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ON A PARAMETRIZATION OF NON-COMPACT WAVELET MATRICES BY WIENER-HOPF FACTORIZATION

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Abstract. A complete parametrization (one-to-one and onto mapping) of a certain class of non-compact wavelet matrices is introduced in terms of coordinates of infinite-dimensional Euclidian space. The developed method relies on Wiener-Hopf factorization of corresponding unitary matrix functions.

1. Introduction

Let $l^2(\mathbb{Z})$ be the standard Hilbert space of two-sided sequences of complex numbers. A matrix $\mathcal{A}$ with m rows and infinitely many columns

$$\mathcal{A}=\begin{pmatrix} \cdots & a^1_{-1} & a^1_0 & a^1_1 & a^1_2 & \cdots \\ \cdots & a^2_{-1} & a^2_0 & a^2_1 & a^2_2 & \cdots \\ & \vdots & \vdots & & & \\ \cdots & a^m_{-1} & a^m_0 & a^m_1 & a^m_2 & \cdots \end{pmatrix}, \quad a^i_j\in\mathbb{C}, \tag{1}$$

where the rows belong to $l^2(\mathbb{Z})$, is called a wavelet matrix (of rank m) if its rows satisfy the so called *shifted orthogonality condition* [4]:

$$\sum_{k=-\infty}^{\infty} a^i_{k+mj}\,\overline{a^r_{k+ms}}=\delta_{ir}\delta_{js} \quad \text{for all} \quad 1\le i,r\le m;\;\; j,s\in\mathbb{Z} \tag{2}$$

(δ stands for the Kronecker delta). Such matrices are a generalization of ordinary $m\times m$ unitary matrices and they play the crucial role in the theory of wavelets [6] and multirate filter banks [7]. Note that if $\mathcal{A}$ is a wavelet matrix and $\mathcal{A}'$ is obtained by shifting some of its rows by a multiple of m, then $\mathcal{A}'$ is a wavelet matrix as well.

In the *polyphase representation* [8] of matrix $\mathcal{A}$,

$$\mathbf{A}(z)=\sum_{k=-\infty}^{\infty} A_k z^k\,, \tag{3}$$

where $\mathcal{A}=(\dots\ A_{-1}\ A_0\ A_1\ A_2\ \dots)$ is the partition of $\mathcal{A}$ into $m\times m$ blocks $A_k=(a^i_{km+j})$, $1\le i\le m$, $0\le j\le m-1$, condition (2) is equivalent to

$$\mathbf{A}(z)\widetilde{\mathbf{A}}(z)=I_m, \tag{4}$$

where $\widetilde{\mathbf{A}}(z)=\sum_{k=-\infty}^{\infty}A_k^* z^{-k}$ is the *adjoint* of $\mathbf{A}(z)$ ($A^*:=\overline{A}^T$ is the Hermitian conjugate, and I_m stands for the $m\times m$ unit matrix). This is easy to see as (2) can be written in the block matrix form $\sum_{k=-\infty}^{\infty} A_k A^*_{l+k}=\delta_{l0}I_m$.

On the other hand, if series (3) is convergent a.e. on $\mathbb{T}:=\{z\in\mathbb{C}:|z|=1\}$, condition (4) means that $\mathbf{A}$ is a unitary matrix function on the unit circle, i.e.,

$$\mathbf{A}(z)\big(\mathbf{A}(z)\big)^*=I_m \;\text{ for }\; z\in\mathbb{T}. \tag{5}$$

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Therefore, wavelet matrices are closely related with unitary matrix functions. There is a natural one-to-one correspondence between them and we will rely on this connection throughout the paper.

