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CERTAIN FRACTIONAL INTEGRAL AND FRACTIONAL DERIVATIVE FORMULAE WITH THEIR IMAGE FORMULAE INVOLVING GENERALIZED MULTI-INDEX MITTAG-LEFFLER FUNCTION

MEHAR CHAND[1], HAMED DAEI KASMAEI[2], AND MEHMET SENOL[3]

Abstract. The main objective of this paper is to establish some image formulas by applying the Riemann–Liouville fractional derivative and integral operators to the product of generalized multi-index Mittag–Leffler function $E^{\gamma,q}_{(\alpha_j,\beta_j)_m}(.)$. Some more image formulas are derived by applying integral transforms. The results obtained here are quite general in nature and capable of yielding a very large number of known and (presumably) new results.

1. Introduction

In fractional calculus several important functions known as special functions are presented via improper integrals or series. Among these vital functions, the Bessel function is widely used in physical sciences and engineering by many authors (see [9, 13, 15, 14, 1, 3, 2, 4, 7, 6, 8]). In recent years, a remarkably sizable amount of research works involving generalizations of Mittag–Leffler function is presented by several researchers.

For our present study we start with recalling the previous work. The Mittag–Leffler function is given as (see Marichev [23]): In this section we recall some known facts about Mittag–Leffler function and its generalizations, and also about the Riemann–Liouville fractional integral and a derivative operator.

Let us begin with few notions and facts related to the Mittag–Leffler function. In this presentation we follow mainly the review article [12] (see also [11]).

The Mittag-Lefler function $E_\alpha(z)$ with $\alpha > 0$ is named in honour of the great Swedish mathematician G.M. Mittag–Leffler who introduced it in the early of this century in a sequence of five notes and defined in the form of series

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}. \tag{1.1}$$

It was noted by Mittag–Leffler himself that for all α; $\Re(\alpha) > 0$ the series in (1.1) converges in the whole complex plane (and thus is an entire function of a complex variable z). For special values of parameter α the function $E_\alpha(z)$ coincides with some elementary and special functions. In particular, $E_1(z) = \exp(z)$. Hence, sometimes, the Mittag–Leffler function is called a generalized exponential. Anyway, the asymptotic behavior at infinity of this function differs of that for exponential function, namely, for all α, $0 < \Re(\alpha) < 2$, $\alpha \neq 1$ there exists an angle of exponential growth, and an angle at which the function is bounded.

First generalization of the function $E_\alpha(z)$ was mentioned by Wiman [34],

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}.$$

For each $\alpha, \beta \in \mathbb{C}$, $\Re(\alpha) > 0$; $E_{\alpha,\beta}(z)$ is an entire function. The function $E_{\alpha,\beta}(z)$ reduces to the classical Mittag–Leffler function if we choose $\beta = 1$.

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Further generalization of the function $E_\alpha(z)$ was proposed by Prabhakar [25]:

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{\Gamma(\alpha k + \beta)},$$

where $\alpha, \beta \in \mathbb{C}$, $\Re(\alpha) > 0$ and $(\gamma)_k$ is the Pochhammer symbol:

$$(\gamma)_k := \begin{cases} 1, & k = 0, \\ \gamma(\gamma+1)\dots(\gamma+k-1), & k \in \mathbb{N}. \end{cases}$$

Extended exposition on the theory and applications of this function is given in [22]. Evidently, the function $E^{\gamma}_{\alpha,\beta}(z)$ is related to the classical Mittag–Leffler function $E_\alpha(z)$ and two-parametric Mittag–Leffler function $E_{\alpha,\beta}(z)$:

$$E^{1}_{\alpha,\beta}(z) = E_{\alpha,\beta}(z); \quad E^{1}_{\alpha,1}(z) = E_{\alpha}(z).$$

Another generalization of two-parametric Mittag–Leffler function is the so-called four-parametric function.

$$E_{\alpha_1,\beta_1;\alpha_2,\beta_2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha_1 k + \beta_1)\,\Gamma(\alpha_1 k + \beta_1)}.$$

For positive $\alpha_1 > 0$; $\alpha_2 > 0$ and real $\beta_1; \beta_2 \in \mathbb{R}$ it was introduced by Djrbashian [10]. It is not hard to see that the convergence conditions for this function can be extended to all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$; $\Re(\alpha_1) > 0$, $\Re(\alpha_2) > 0$. Besides, $E_{\alpha,\beta;0,1}(z) = E_{\alpha,\beta}(z)$.

Generalizing the four-parametric Mittag–Leffler function, Al-Bassam and Luchko [5] introduced the Mittag–Leffler type function

$$E_{(\alpha_j,\beta_j)_m}(z) = E\big((\alpha_j,\beta_j)_{j=1}^m; z\big) = \sum_{k=0}^{\infty} \frac{z^k}{\prod_{j=1}^m \Gamma(\alpha_j k + \beta_j)}. \tag{1.2}$$

with $2m$ real parameters $\alpha_j > 0$; $\beta_j \in \mathbb{R}$ $(j = 1, \dots, m)$ and with complex $z \in \mathbb{C}$. In [5], an explicit solution to the Cauchy type problem for a fractional differential equation is given in terms of (1.2). The theory of this class of functions was developed in a series of articles by Kiryakova et al. [18, 19, 20, 22, 21] (see also [16]).

Generalization of the Prabhakar type function was done by Shukla and Prajapati [29]:

$$E^{\gamma,q}_{(\alpha,\beta)}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_{qk} z^k}{\Gamma(\alpha k + \beta)}, \quad (n \in \mathbb{N}). \tag{1.3}$$

under the following assumptions on parameters: $q \in (0,1) \cup \mathbb{N}$ and $\min\{\Re(\beta); \Re(\gamma)\}$. In [33], it is shown the existence of the function (1.3) for a wider set of parameters:

$$\big\{\alpha, \beta, \gamma \in \mathbb{C}; \Re(\alpha) > \max\{0; \Re(q-1)\}; \Re(q) > 0\big\}.$$

The definition (1.3) was combined with (1.2) in [27] (see also [28]). As a result, there appeared the following definition of generalized multi-index Mittag–Leffler function

$$E^{\gamma,q}_{(\alpha_j,\beta_j)_m}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_{qk}}{\prod_{j=1}^m \Gamma(\alpha_j k + \beta_j)} \frac{(z)^k}{k!},$$

where $m \in \mathbb{N}$, $\alpha_j, \beta_j, \gamma, q, z \in \mathbb{C}$ $(j = 1, \dots, m)$ such that $\sum_{j=1}^{m} \Re(\alpha_j) > \max\{0; \Re(q) - 1\}$; $\Re(q) > 0$.

The results given by Kiryakova [17], Miller and Ross [24], Srivastava et. al., [32] can be referred for some basic results on fractional calculus. The Fox-Wright function ${}_p\Psi_q$ is defined as (see, for details, Srivastava and Karlsson 1985, [31])

$${}_p\Psi_q[z] = {}_p\Psi_q\left[\begin{matrix} (a_1,\alpha_1),\dots,(a_p,\alpha_p); \\ (b_1,\beta_1),\dots,(b_q,\beta_q); \end{matrix}\; z\right] = {}_p\Psi_q\left[\begin{matrix} (a_i,\alpha_i)_{1,p}; \\ (b_j,\beta_j)_{1,q}; \end{matrix}\; z\right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^p \Gamma(a_i + \alpha_i n)}{\prod_{j=1}^q \Gamma(b_j + \beta_j n)} \frac{z^n}{n!}, \tag{1.4}$$

where the coefficients $\alpha_1, \dots, \alpha_p$, $\beta_1, \dots, \beta_q \in \mathbb{R}^+$ such that

$$1 + \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i \geq 0.$$

2. Fractional Derivative and Integral Operators

The right-sided Riemann–Liouville fractional integral operator I^{σ}_{a+} and the left-sided Riemann–Liouville fractional integral operator I^{σ}_{a-} and the corresponding Riemann–Liouville fractional derivative operator D^{σ}_{a+} and D^{σ}_{a-} are given as follows [26].

