\frac{n}{s}}\|f_1\|_{\mathcal{M}^{p_1}_{q_1}}\|f_2\|_{\mathcal{M}^{p_2}_{q_2}}.$$

In particular, choose the constant $L=L(x)$ to optimize the right-hand side:

$$L=\left(\frac{\|f_1\|_{\mathcal{M}^{p_1}_{q_1}}\|f_2\|_{\mathcal{M}^{p_2}_{q_2}}}{Mf_1(x)\,Mf_2(x)}\right)^{\frac{p}{n}}.$$

Then we have

$$\sum_{Q\in\mathcal{D}}\frac{\chi_Q(x)}{\ell(Q)^{2n-\alpha}}\int_{(3Q)^2} f_1(y_1)\,f_2(y_2)\,dy_1\,dy_2 \lesssim (Mf_1(x)\,Mf_2(x))^{\frac{p}{s}}\left(\|f_1\|_{\mathcal{M}^{p_1}_{q_1}}\|f_2\|_{\mathcal{M}^{p_2}_{q_2}}\right)^{1-\frac{p}{s}}.$$

Therefore, using Hölder's inequality for Morrey spaces, the $\mathcal{M}^{p_1}_{q_1}(\mathbb{R}^n)$-boundedness of M and the $\mathcal{M}^{p_1}_{q_1}(\mathbb{R}^n)$-boundedness of M, we have

$$\begin{aligned}
&|R|^{\frac{1}{s}-\frac{1}{t}}\left\|\sum_{Q\in\mathcal{D}}\frac{\chi_Q}{\ell(Q)^{2n-\alpha}}\int\limits_{(3Q)^2} f_1(y_1)\,f_2(y_2)\,dy_1\,dy_2\right\|_{L^t(R)}\\
&\qquad\lesssim\left\|(Mf_1\cdot Mf_2)^{\frac{p}{s}}\right\|_{\mathcal{M}^s_t}\left(\|f_1\|_{\mathcal{M}^{p_1}_{q_1}}\|f_2\|_{\mathcal{M}^{p_2}_{q_2}}\right)^{1-\frac{p}{s}}=\|Mf_1\cdot Mf_2\|^{\frac{p}{s}}_{\mathcal{M}^p_q}\left(\|f_1\|_{\mathcal{M}^{p_1}_{q_1}}\|f_2\|_{\mathcal{M}^{p_2}_{q_2}}\right)^{1-\frac{p}{s}}\\
&\qquad\le\left(\|Mf_1\|_{\mathcal{M}^{p_1}_{q_1}}\|Mf_2\|_{\mathcal{M}^{p_2}_{q_2}}\right)^{\frac{p}{s}}\left(\|f_1\|_{\mathcal{M}^{p_1}_{q_1}}\|f_2\|_{\mathcal{M}^{p_2}_{q_2}}\right)^{1-\frac{p}{s}}\lesssim\|f_1\|_{\mathcal{M}^{p_1}_{q_1}}\|f_2\|_{\mathcal{M}^{p_2}_{q_2}}. \qquad\square
\end{aligned}$$

3. Proof of Theorem 1.2

Let $v\in(s,\min(q_1,q_2))$. Let $x\in Q\in\mathcal{D}_l$. By the Minkowski inequality and the Hölder inequality,

$$\begin{aligned}
&\left\|\int\limits_{B(2^{-l})} f_1(\cdot+y)\,f_2(\cdot-y)dy\right\|_{L^v(Q)}\\
&\le\int\limits_{B(2^{-l})}\|f_1(\cdot+y)\,f_2(\cdot-y)\|_{L^v(Q)}dy\le|B(2^{-l})|^{\frac{1}{v'}}\left(\int\limits_{B(2^{-l})}\|f_1(\cdot+y)\,f_2(\cdot-y)\|_{L^v(Q)}{}^v dy\right)^{\frac{1}{v}}\\
&\lesssim|B(2^{-l})|^{\frac{1}{v'}}\|f_1\|_{L^v(Q(x,3\cdot2^{-l}))}\|f_2\|_{L^v(Q(x,3\cdot2^{-l}))}\lesssim|B(2^{-l})|^{1+\frac{1}{v}}\inf_{y_1\in Q}M^{(v)}f_1(y_1)\inf_{y_2\in Q}M^{(v)}f_2(y_2).
\end{aligned}$$

Then owing to Theorem 2.2,

$$\begin{aligned}
\|\mathcal{J}_\alpha[f_1,f_2]\|_{\mathcal{M}^s_t}&\lesssim\left\|\sum_{l=-\infty}^{\infty}\sum_{Q\in\mathcal{D}_l}2^{l(n-\alpha)}\frac{\chi_Q}{|Q|^{\frac{1}{v}}}\left\|\int\limits_{B(2^{-l})}f_1(\cdot+y)\,f_2(\cdot-y)\,dy\right\|_{L^v(Q)}\right\|_{\mathcal{M}^s_t}\\
&\lesssim\left\|\sum_{l=-\infty}^{\infty}\sum_{Q\in\mathcal{D}_l}2^{-l\alpha}\frac{\chi_Q}{|Q|}\int\limits_{3Q}M^{(v)}f_1(y_1)\,dy_1\cdot\frac{1}{|Q|}\int\limits_{3Q}M^{(v)}f_2(y_2)\,dy_2\right\|_{\mathcal{M}^s_t}.
\end{aligned}$$

Thus, we are again in the position of using (2.5) to have

$$\|\mathcal{J}_\alpha[f_1,f_2]\|_{\mathcal{M}^s_t}\lesssim\|M^{(v)}f_1\|_{\mathcal{M}^{p_1}_{q_1}}\|M^{(v)}f_2\|_{\mathcal{M}^{p_2}_{q_2}}.$$

Since $v<q_1,q_2$, we are in the position of using the boundedness of M on Morrey spaces obtained by Chiarenza and Frasca [2]. If we use the boundedness of the Hardy–Littlewood maximal operator, then we obtain

$$\|\mathcal{J}_\alpha[f_1,f_2]\|_{\mathcal{M}^s_t}\lesssim\|f_1\|_{\mathcal{M}^{p_1}_{q_1}}\|f_2\|_{\mathcal{M}^{p_2}_{q_2}}.$$

This is the desired result.

To conclude the paper, we remark that Fan and Gao obtained an estimate to control

$$\left\|\int\limits_{B(2^{-l})}f_1(\cdot+y)\,f_2(\cdot-y)\,dy\right\|_{L^v(Q)}$$

in [4, Lemma 2.1].

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