Our notion of a wavelet matrix is somewhat different from the standard one. Namely, the *linear condition* $\mathbf{A}(1)\,\mathbf{e} = \sqrt{m}\,\mathbf{e}_1$, where $\mathbf{e} = (1, 1, \dots, 1)^T$ and $\mathbf{e}_1 = (1, 0, \dots, 0)^T$, must be satisfied in the usual definition (see [6, Eq. 4.9]) in order the corresponding orthogonal basis of $L^2(\mathbb{R})$ can be constructed by means of $\mathbf{A}$ (see [6, Ch-s 4, 5]). In our consideration, the linear condition is irrelevant. Furthermore, since the structure of coefficients of unitary matrix functions $\mathbf{A}(z)$ and $\mathbf{A}(z)\cdot U$, where U is a constant unitary matrix, are closely related, we introduce the equivalent classes of wavelet matrices as follows:

$$\mathcal{A} \sim \mathcal{A}' \iff A_j = A'_j U \text{ for some constant unitary matrix } U \text{ and every } j \in \mathbb{Z}. \tag{6}$$

We get a unique representative with a corresponding linear condition in each class in this way.

If the number of non-zero columns in (1) is finite, then the wavelet matrix $\mathcal{A}$ is called compact. Otherwise, it is non-compact.

For a compact wavelet matrix

$$\mathbf{A}(z) = \sum_{k=0}^{N} A_k z^k\,, \tag{7}$$

in order to avoid a chaotic rearrangement of the rows of $\mathcal{A}$, we assume that not only $A_0 \neq 0$ and $A_N \neq 0$ (N is called the *order* of (7) in this case) but also

$$\det \mathbf{A}(z) = cz^N. \tag{8}$$

Since it follows from (5) that $\det \mathbf{A}(z)$ is a monomial for compact wavelet matrices, it has necessarily form (8) and the power of z is called the *degree* of (7). It is proved in [1] that the degree of (7) is N if and only if $\operatorname{rank} A_0 = m-1$ (see Lemma 1 therein). This is the maximal possible value for the rank of A_0 and such situation is naturally called nonsingular.

In [1], a complete parametrization (one-to-one and onto mapping) of compact wavelet matrices of rank m and of order and degree N, with a minor restriction that the last row of A_N is not all zeros (this set is denoted by $\mathcal{CWM}_1[m,N,N]$), is proposed in terms of coordinates in the Euclidian space $\mathbb{C}^{(m-1)N}$. Namely, we have

$$\mathcal{CWM}_1[m,N,N] \longleftrightarrow \underbrace{\mathcal{P}_N^- \times \mathcal{P}_N^- \times \cdots \times \mathcal{P}_N^-}_{m-1} \cong \underbrace{\mathbb{C}^N \times \mathbb{C}^N \times \cdots \times \mathbb{C}^N}_{m-1} \tag{9}$$

in the following sence: For each $\mathbf{A} \in \mathcal{CWM}_1[m,N,N]$ there exists a unique Laurent matrix polynomial $F(z)$ of the form

$$F(z) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \zeta_1^-(z) & \zeta_2^-(z) & \zeta_3^-(z) & \cdots & \zeta_{m-1}^-(z) & 1 \end{pmatrix}, \tag{10}$$

where $\zeta_i^-(z) \in \mathcal{P}_N^-$, $j = 1, 2, \dots, m-1$, such that

$$F(z)U(z) \in \mathcal{P}_N^+(m \times m),$$

where

$$U(z) = \operatorname{diag}[1, \dots, 1, z^{-N}]\mathbf{A}(z) \tag{11}$$

(the last row of $\mathcal{A}$ is shifted to the left by mN), and

$$\mathcal{P}_N^+ := \left\{ \sum_{k=0}^{N} c_k z^k : c_k \in \mathbb{C},\ k = 0, \dots, N \right\}; \quad \mathcal{P}_N^- := \left\{ \sum_{k=1}^{N} c_k z^{-k} : c_k \in \mathbb{C},\ k = 1, \dots, N \right\}.$$

In other words

$$U(z) = U_-(z)U_+(z),$$

where

$$U_-(z) = F^{-1}(z) \text{ and } U_+(z) = F(z)U(z),$$

is the (right) Wiener-Hopf factorization of U. Note that F^{-1} can be obtained from F if we replace each ζ_i^- in (10) by $-\zeta_i^-$.