Lemma 1 (Riemann–Liouville fractional integral operators [26])**.** *Let $\Psi = [a, b]$ $(-\infty < a < b < \infty)$ be a finite interval on the real axis $\mathbb{R}$. The left-sided Riemann–Liouville fractional integral operators I^{σ}_{a+} and the right-sided Riemann–Liouville fractional integral operators I^{σ}_{b-} of order $\sigma \in \mathbb{C}$ are defined as*

$$\left(I^{\sigma}_{a+} f\right)(x) = \frac{1}{\Gamma(\sigma)} \int_{a}^{x} \frac{f(t)}{(x-t)^{1-\sigma}} dt \quad (x > a;\ \Re(\sigma) > 0), \tag{2.1}$$

$$\left(I^{\sigma}_{b-} f\right)(x) = \frac{1}{\Gamma(\sigma)} \int_{x}^{b} \frac{f(t)}{(t-x)^{1-\sigma}} dt \quad (x < b;\ \Re(\sigma) > 0).$$

Lemma 2 (Riemann–Liouville fractional derivative operators [26])**.** *Let $\Psi = [a, b]$ $(-\infty < a < b < \infty)$ be a finite interval on the real axis $\mathbb{R}$. The Riemann–Liouville fractional derivative operators D^{σ}_{a+} and D^{σ}_{b-} of order $\sigma \in \mathbb{C}$ are defined as*

$$\left(D^{\sigma}_{a+} f\right)(x) = \frac{d^n}{dx^n} \left(I^{n-\sigma}_{a+} f\right)(x), \quad \big(\Re(\sigma) \geq 0;\ \ n = 1 + [\Re(\sigma)]\big), \tag{2.2}$$

$$\left(D^{\sigma}_{b-} f\right)(x) = (-1)^n \frac{d^n}{dx^n} \left(I^{n-\sigma}_{b-} f\right)(x), \quad \big(\Re(\sigma) \geq 0;\ \ n = 1 + [\Re(\sigma)]\big),$$

where the function is locally integrable, $\Re(\sigma)$ denotes real part of the complex number and $[\Re(\sigma)]$ means the greatest integer in $\Re(\sigma)$. Also, the following n^{th} order derivative of x^{α} is defined as:

$$\frac{d^n}{dx^n}(x^{\alpha}) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-n)} x^{\alpha-n}, \quad \Re(\alpha) > 0. \tag{2.3}$$

For our present work, the following result is also required:

$$\int_{b}^{a} (a-t)^{\beta-1}(t-b)^{\alpha-1} dt = (a-b)^{\alpha+\beta-1} B(\alpha, \beta) \quad \big(\Re(\alpha) > 0;\ \ \Re(\beta) > 0;\ \ b < a\big). \tag{2.4}$$

3. Fractional Integral and Derivative Formulae Involving Generalized Multi-index Mittag–Leffler Function

In this section, we derive the formulae by using the Riemann–Liouville fractional integral and derivative operator involving generalized multi-index Mittag–Leffler function.

Theorem 1. *Let $m, r \in \mathbb{N}$, $\alpha_{ij}, \beta_{ij}, \gamma_i, q_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \dots, r;\ j = 1, \dots, m)$ such that $\sum_{j=1}^{m} \Re(\alpha_{ij}) > \max\{0; \Re(q_i) - 1\}$; $\Re(q_i) > 0$ and $\Re(\sigma) > 0$. For $x > a$, the following integral formula*

$$\Big(I^{\sigma}_{a+} (t-a)^{\rho} \prod_{i=1}^{r} E^{\gamma_i, q_i}_{(\alpha_{ij}, \beta_{ij})_m} \left(\xi (t-a)^{\mu}\right)\Big)(x) = \frac{(x-a)^{\sigma+\rho}}{\Gamma(\gamma_1) \dots \Gamma(\gamma_r)}$$

$$\times {}_{r+1}\Psi_{mr+r} \left[\begin{matrix} (\gamma_1, q_1), \dots, (\gamma_r, q_r), (\rho+1, \mu r) \\ (\beta_{1j}, \alpha_{1j})_{1,m}, \dots, (\beta_{rj}, \alpha_{rj})_{1,m}, (\sigma+\rho+1, \mu r), (-1, 1)_{1,r-1} \end{matrix} \,\middle|\, (\xi (x-a)^{\mu})^{r} \right] \tag{3.1}$$

holds.

Proof. Let the left-hand side of equation (3.1) be denoted by $\mathcal{I}$. Applying (1.4) and using the definition in equation (2.1) and interchanging the order of integration and summation, we have

$$\mathcal{I} = \frac{1}{\Gamma(\sigma)} \prod_{i=1}^{r} \left\{ \sum_{k=0}^{\infty} \frac{(\gamma_i)_{kq_i}}{\prod_{j=1}^{m} \Gamma(k\alpha_{ij} + \beta_{ij})} \frac{\xi^k}{k!} \right\} \int_a^x (x-t)^{\sigma-1} (t-a)^{\rho+\mu kr} dt, \tag{3.2}$$

applying the result (2.4), the above equation (3.2) reduces to

$$\mathcal{I} = \frac{1}{\Gamma(\sigma)} \prod_{i=1}^{r} \left\{ \sum_{k=0}^{\infty} \frac{(\gamma_i)_{kq_i}}{\prod_{j=1}^{m} \Gamma(k\alpha_{ij} + \beta_{ij})} \frac{\xi^k}{k!} \right\} (x-a)^{\sigma+\rho+\mu kr} B(\sigma, \rho + \mu kr + 1). \tag{3.3}$$

After simplification, the above equation (3.3) reduces to

$$\mathcal{I} = (x-a)^{\sigma+\rho} \prod_{i=1}^{r} \frac{1}{\Gamma(\gamma_i)} \left\{ \sum_{k=0}^{\infty} \frac{\Gamma(\gamma_i + kq_i)}{\prod_{j=1}^{m} \Gamma(k\alpha_{ij} + \beta_{ij})} \frac{\Gamma(\rho + \mu kr + 1)}{\Gamma(\sigma + \rho + \mu kr + 1)} \frac{(\xi(x-a)^{\mu})^{kr}}{k!} \right\}, \tag{3.4}$$

the above equation (3.4) can be written as

$$\mathcal{I} = \frac{(x-a)^{\sigma+\rho}}{\Gamma(\gamma_1) \ldots \Gamma(\gamma_r)} \left\{ \sum_{k=0}^{\infty} \frac{\Gamma(\gamma_1 + kq_1) \ldots \Gamma(\gamma_r + kq_r)}{\prod_{j=1}^{m} \Gamma(k\alpha_{1j} + \beta_{1j}) \cdots \prod_{j=1}^{m} \Gamma(k\alpha_{rj} + \beta_{rj})} \right.$$
$$\left. \times \frac{\Gamma(\rho + \mu kr + 1)}{\Gamma(\sigma + \rho + \mu kr + 1)(\Gamma(k-1))^{r-1}} \frac{(\xi(x-a)^{\mu})^{kr}}{k!} \right\}, \tag{3.5}$$

interpret the above equation (3.5), in the view of (1.4), we have the required result (3.1) □