It readily follows from (11) and properties of $\mathbf{A}$ that the unitary Laurent matrix polynomial U has the following properties:

$$\det U(z) = \text{Const}, \quad \text{and} \quad \sum_{j=1}^{m} |u_{mj}(0)| > 0.$$

In the present paper, we are going to extend parametrization (9) to a certain class of non-compact wavelet matrices by letting $N \to \infty$ in the above formulations. To this end, we introduce some additional definitions.

Let $L_p^+ = H_p$, where $0 < p \leq \infty$, be the Hardy space of analytic functions (we usually identify analytic functions in the unit disk and their boundary values on $\mathbb{T}$) and $L_p^- := \{f : \overline{f} \in L_p^+\}$ be the corresponding set of anti-analytic functions. Denote also

$$L^{\pm} := \bigcap_{0<p<\infty} L_p^{\pm}.$$

Obviously, both of the sets L^+ and L^- are closed under multiplication:

$$f, g \in L^{\pm} \Longrightarrow fg \in L^{\pm}. \tag{12}$$

Let $\mathcal{WM}^{\pm}[m]$ be the set of equivalent classes (see (6)) of wavelet matrices (1) with $a_j^i = 0$ for $i = 1, 2, \dots, m-1$ and $j < 0$ or $i = m$ and $j \geq m$ (i.e., the entries in the first $m-1$ rows in the polyphase representation (3) are from L_∞^+ and the entries in the last row are from L_∞^-) such that

$$\det \mathbf{A}(z) = \text{Const} \quad \text{for a.a.} \quad z \in \mathbb{T}, \tag{13}$$

and the analytic functions $f_j(z) := \widetilde{\mathbf{A}}_{m,j}(z) = \sum_{k=0}^{\infty} \overline{a_{j-1-mk}^m} z^k$, $j = 1, 2, \dots, m$ (the adjoints of the entries in the last row of $\mathbf{A}(z)$) are not simultaneously equal to 0 in the space of maximal ideals of H_∞, i.e.,

$$\sum_{j=0}^{m} |f_j(z)| > \delta, \quad |z| < 1, \quad \text{for some} \quad \delta > 0;$$

and let $\mathcal{P}_\infty^-$ be the projection of L_∞ on the set of anti-analytic functions vanishing at the infinity, i.e.,

$$\mathcal{P}_\infty^- := \left\{ \sum_{k=-\infty}^{-1} c_k t^k : \text{ there exist } f \in L_\infty \text{ such that } \hat{f}(k) = c_k \text{ for } k < 0 \right\} \subset L^-,$$

where $\hat{f}(k)$ stands for the k-th Fourier coefficient of f. Then we have a ono-to-one and onto mapping similar to (9):

$$\mathcal{WM}^{\pm}[m] \longleftrightarrow \underbrace{\mathcal{P}_\infty^- \times \mathcal{P}_\infty^- \times \cdots \times \mathcal{P}_\infty^-}_{m-1},$$

which is the claim of the following

Theorem 1. *Let $\mathcal{A} = \mathbf{A}(z) \in \mathcal{WM}^{\pm}[m]$. Then there exists a unique matrix function $F(z)$ of the form* (10)*, where*

$$\zeta_i^- \in \mathcal{P}_\infty^-, \tag{14}$$

$j = 1, 2, \dots, m-1$, such that

$$F(z)\mathbf{A}(z) \in L^+(m \times m). \tag{15}$$

Conversely, for each matrix function (10), (14) *there exists a unique $\mathbf{A}(z) \in \mathcal{WM}^{\pm}[m]$ such that* (15) *holds.*

The inclusion (15) means again that the representation

$$\mathbf{A}(z) = \mathbf{A}_-(z)\mathbf{A}_+(z),$$

where

$$\mathbf{A}_-(z) = F^{-1}(z) \text{ and } \mathbf{A}_+(z) = F(z)\mathbf{A}(z),$$

is the (right) Wiener-Hopf factorization of $\mathbf{A}(z)$.

2. Proof of Theorem 1

Proof of Theorem 1 is based on the technique developed in [2].

Since $\mathbf{A}(z) \in L_\infty(m \times m)$ is a unitary matrix function, we have

$$\mathbf{A}^{-1}(z) = \mathbf{A}^*(z) \text{ a.e. on } \mathbb{T}. \tag{16}$$

Because of the Carleson Corona Theorem (see, e.g. [5]) there exist functions $g_1, g_2, \ldots, g_m$ from H_∞ such that

$$\sum_{j=1}^{m} f_j(z)g_j(z) = 1 \quad \text{for} \quad |z| < 1. \tag{17}$$

Let $\mathbf{B} \in L^+_\infty(m \times m)$ be the matrix function $\mathbf{A}$ with its last row replaced by $(g_1, g_2, \ldots, g_m)$. Then, since the last column of $\mathbf{A}$ is $(f_1, f_2, \ldots, f_m)^T$ and (16), (17) hold, we have

$$\mathbf{B}\mathbf{A}^* = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ \zeta_1 & \zeta_2 & \cdots & \zeta_{m-1} & 1 \end{pmatrix} =: \Phi \in L_\infty(m \times m),$$

where $\zeta_i = \sum_{k=1}^m g_k \widetilde{\mathbf{A}}_{ik}$. Thus, it follows from (16) that

$$\Phi\mathbf{A} = \mathbf{B}. \tag{18}$$

Let

$$\zeta_i = \zeta_i^+ + \zeta_i^-, \text{ where } \zeta^\pm \in \mathcal{P}^\pm_\infty, \ i = 1, 2, \ldots, m-1 \tag{19}$$

(the definition of $\mathcal{P}^+_\infty$ and the inclusion $\mathcal{P}^+_\infty \subset L^+$ are obvious). Then

$$\Phi = \Phi^+\Phi^-, \tag{20}$$

where $\Phi^\pm \in \mathcal{P}^\pm$ is the matrix Φ whith its last row replaced by $(\zeta_1^\pm, \zeta_2^\pm, \ldots, \zeta_{m-1}^\pm, 1)$. The equations (18) and (20) imply that

$$\Phi^-\mathbf{A} = (\Phi^+)^{-1}\mathbf{B} \in L^+(m \times m), \tag{21}$$

which proves (15) if we observe that $F(z) = \Phi^-(z)$ and $(\Phi^+)^{-1}$ is the matrix Φ^+ whith its last row replaced by $(-\zeta_1^+, -\zeta_2^+, \ldots, -\zeta_{m-1}^+, 1)$.

Let us now prove the uniqueness of F.

Assume

$$F_i(z)\mathbf{A}(z) = \Phi_i^+(z) \in L^+(m \times m), \quad i = 1, 2, \tag{22}$$

are two representations of type (10), (14), where $F_1 = F$ and F_2 is the matrix F with its last row replaced by $(\zeta_1', \zeta_2', \ldots, \zeta_{m-1}', 1)$.

Since $\Phi_i^+ \in L^+(m \times m) \Longrightarrow \det \Phi_i^+ \in L^+$ (see (12)) and $\det \Phi_i^+(z) = C$ a.e. on $\mathbb{T}$ (see (13), (22)), it follows that $\det \Phi_i^+(z) = C$ for $|z| < 1$. Therefore $(\Phi_i^+(z))^{-1} \in L^+(m \times m)$ because of Cramer's formula.

Equations in (22) imply that

$$\mathcal{P}^-_\infty(m \times m) \ni F_2^{-1}(z)F_1(z) = (\Phi_2^+(z))^{-1}\Phi_1^+(z) \in L^+(m \times m).$$