Theorem 2. *Let $m, r \in \mathbb{N}$, $\alpha_{ij}, \beta_{ij}, \gamma_i, q_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r;\, j = 1, \ldots, m)$ such that $\sum_{j=1}^{m} \Re(\alpha_{ij}) > \max\{0; \Re(q_i) - 1\}$; $\Re(q_i) > 0$ and $\Re(\sigma) > 0$. For $b > x$, the following integral formula*

$$\left(I_{b-}^{\sigma} (b-t)^{\rho} \prod_{i=1}^{r} E_{(\alpha_{ij}, \beta_{ij})_m}^{\gamma_i, q_i} \left(\xi (b-t)^{\mu} \right) \right)(x) = \frac{(b-x)^{\sigma+\rho}}{\Gamma(\gamma_1) \ldots \Gamma(\gamma_r)}$$
$$\times {}_{r+1}\Psi_{mr+r} \left[\begin{array}{l} (\gamma_1, q_1), \ldots, (\gamma_r, q_r), (\rho + 1, \mu r) \\ (\beta_{1j}, \alpha_{1j})_{1,m}, \ldots, (\beta_{rj}, \alpha_{rj})_{1,m}, (\sigma + \rho + 1, \mu r), (-1, 1)_{1, r-1} \end{array} \middle| (\xi (b-x)^{\mu})^r \right] \tag{3.6}$$

holds.

Proof. The proof of Theorem 2 is similar to that of Theorem 1, therefore we omit the details. □

Theorem 3. *Let $m, r \in \mathbb{N}$, $\alpha_{ij}, \beta_{ij}, \gamma_i, q_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r;\, j = 1, \ldots, m)$ such that $\sum_{j=1}^{m} \Re(\alpha_{ij}) > \max\{0; \Re(q_i) - 1\}$; $\Re(q_i) > 0$ and $\Re(\sigma) \geq 0$. For $x > a$, the following integral formula*

$$\left(D_{a+}^{\sigma} (t-a)^{\rho} \prod_{i=1}^{r} E_{(\alpha_{ij}, \beta_{ij})_m}^{\gamma_i, q_i} \left(\xi (t-a)^{\mu} \right) \right)(x) = \frac{(x-a)^{\sigma+\rho}}{\Gamma(\gamma_1) \ldots \Gamma(\gamma_r)}$$
$$\times {}_{r+1}\Psi_{mr+r} \left[\begin{array}{l} (\gamma_1, q_1), \ldots, (\gamma_r, q_r), (\rho + 1, \mu r) \\ (\beta_{1j}, \alpha_{1j})_{1,m}, \ldots, (\beta_{rj}, \alpha_{rj})_{1,m}, (\rho - \sigma + 1, \mu r), (-1, 1)_{1, r-1} \end{array} \middle| (\xi (x-a)^{\mu})^r \right] \tag{3.7}$$

holds.

Proof. Let the left-hand side of equation (3.7) be denoted by $\mathcal{D}$, then interchanging the order of differentiation and summation, we have

$$\mathcal{D} = \prod_{i=1}^{r} \left\{ \sum_{k=0}^{\infty} \frac{(\gamma_i)_{kq_i}}{\prod_{j=1}^{m} \Gamma(k\alpha_{ij} + \beta_{ij})} \frac{\xi^k}{k!} \right\} \left(D_{a+}^{\sigma} (t-a)^{\rho+\mu kr} \right)(x), \tag{3.8}$$

now using the result given in equation (2.2) and further applying the result (2.1), the above equation (3.8) reduces to the following form

$$\mathcal{D}=\frac{1}{\Gamma(n-\sigma)}\prod_{i=1}^{r}\left\{\sum_{k=0}^{\infty}\frac{(\gamma_i)_{kq_i}}{\prod_{j=1}^{m}\Gamma(k\alpha_{ij}+\beta_{ij})}\frac{\xi^k}{k!}\right\}\frac{d^n}{dx^n}\int_a^x (x-t)^{n-\sigma-1}(t-a)^{\rho+\mu kr}dt, \tag{3.9}$$

substituting the result (2.4) into the above equation (3.9) and after simplification, we get

$$\mathcal{D}=\prod_{i=1}^{r}\left\{\sum_{k=0}^{\infty}\frac{(\gamma_i)_{kq_i}}{\prod_{j=1}^{m}\Gamma(k\alpha_{ij}+\beta_{ij})}\frac{\xi^k}{k!}\right\}\frac{\Gamma(\rho+\mu kr+1)}{\Gamma(k-\sigma+\rho+\mu kr+1)}\frac{d^n}{dx^n}(x-a)^{\rho-\sigma+n+\mu kr}, \tag{3.10}$$

using the result given in equation (2.3) into the above equation (3.10), we have

$$\mathcal{I}=\frac{(x-a)^{\sigma+\rho}}{\Gamma(\gamma_1)\dots\Gamma(\gamma_r)}\left\{\sum_{k=0}^{\infty}\frac{\Gamma(\gamma_1+kq_1)\dots\Gamma(\gamma_r+kq_r)}{\prod_{j=1}^{m}\Gamma(k\alpha_{1j}+\beta_{1j})\cdots\prod_{j=1}^{m}\Gamma(k\alpha_{rj}+\beta_{rj})}\right.$$
$$\left.\times\frac{\Gamma(\rho+\mu kr+1)}{\Gamma(\rho-\sigma+\mu kr+1)(\Gamma(k-1))^{r-1}}\frac{(\xi(x-a)^{\mu})^{kr}}{k!}\right\}, \tag{3.11}$$

further, interpret the above equation (3.11) with the view of (1.4), we obtain the required result (3.7). □

Theorem 4. *Let* $m,r\in\mathbb{N}$, $\alpha_{ij},\beta_{ij},\gamma_i,q_i,\rho,\xi,\mu\in\mathbb{C}$ $(i=1,\dots,r;\, j=1,\dots,m)$ *such that* $\sum_{j=1}^{m}\Re(\alpha_{ij})>\max\{0;\Re(q_i)-1\}$; $\Re(q_i)>0$ *and* $\Re(\sigma)\geq 0$. *For* $b>x$, *the following integral formula*

$$\left(D_{b-}^{\sigma}(b-t)^{\rho}\prod_{i=1}^{r}E_{(\alpha_{ij},\beta_{ij})_m}^{\gamma_i,q_i}\left(\xi(b-t)^{\mu}\right)\right)(x)=\frac{(b-x)^{\sigma+\rho}}{\Gamma(\gamma_1)\dots\Gamma(\gamma_r)}$$
$$\times{}_{r+1}\Psi_{mr+r}\left[\begin{array}{c}(\gamma_1,q_1),\dots,(\gamma_r,q_r),(\rho+1,\mu r)\\(\beta_{1j},\alpha_{1j})_{1,m},\dots,(\beta_{rj},\alpha_{rj})_{1,m},(\rho-\sigma+1,\mu r),(-1,1)_{1,r-1}\end{array}\middle|(\xi(b-x)^{\mu})^{r}\right] \tag{3.12}$$

holds.