Hence the matrix function $F_2^{-1}F_1$ is constant, while it has form (10) whith its last row replaced by $(\zeta_1^- - \zeta_1', \zeta_2^- - \zeta_2', \ldots, \zeta_{m-1}^- - \zeta_{m-1}', 1)$. Consequently

$$\zeta_i^- = \zeta_i' \quad \text{for} \quad i = 1, 2, \ldots, m-1.$$

Let us now show the converse part of Theorem 1. The essential part of the claim is proved in [3, Lemma 4]: For each matrix of form (10), where $\zeta_i^- \in L_2^-$, $i = 1, 2, \dots, m-1$, there exists a unique (up to a constant right factor) unitary matrix function

$$U(t) = \begin{pmatrix} u_{11}^+(t) & u_{12}^+(t) & \cdots & u_{1,m-1}^+(t) & u_{1m}^+(t) \\ u_{21}^+(t) & u_{22}^+(t) & \cdots & u_{2,m-1}^+(t) & u_{2m}^+(t) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_{m-1,1}^+(t) & u_{m-1,2}^+(t) & \cdots & u_{m-1,m-1}^+(t) & u_{m-1,m}^+(t) \\ \overline{u_{m1}^+(t)} & \overline{u_{m2}^+(t)} & \cdots & \overline{u_{m,m-1}^+(t)} & \overline{u_{mm}^+(t)} \end{pmatrix}, \quad u_{ij}^+ \in L_\infty^+,$$

with constant determinant

$$\det U(t) = \text{Const} \quad \text{for a.a.} \quad t \in \mathbb{T}, \tag{23}$$

such that

$$F(t)U(t) \in L_2^+(m \times m).$$

It remains to prove that if (14) holds, then

$$\sum_{j=0}^{m} |u_{mj}^+(z)| > \delta, \quad |z| < 1, \ \text{ for some } \quad \delta > 0, \tag{24}$$

and

$$F(t)U(t) \in L^+(m \times m). \tag{25}$$

We obtain both relations simultaneously.

Since (14) holds, there exist bounded functions $\zeta_i \in L_\infty$ such that (19) holds. Let $\Phi^\pm$ be defined as in (20). Then $\Phi^+ F = \Phi^+\Phi^- = \Phi$ is bounded and therefore

$$\Phi^+ FU =: \Psi^+ \in L_\infty^+(m \times m). \tag{26}$$

Hence

$$FU = (\Phi^+)^{-1}\Psi^+ \in L^+(m \times m)$$

and (25) holds.

To show (24), let us first observe that $\det \Psi^+(z) = \text{Const}$ for $|z| < 1$ since $\det \Psi^+ \in H_\infty$ and it is constant a.e. on the boundary (see (20), (23), and (26)). Therefore

$$\sum_{j=1}^{m} \Psi_{mi}^+(z) \operatorname{Cof}(\Psi_{mi}^+)(z) = C, \tag{27}$$

where Cof stands for the cofactor. However, the first $m-1$ rows of U and Ψ^+ coincide. So that

$$\operatorname{Cof}(\Psi_{mi}^+) = \operatorname{Cof}(U_{mi}), \ \ j = 1, 2, \dots, m. \tag{28}$$

In addition, since U is unitary, i.e., $U^{-1} = U^*$, the formula for the inverse matrix implies that

$$u_{mj}^+ = \frac{1}{C} \operatorname{Cof}(U_{mj}). \tag{29}$$

Therefore, substituting (28) and (29) in (27), we get

$$\sum_{j=1}^{m} \Psi_{mi}^+(z) u_{mj}^+(z) = 1,$$

and, because of boundedness of the functions Ψ_{mi}^+ (see (26)), relation (24) holds.

3. Open Problems

For compact wavelet matrices, it is proved in [1] that the entries ζ_i^- of the matrix (10) in Theorem 1 can be computed by the formula

$$\zeta_i^-(z) = \mathbb{P}_N^-\big(\widetilde{\mathbf{A}}_{ij}(z)/\mathbf{A}_{mj}(z)\big), \quad \text{if } \mathbf{A}_{mj}(0) \neq 0, \tag{30}$$

where $\mathbb{P}_N^-$ is the projection of a (formal) Fourier series $\sum_{k=-N}^{\infty} c_k t^k$ on $\mathcal{P}_N^-$ (see [1, Eq. (25)]). To describe the conditions under which we can let $N \to \infty$ in equation (30) and to determine in which sense the limit exists is an interesting problem. It is related to the computation of partial indices of Wiener-Hopf factorization for a certain class of matrix functions which is the subject of a forthcoming paper.

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