Proof. The proof of Theorem 4 is similar to that of Theorem 3, therefore we omit the details. □

3.1. **Special cases of fractional integral and derivative formulae.** Choose $m=1$, the generalized multi-index Mittag–Leffler function $E_{(\alpha_j,\beta_j)_m}^{\gamma,q}(.)$ reduces to the generalized Mittag–Leffler function $E_{(\alpha,\beta)}^{\gamma,q}(.)$ defined by Shukla and Prajapati. Using this concept, the results established in equations (3.1), (3.6), (3.7) and (3.12) are reduced to the following form

Corollary 1. *Let* $r\in\mathbb{N}$, $\alpha_i,\beta_i,\gamma_i,q_i,\rho,\xi,\mu\in\mathbb{C}$ $(i=1,\dots,r)$ *such that* $\Re(\alpha_i)>\max\{0;\Re(q_i)-1\}$; $\Re(q_i)>0$ *and* $\Re(\sigma)>0$. *For* $x>a$, *the following integral formula*

$$\left(I_{a+}^{\sigma}(t-a)^{\rho}\prod_{i=1}^{r}E_{(\alpha_i,\beta_i)}^{\gamma_i,q_i}\left(\xi(t-a)^{\mu}\right)\right)(x)=\frac{(x-a)^{\sigma+\rho}}{\Gamma(\gamma_1)\dots\Gamma(\gamma_r)}$$
$$\times{}_{r+1}\Psi_{2r}\left[\begin{array}{c}(\gamma_1,q_1),\dots,(\gamma_r,q_r),(\rho+1,\mu r)\\(\beta_1,\alpha_1),\dots,(\beta_r,\alpha_r),(\sigma+\rho+1,\mu r),(-1,1)_{1,r-1}\end{array}\middle|(\xi(x-a)^{\mu})^{r}\right]$$

holds.

Corollary 2. *Let* $r\in\mathbb{N}$, $\alpha_i,\beta_i,\gamma_i,q_i,\rho,\xi,\mu\in\mathbb{C}$ $(i=1,\dots,r)$ *such that* $\Re(\alpha_i)>\max\{0;\Re(q_i)-1\}$; $\Re(q_i)>0$ *and* $\Re(\sigma)>0$. *For* $b>x$, *the following integral formula*

$$\left(I_{b-}^{\sigma}(b-t)^{\rho}\prod_{i=1}^{r}E_{(\alpha_i,\beta_i)}^{\gamma_i,q_i}\left(\xi(b-t)^{\mu}\right)\right)(x)=\frac{(b-x)^{\sigma+\rho}}{\Gamma(\gamma_1)\dots\Gamma(\gamma_r)}$$
$$\times{}_{r+1}\Psi_{2r}\left[\begin{array}{c}(\gamma_1,q_1),\dots,(\gamma_r,q_r),(\rho+1,\mu r)\\(\beta_1,\alpha_1),\dots,(\beta_r,\alpha_r),(\sigma+\rho+1,\mu r),(-1,1)_{1,r-1}\end{array}\middle|(\xi(b-x)^{\mu})^{r}\right]$$

holds.

Corollary 3. *Let $r \in \mathbb{N}$, $\alpha_i, \beta_i, \gamma_i, q_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r)$ such that $\Re(\alpha_i) > \max\{0; \Re(q_i) - 1\}$; $\Re(q_i) > 0$ and $\Re(\sigma) > 0$. For $x > a$, the following integral formula*

$$\left(D^{\sigma}_{a+}(t-a)^{\rho} \prod_{i=1}^{r} E^{\gamma_i, q_i}_{(\alpha_i, \beta_i)}\left(\xi(t-a)^{\mu}\right)\right)(x) = \frac{(x-a)^{\sigma+\rho}}{\Gamma(\gamma_1)\ldots\Gamma(\gamma_r)}$$
$$\times {}_{r+1}\Psi_{2r}\left[\begin{array}{l} (\gamma_1, q_1), \ldots, (\gamma_r, q_r), (\rho+1, \mu r) \\ (\beta_1, \alpha_1), \ldots, (\beta_r, \alpha_r), (\rho-\sigma+1, \mu r), (-1, 1)_{1, r-1} \end{array} \middle| (\xi(x-a)^{\mu})^{r}\right]$$

holds.

Corollary 4. *Let $r \in \mathbb{N}$, $\alpha_i, \beta_i, \gamma_i, q_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r)$ such that $\Re(\alpha_i) > \max\{0; \Re(q_i) - 1\}$; $\Re(q_i) > 0$ and $\Re(\sigma) > 0$. For $b > x$, the following integral formula*

$$\left(D^{\sigma}_{b-}(b-t)^{\rho} \prod_{i=1}^{r} E^{\gamma_i, q_i}_{(\alpha_i, \beta_i)}\left(\xi(b-t)^{\mu}\right)\right)(x) = \frac{(b-x)^{\sigma+\rho}}{\Gamma(\gamma_1)\ldots\Gamma(\gamma_r)}$$
$$\times {}_{r+1}\Psi_{2r}\left[\begin{array}{l} (\gamma_1, q_1), \ldots, (\gamma_r, q_r), (\rho+1, \mu r) \\ (\beta_1, \alpha_1), \ldots, (\beta_r, \alpha_r), (\rho-\sigma+1, \mu r), (-1, 1)_{1, r-1} \end{array} \middle| (\xi(b-x)^{\mu})^{r}\right]$$

holds.

Choosing $q = m = 1$, the generalized multi-index Mittag–Leffler function $E^{\gamma,q}_{(\alpha_j,\beta_j)_m}(.)$ reduces to the generalized Mittag–Leffler function $E^{\gamma}_{(\alpha,\beta)}(.)$ defined by Prabhakar. Using this concept, the results established in equations (3.1), (3.6), (3.7) and (3.12) are reduced to the following form.

Corollary 5. *Let $r \in \mathbb{N}$, $\alpha_i, \beta_i, \gamma_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r)$ such that $\Re(\alpha_i) > 0$; $\Re(q_i) > 0$ and $\Re(\sigma) > 0$. For $x > a$, the following integral formula*

$$\left(I^{\sigma}_{a+}(t-a)^{\rho} \prod_{i=1}^{r} E^{\gamma_i}_{(\alpha_i, \beta_i)}\left(\xi(t-a)^{\mu}\right)\right)(x) = \frac{(x-a)^{\sigma+\rho}}{\Gamma(\gamma_1)\ldots\Gamma(\gamma_r)}$$
$$\times {}_{r+1}\Psi_{2r}\left[\begin{array}{l} (\gamma_1, 1), \ldots, (\gamma_r, 1), (\rho+1, \mu r) \\ (\beta_1, \alpha_1), \ldots, (\beta_r, \alpha_r), (\sigma+\rho+1, \mu r), (-1, 1)_{1, r-1} \end{array} \middle| (\xi(x-a)^{\mu})^{r}\right]$$

holds.

Corollary 6. *Let $r \in \mathbb{N}$, $\alpha_i, \beta_i, \gamma_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r)$ such that $\Re(\alpha_i) > 0$; $\Re(q_i) > 0$ and $\Re(\sigma) > 0$. For $b > x$, the following integral formula*

$$\left(I^{\sigma}_{b-}(b-t)^{\rho} \prod_{i=1}^{r} E^{\gamma_i}_{(\alpha_i, \beta_i)}\left(\xi(b-t)^{\mu}\right)\right)(x) = \frac{(b-x)^{\sigma+\rho}}{\Gamma(\gamma_1)\ldots\Gamma(\gamma_r)}$$
$$\times {}_{r+1}\Psi_{2r}\left[\begin{array}{l} (\gamma_1, 1), \ldots, (\gamma_r, 1), (\rho+1, \mu r) \\ (\beta_1, \alpha_1), \ldots, (\beta_r, \alpha_r), (\sigma+\rho+1, \mu r), (-1, 1)_{1, r-1} \end{array} \middle| (\xi(b-x)^{\mu})^{r}\right]$$

holds.

Corollary 7. *Let $r \in \mathbb{N}$, $\alpha_i, \beta_i, \gamma_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r)$ such that $\Re(\alpha_i) > 0$; $\Re(q_i) > 0$ and $\Re(\sigma) > 0$. For $x > a$, the following integral formula*

$$\left(D^{\sigma}_{a+}(t-a)^{\rho} \prod_{i=1}^{r} E^{\gamma_i}_{(\alpha_i, \beta_i)}\left(\xi(t-a)^{\mu}\right)\right)(x) = \frac{(x-a)^{\sigma+\rho}}{\Gamma(\gamma_1)\ldots\Gamma(\gamma_r)}$$
$$\times {}_{r+1}\Psi_{2r}\left[\begin{array}{l} (\gamma_1, 1), \ldots, (\gamma_r, 1), (\rho+1, \mu r) \\ (\beta_1, \alpha_1), \ldots, (\beta_r, \alpha_r), (\rho-\sigma+1, \mu r), (-1, 1)_{1, r-1} \end{array} \middle| (\xi(x-a)^{\mu})^{r}\right]$$

holds.

Corollary 8. *Let $r \in \mathbb{N}$, $\alpha_i, \beta_i, \gamma_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r)$ such that $\Re(\alpha_i) > 0$; $\Re(q_i) > 0$ and $\Re(\sigma) > 0$. For $b > x$, the following integral formula*

$$\left(D_{b-}^{\sigma}(b-t)^{\rho}\prod_{i=1}^{r}E_{(\alpha_i,\beta_i)}^{\gamma_i}\left(\xi(b-t)^{\mu}\right)\right)(x) = \frac{(b-x)^{\sigma+\rho}}{\Gamma(\gamma_1)\ldots\Gamma(\gamma_r)}$$
$$\times_{r+1}\Psi_{2r}\left[\begin{array}{l}(\gamma_1,1),\ldots,(\gamma_r,1),(\rho+1,\mu r)\\(\beta_1,\alpha_1),\ldots,(\beta_r,\alpha_r),(\rho-\sigma+1,\mu r),(-1,1)_{1,r-1}\end{array}\middle|(\xi(b-x)^{\mu})^{r}\right]$$

holds.

Choosing $\gamma = q = m = 1$, the generalized multi-index Mittag–Leffler function $E_{(\alpha_j,\beta_j)_m}^{\gamma,q}(.)$ reduces to the generalized Mittag–Leffler function $E_{(\alpha,\beta)}(.)$ defined by Prabhakar. Using this concept, the results established in equations (3.1), (3.6), (3.7) and (3.12) are reduced to the following form.

Corollary 9. *Let $r \in \mathbb{N}$, $\alpha_i, \beta_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r)$ such that $\Re(\alpha_i) > 0$; and $\Re(\sigma) > 0$. For $x > a$, the following integral formula*

$$\left(I_{a+}^{\sigma}(t-a)^{\rho}\prod_{i=1}^{r}E_{(\alpha_i,\beta_i)}\left(\xi(t-a)^{\mu}\right)\right)(x)$$
$$= (x-a)^{\sigma+\rho}{}_1\Psi_{2r}\left[\begin{array}{l}(\rho+1,\mu r)\\(\beta_1,\alpha_1),\ldots,(\beta_r,\alpha_r),(\sigma+\rho+1,\mu r),(-1,1)_{1,r-1}\end{array}\middle|(\xi(x-a)^{\mu})^{r}\right]$$

holds.

Corollary 10. *Let $r \in \mathbb{N}$, $\alpha_i, \beta_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r)$ such that $\Re(\alpha_i) > 0$; and $\Re(\sigma) > 0$. For $b > x$, the following integral formula*

$$\left(I_{b-}^{\sigma}(b-t)^{\rho}\prod_{i=1}^{r}E_{(\alpha_i,\beta_i)}\left(\xi(b-t)^{\mu}\right)\right)(x)$$
$$= (b-x)^{\sigma+\rho}{}_1\Psi_{2r}\left[\begin{array}{l}(\rho+1,\mu r)\\(\beta_1,\alpha_1),\ldots,(\beta_r,\alpha_r),(\sigma+\rho+1,\mu r),(-1,1)_{1,r-1}\end{array}\middle|(\xi(b-x)^{\mu})^{r}\right]$$

holds.

Corollary 11. *Let $r \in \mathbb{N}$, $\alpha_i, \beta_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r)$ such that $\Re(\alpha_i) > 0$; and $\Re(\sigma) > 0$. For $x > a$, the following integral formula*

$$\left(D_{a+}^{\sigma}(t-a)^{\rho}\prod_{i=1}^{r}E_{(\alpha_i,\beta_i)}\left(\xi(t-a)^{\mu}\right)\right)(x)$$
$$= (x-a)^{\sigma+\rho}{}_1\Psi_{2r}\left[\begin{array}{l}(\rho+1,\mu r)\\(\beta_1,\alpha_1),\ldots,(\beta_r,\alpha_r),(\rho-\sigma+1,\mu r),(-1,1)_{1,r-1}\end{array}\middle|(\xi(x-a)^{\mu})^{r}\right]$$

holds.

Corollary 12. *Let $r \in \mathbb{N}$, $\alpha_i, \beta_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r)$ such that $\Re(\alpha_i) > 0$; and $\Re(\sigma) > 0$. For $b > x$, the following integral formula*

$$\left(D_{b-}^{\sigma}(b-t)^{\rho}\prod_{i=1}^{r}E_{(\alpha_i,\beta_i)}\left(\xi(b-t)^{\mu}\right)\right)(x)$$
$$= (b-x)^{\sigma+\rho}{}_1\Psi_{2r}\left[\begin{array}{l}(\rho+1,\mu r)\\(\beta_1,\alpha_1),\ldots,(\beta_r,\alpha_r),(\rho-\sigma+1,\mu r),(-1,1)_{1,r-1}\end{array}\middle|(\xi(b-x)^{\mu})^{r}\right]$$

holds.

Choosing $\beta_j = \gamma = q = m = 1$, the generalized multi-index Mittag–Leffler function $E_{(\alpha_j,\beta_j)_m}^{\gamma,q}(.)$ reduces to the Mittag–Leffler function $E_{\alpha}(.)$ defined by the great Swedish mathematician G.M. Using this concept, the results established in equations (3.1), (3.6), (3.7) and (3.12) are reduced to the following form.

Corollary 13. *Let $r \in \mathbb{N}$, $\alpha_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r)$ such that $\Re(\alpha_i) > 0$; and $\Re(\sigma) > 0$. For $x > a$, the following integral formula*

$$\left(I_{a+}^{\sigma}(t-a)^{\rho}\prod_{i=1}^{r}E_{\alpha_i}\left(\xi(t-a)^{\mu}\right)\right)(x)$$
$$= (x-a)^{\sigma+\rho}{}_1\Psi_{2r}\left[\begin{array}{l}(\rho+1,\mu r)\\(1,\alpha_1),\ldots,(1,\alpha_r),(\sigma+\rho+1,\mu r),(-1,1)_{1,r-1}\end{array}\middle|(\xi(x-a)^{\mu})^{r}\right]$$

holds.

Corollary 14. *Let $r \in \mathbb{N}$, $\alpha_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r)$ such that $\Re(\alpha_i) > 0$; and $\Re(\sigma) > 0$. For $b > x$, the following integral formula*

$$\left(I_{b-}^{\sigma}(b-t)^{\rho}\prod_{i=1}^{r}E_{\alpha_i}\left(\xi(b-t)^{\mu}\right)\right)(x)$$
$$= (b-x)^{\sigma+\rho}{}_1\Psi_{2r}\left[\begin{array}{l}(\rho+1,\mu r)\\(1,\alpha_1),\ldots,(1,\alpha_r),(\sigma+\rho+1,\mu r),(-1,1)_{1,r-1}\end{array}\middle|(\xi(b-x)^{\mu})^{r}\right]$$

holds.

Corollary 15. *Let $r \in \mathbb{N}$, $\alpha_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r)$ such that $\Re(\alpha_i) > 0$; and $\Re(\sigma) > 0$. For $x > a$, the following integral formula*

$$\left(D_{a+}^{\sigma}(t-a)^{\rho}\prod_{i=1}^{r}E_{\alpha_i}\left(\xi(t-a)^{\mu}\right)\right)(x)$$
$$= (x-a)^{\sigma+\rho}{}_1\Psi_{2r}\left[\begin{array}{l}(\rho+1,\mu r)\\(1,\alpha_1),\ldots,(1,\alpha_r),(\rho-\sigma+1,\mu r),(-1,1)_{1,r-1}\end{array}\middle|(\xi(x-a)^{\mu})^{r}\right]$$

holds.

Corollary 16. *Let $r \in \mathbb{N}$ $\alpha_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r)$ such that $\Re(\alpha_i) > 0$; and $\Re(\sigma) > 0$. For $b > x$, the following integral formula*

$$\left(D_{b-}^{\sigma}(b-t)^{\rho}\prod_{i=1}^{r}E_{\alpha_i}\left(\xi(b-t)^{\mu}\right)\right)(x)$$
$$= (b-x)^{\sigma+\rho}{}_1\Psi_{2r}\left[\begin{array}{l}(\rho+1,\mu r)\\(1,\alpha_1),\ldots,(1,\alpha_r),(\rho-\sigma+1,\mu r),(-1,1)_{1,r-1}\end{array}\middle|(\xi(b-x)^{\mu})^{r}\right]$$

holds.

Choosing $r = m = 1$, the generalized multi-index Mittag–Leffler function $E_{(\alpha_j,\beta_j)_m}^{\gamma,q}(.)$ reduces to the Mittag–Leffler function $E_{(\alpha,\beta)}^{\gamma,q}(.)$ defined by the great Swedish mathematician G.M. Using this concept, the results established in equations (3.1), (3.6), (3.7) and (3.12) are reduced to the following form.

Corollary 17. *Let $\alpha, \beta, \gamma, q, \rho, \xi, \mu \in \mathbb{C}$ such that $\Re(\alpha) > \max\{0; \Re(q) - 1\}$; $\Re(q) > 0$ and $\Re(\sigma) > 0$. For $x > a$, the following integral formula*

$$\left(I_{a+}^{\sigma}(t-a)^{\rho}E_{(\alpha,\beta)}^{\gamma,q}\left(\xi(t-a)^{\mu}\right)\right)(x) = \frac{(x-a)^{\sigma+\rho}}{\Gamma(\gamma)}\,{}_2\Psi_2\left[\begin{array}{l}(\gamma,q),(\rho+1,\mu)\\(\beta,\alpha),(\sigma+\rho+1,\mu)\end{array}\middle|\xi(x-a)^{\mu}\right]$$

holds.

Corollary 18. *Let $\alpha, \beta, \gamma, q, \rho, \xi, \mu \in \mathbb{C}$ such that $\Re(\alpha) > \max\{0; \Re(q) - 1\}$; $\Re(q) > 0$ and $\Re(\sigma) > 0$. For $b > x$, the following integral formula*

$$\left(I_{b-}^{\sigma}(b-t)^{\rho}E_{(\alpha,\beta)}^{\gamma,q}\left(\xi(b-t)^{\mu}\right)\right)(x) = \frac{(b-x)^{\sigma+\rho}}{\Gamma(\gamma)}\,{}_2\Psi_2\left[\begin{array}{l}(\gamma,q),(\rho+1,\mu)\\(\beta,\alpha),(\sigma+\rho+1,\mu)\end{array}\middle|\xi(b-x)^{\mu}\right].$$

holds.

Corollary 19. *Let $\alpha, \beta, \gamma, q, \rho, \xi, \mu \in \mathbb{C}$ such that $\Re(\alpha) > \max\{0; \Re(q) - 1\}$; $\Re(q) > 0$ and $\Re(\sigma) \geq 0$. For $x > a$, the following integral formula*

$$\left(D_{a+}^{\sigma}(t-a)^{\rho} E_{(\alpha,\beta)}^{\gamma,q}\left(\xi(t-a)^{\mu}\right)\right)(x) = \frac{(x-a)^{\sigma+\rho}}{\Gamma(\gamma)}\, {}_2\Psi_2\left[\begin{array}{c}(\gamma, q), (\rho+1, \mu) \\ (\beta, \alpha), (\rho-\sigma+1, \mu)\end{array}\middle|\, \xi(x-a)^{\mu}\right]$$

holds.

Corollary 20. *Let $\alpha, \beta, \gamma, q, \rho, \xi, \mu \in \mathbb{C}$ such that $\Re(\alpha) > \max\{0; \Re(q) - 1\}; \Re(q) > 0$ and $\Re(\sigma) \geq 0$. For $b > x$, the following integral formula*

$$\left(D_{b-}^{\sigma}(b-t)^{\rho} E_{(\alpha,\beta)}^{\gamma,q}\left(\xi(b-t)^{\mu}\right)\right)(x) = \frac{(b-x)^{\sigma+\rho}}{\Gamma(\gamma)}\, {}_2\Psi_2\left[\begin{array}{c}(\gamma, q), (\rho+1, \mu) \\ (\beta, \alpha), (\rho-\sigma+1, \mu)\end{array}\middle|\, \xi(b-x)^{\mu}\right]$$

holds.

Choosing $r = m = \gamma = q = \alpha = \beta = 1$, the generalized multi-index Mittag–Leffler function $E_{(\alpha_j,\beta_j)_m}^{\gamma,q}(.)$ reduces to the Mittag–Leffler function $E_{(1,1)_1}^{1,1}(.) = E_{(1,1)}^{1,1}(.) = \exp(.)$ defined by the great Swedish mathematician G.M. Using this concept, the results established in equations (3.1), (3.6), (3.7) and (3.12) are reduced to the following form.

Corollary 21. *Let $\rho, \xi, \mu \in \mathbb{C}$ such that $\Re(\sigma) > 0$. For $x > a$, the following integral formula*

$$\left(I_{a+}^{\sigma}(t-a)^{\rho} \exp\left(\xi(t-a)^{\mu}\right)\right)(x) = (x-a)^{\sigma+\rho}\, {}_1\Psi_1\left[\begin{array}{c}(\rho+1, \mu) \\ (\sigma+\rho+1, \mu)\end{array}\middle|\, \xi(x-a)^{\mu}\right]$$

holds.

Corollary 22. *Let $\rho, \xi, \mu \in \mathbb{C}$ such that $\Re(\sigma) > 0$. For $b > x$, the following integral formula*

$$\left(I_{b-}^{\sigma}(b-t)^{\rho} \exp\left(\xi(b-t)^{\mu}\right)\right)(x) = (b-x)^{\sigma+\rho}\, {}_1\Psi_1\left[\begin{array}{c}(\rho+1, \mu) \\ (\sigma+\rho+1, \mu)\end{array}\middle|\, \xi(b-x)^{\mu}\right]$$

holds.

Corollary 23. *Let $\rho, \xi, \mu \in \mathbb{C}$ such that $\Re(\sigma) \geq 0$. For $x > a$, the following integral formula*

$$\left(D_{a+}^{\sigma}(t-a)^{\rho} \exp\left(\xi(t-a)^{\mu}\right)\right)(x) = (x-a)^{\sigma+\rho}\, {}_1\Psi_1\left[\begin{array}{c}(\rho+1, \mu) \\ (\rho-\sigma+1, \mu)\end{array}\middle|\, \xi(x-a)^{\mu}\right]$$

holds.

Corollary 24. *Let $\rho, \xi, \mu \in \mathbb{C}$ such that $\Re(\sigma) \geq 0$. For $b > x$, the following integral formula*

$$\left(D_{b-}^{\sigma}(b-t)^{\rho} \exp\left(\xi(b-t)^{\mu}\right)\right)(x) = (b-x)^{\sigma+\rho}\, {}_1\Psi_1\left[\begin{array}{c}(\rho+1, \mu) \\ (\rho-\sigma+1, \mu)\end{array}\middle|\, \xi(b-x)^{\mu}\right]$$

holds.

Choosing $r = m = \gamma = q = \beta = 1; \alpha = 2$, the generalized multi-index Mittag–Leffler function $E_{(\alpha_j,\beta_j)_m}^{\gamma,q}(.)$ reduces to the Mittag–Leffler function $E_{(2,1)_1}^{1,1}(.) = E_{(2,1)}^{1,1}(.) = \cosh\sqrt{(.)}$ defined by the great Swedish mathematician G.M. Using this concept, the results established in equations (3.1), (3.6), (3.7) and (3.12) are reduced to the following form.

Corollary 25. *Let $\rho, \xi, \mu \in \mathbb{C}$ such that $\Re(\sigma) > 0$. For $x > a$, the following integral formula*

$$\left(I_{a+}^{\sigma}(t-a)^{\rho} \cosh\sqrt{(\xi(t-a)^{\mu})}\right)(x) = (x-a)^{\sigma+\rho}\, {}_1\Psi_2\left[\begin{array}{l}(\rho+1, \mu) \\ (1, 2), (\sigma+\rho+1, \mu)\end{array}\middle|\, \xi(x-a)^{\mu}\right]$$

holds.

Corollary 26. *Let $\rho, \xi, \mu \in \mathbb{C}$ such that $\Re(\sigma) > 0$. For $b > x$, the following integral formula*

$$\left(I_{b-}^{\sigma}(b-t)^{\rho} \cosh\sqrt{(\xi(b-t)^{\mu})}\right)(x) = (b-x)^{\sigma+\rho}\, {}_1\Psi_2\left[\begin{array}{l}(\rho+1, \mu) \\ (1, 2), (\sigma+\rho+1, \mu)\end{array}\middle|\, \xi(b-x)^{\mu}\right]$$

holds.

Corollary 27. *Let $\rho, \xi, \mu \in \mathbb{C}$ such that $\Re(\sigma) \geq 0$. For $x > a$, the following integral formula*

$$\left(D^{\sigma}_{a+}(t-a)^{\rho}\cosh\sqrt{(\xi(t-a)^{\mu})}\right)(x) = (x-a)^{\sigma+\rho}\,{}_1\Psi_2\left[\begin{array}{l}(\rho+1,\mu)\\(1,2),(\rho-\sigma+1,\mu)\end{array}\middle|\xi(x-a)^{\mu}\right]$$

holds.

Corollary 28. *Let $\rho, \xi, \mu \in \mathbb{C}$ such that $\Re(\sigma) \geq 0$. For $b > x$, the following integral formula*

$$\left(D^{\sigma}_{b-}(b-t)^{\rho}\cosh\sqrt{(\xi(b-t)^{\mu})}\right)(x) = (b-x)^{\sigma+\rho}\,{}_1\Psi_2\left[\begin{array}{l}(\rho+1,\mu)\\(1,2),(\rho-\sigma+1,\mu)\end{array}\middle|\xi(b-x)^{\mu}\right]$$

holds.

Choosing $r = m = \gamma = q = \beta = 1$; $\alpha = 0$, the generalized multi-index Mittag–Leffler function $E^{\gamma,q}_{(\alpha_j,\beta_j)_m}(z)$ reduces to the Mittag–Leffler function $E^{1,1}_{(0,1)_1}(z) = E^{1,1}_{(0,1)}(z) = (1-z)^{-1}$ (where $|z| < 1$). Using this concept, the results established in equations (3.1), (3.6), (3.7) and (3.12) are reduced to the following form.

Corollary 29. *Let $\rho, \xi, \mu \in \mathbb{C}$ such that $\Re(\sigma) > 0$. For $|\xi(t-a)^{\mu}| < 1$, the following integral formula*

$$\left(I^{\sigma}_{a+}\frac{(t-a)^{\rho}}{1-\xi(t-a)^{\mu}}\right)(x) = (x-a)^{\sigma+\rho}\,{}_1\Psi_1\left[\begin{array}{l}(\rho+1,\mu)\\(\sigma+\rho+1,\mu)\end{array}\middle|\xi(x-a)^{\mu}\right]$$

holds.

Corollary 30. *Let $\rho, \xi, \mu \in \mathbb{C}$ such that $\Re(\sigma) > 0$. For $|\xi(b-t)^{\mu}| < 1$, the following integral formula*

$$\left(I^{\sigma}_{b-}\frac{(b-t)^{\rho}}{1-\xi(b-t)^{\mu}}\right)(x) = (b-x)^{\sigma+\rho}\,{}_1\Psi_1\left[\begin{array}{l}(\rho+1,\mu)\\(\sigma+\rho+1,\mu)\end{array}\middle|\xi(b-x)^{\mu}\right].$$

holds.

Corollary 31. *Let $\rho, \xi, \mu \in \mathbb{C}$ such that $\Re(\sigma) \geq 0$. For $|\xi(t-a)^{\mu}| < 1$, the following integral formula*

$$\left(D^{\sigma}_{a+}\frac{(t-a)^{\rho}}{1-\xi(t-a)^{\mu}}\right)(x) = (x-a)^{\sigma+\rho}\,{}_1\Psi_1\left[\begin{array}{l}(\rho+1,\mu)\\(\rho-\sigma+1,\mu)\end{array}\middle|\xi(x-a)^{\mu}\right]$$

holds.

Corollary 32. *Let $\rho, \xi, \mu \in \mathbb{C}$ such that $\Re(\sigma) \geq 0$. For $|\xi(b-t)^{\mu}| < 1$, the following integral formula*

$$\left(D^{\sigma}_{b-}\frac{(b-t)^{\rho}}{1-\xi(b-t)^{\mu}}\right)(x) = (b-x)^{\sigma+\rho}\,{}_1\Psi_1\left[\begin{array}{l}(\rho+1,\mu)\\(\rho-\sigma+1,\mu)\end{array}\middle|\xi(b-x)^{\mu}\right]$$

holds.

4. Numerical Results and Graphical Interpretation

In this section, the numerical results of the formulae established in equations (3.1), (3.6), (3.7) and (3.12) are presented in Tables 1, 2, 3 and 4, respectively. The graphs of the formulae are plotted in Figures 1–7. All these numerical values are selected for $r = m = 2$. The product of Mittag–Leffler function for these values reduces to the form

$$\prod_{i=1}^{2} E^{\gamma_i,q_i}_{(\alpha_{ij},\beta_{ij})_2}(.) = E^{\gamma_1,q_1}_{(\alpha_{11},\beta_{11};\alpha_{12},\beta_{12})}(.)\,E^{\gamma_2,q_2}_{(\alpha_{21},\beta_{21};\alpha_{22},\beta_{22})}(.)$$

We select the values of parameters $\gamma_1 = 0.2$, $\gamma_2 = 0.3$, $q_1 = 0.05$, $q_2 = 0.06$; $\alpha_{11} = 0.3$, $\beta_{11} = 0.5$; $\alpha_{12} = 0.4$, $\beta_{12} = 0.6$; $\alpha_{21} = 0.5$, $\beta_{21} = 0.7$; $\alpha_{22} = 0.6$, $\beta_{22} = 0.8$; $\nu = 0.1$; $\xi = 0.5$; $\mu = 0.2$; in all Figures 1–7 and Tables 1–4, which are fixed for our investigation for generalized multi-index Mittag–Leffler function. It is also noted that imaginary part of complex numerical values of fractional integrals and fractional derivatives arguments are ignored in each case. The generalized multi-index Mittag–Leffler function $E^{\gamma_i,q_i}_{(\alpha_{ij},\beta_{ij})_m}(.)$ given in equation (1.4) is in summation form and all the results established in equations (3.1), (3.6), (3.7) and (3.12) involve the generalized multi-index Mittag–Leffler

Table 1. Numerical values of equation (3.1)

x	$\sigma = 0.1$	$\sigma = 0.3$	$\sigma = 0.5$	$\sigma = 0.7$
0.00	7.38	-1.90	-12.39	-19.64
0.50	5.95	-1.21	-9.08	-14.33
1.00	4.68	-0.69	-6.40	-10.08
1.50	3.56	-0.32	-4.31	-6.77
2.00	2.59	-0.08	-2.72	-4.28
2.50	1.78	0.05	-1.58	-2.49
3.00	1.13	0.10	-0.81	-1.28
3.50	0.63	0.10	-0.34	-0.55
4.00	0.27	0.06	-0.10	-0.17
4.50	0.07	0.02	-0.01	-0.02
5.00	0.00	0.00	0.00	0.00
5.50	0.11	0.08	0.05	0.04
6.00	0.50	0.41	0.33	0.26
6.50	1.25	1.10	0.95	0.82
7.00	2.41	2.24	2.05	1.86
7.50	4.01	3.89	3.73	3.53
8.00	6.10	6.14	6.10	5.98
8.50	8.70	19.03	9.24	9.35
9.00	11.85	12.62	13.27	13.78
9.50	15.58	16.98	18.27	19.41
10.00	19.91	22.15	24.33	26.40

Table 2. Numerical values of the equation (3.6)

x	$\sigma = 0.1$	$\sigma = 0.3$	$\sigma = 0.5$	$\sigma = 0.7$
0.00	19.91	22.15	24.33	26.40
0.50	15.58	16.98	18.27	19.41
1.00	11.85	12.62	13.27	13.78
1.50	8.70	9.03	9.24	9.35
2.00	6.10	6.14	6.10	5.98
2.50	4.01	3.89	3.73	3.53
3.00	2.41	2.24	2.05	1.86
3.50	1.25	1.10	0.95	0.82
4.00	0.50	0.41	0.33	0.26
4.50	0.11	0.08	0.05	0.04
5.00	0.00	0.00	0.00	0.00
5.50	0.07	0.02	-0.01	-0.02
6.00	0.27	0.06	-0.10	-0.17
6.50	0.63	0.10	-0.34	-0.55
7.00	1.13	0.10	-0.81	-1.28
7.50	1.78	0.05	-1.58	-2.49
8.00	2.59	-0.08	-2.72	-4.28
8.50	3.56	-0.32	-4.31	-6.77
9.00	4.68	-0.69	-6.40	-10.08
9.50	5.95	-1.21	-9.08	-14.33
10.00	7.38	-1.90	-12.39	-19.64

function. To establish the graphs and data-base of the results, we take a sum of the first 500 terms of the summation involved in each result.

In Figure 1, we have opted the parametric value as $\rho = .01$; $a = b = .5$ and $\sigma = 0.01 : 0.02 : 0.07$. $\rho = 1.8$; $a = b = 2$ and $\sigma = 0.1 : 0.2 : 0.7$ are opted for Figure 2. $\rho = 2.5$; $a = b = 2$ and $\sigma = 0.1 : 0.2 : 0.7$ are opted for Figure 3. Tables 1–2 are established on the basis of parametric values as those of Figure 4. Figures 5-7 are plotted for the values of the parameters as those of the values of the parameters of the Figures 1–3, which depict that the graphs of formulae (3.1) and (3.6) are reflected identically in each figure, respectively.

5. Image Formulas Associated With Integral Transform

In this section, we establish certain theorems involving the results obtained in previous section associated with the integral transforms like Beta transform, Laplace transform and Whittaker transform.

5.1. **Beta transform.** The Beta transform of $f(z)$ is defined as [30]

$$B\{f(z) : \alpha, \beta\} = \int_0^1 z^{\alpha-1}(1-z)^{\beta-1} f(z)dz,$$

Theorem 5. *Let* $m, r \in \mathbb{N}$, $\alpha_{ij}, \beta_{ij}, \gamma_i, q_i, \rho, \xi, \mu \in \mathbb{C}\,(i = 1, \ldots, r; j = 1, \ldots, m)$ *such that* $\sum_{j=1}^{m} \Re(\alpha_{ij}) > \max\{0; \Re(q_i) - 1\}; \Re(q_i) > 0$ *and* $\Re(\sigma) > 0$. *For* $x > a$, *the following integral formula*

$$B\left\{\left(I^{\sigma}_{a+}(t-a)^{\rho} \prod_{i=1}^{r} E^{\gamma_i, q_i}_{(\alpha_{ij},\beta_{ij})_m}\left[\xi\big(z(t-a)\big)^{\mu}\right]\right)(x) : \alpha, \beta\right\}$$
$$= \frac{\Gamma(\beta)(x-a)^{\sigma+\rho}}{\Gamma(\gamma_1)\ldots\Gamma(\gamma_r)}\, {}_{r+2}\Psi_{(m+1)r+1}\left[\begin{array}{c} A_1 \\ B_1 \end{array}\middle| (\xi(x-a)^{\mu})^{r}\right], \tag{5.1}$$

$$\begin{aligned} A_1 &= (\gamma_1, q_1), \ldots, (\gamma_r, q_r), (\rho+1, \mu r), (\alpha, \mu r), \\ B_1 &= (\beta_{1j}, \alpha_{1j})_{1,m}, \ldots, (\beta_{rj}, \alpha_{rj})_{1,m}, (\sigma+\rho+1, \mu r), (\alpha+\beta, \mu r), (-1, 1)_{1,r-1} \end{aligned} \tag{5.2}$$

holds.

Proof. For convenience, we denote the left-hand side of the result (5.1) by $\mathcal{B}$, then using the definition of beta transform, we have

$$\mathcal{B} = \int_0^1 z^{\alpha-1}(1-z)^{\beta-1}\left(I^{\sigma}_{a+}(t-a)^{\rho} \prod_{i=1}^{r} E^{\gamma_i, q_i}_{(\alpha_{ij},\beta_{ij})_m}\left[\xi\big(z(t-a)\big)^{\mu}\right]\right)(x)dz, \tag{5.3}$$

further applying the result from equation (3.5) to the above equation (5.3), then interchanging the order of integration and summation, we have

$$\mathcal{B} = \frac{(x-a)^{\sigma+\rho}}{\Gamma(\gamma_1)\ldots\Gamma(\gamma_r)}\left\{\sum_{k=0}^{\infty} \frac{\Gamma(\gamma_1+kq_1)\ldots\Gamma(\gamma_r+kq_r)}{\prod_{j=1}^{m}\Gamma(k\alpha_{1j}+\beta_{1j})\cdots\prod_{j=1}^{m}\Gamma(k\alpha_{rj}+\beta_{rj})}\right.$$
$$\left.\times\frac{\Gamma(\rho+\mu kr+1)}{\Gamma(\sigma+\rho+\mu kr+1)(\Gamma(k-1))^{r-1}}\,\frac{(\xi(x-a)^{\mu})^{kr}}{k!}\right\}\int_0^1 z^{\alpha+\mu kr-1}(1-z)^{\beta-1}dz,$$

applying the definition of beta transform and after little simplification, we have

$$\mathcal{B} = \frac{\Gamma(\beta)(x-a)^{\sigma+\rho}}{\Gamma(\gamma_1)\ldots\Gamma(\gamma_r)}\left\{\sum_{k=0}^{\infty} \frac{\Gamma(\gamma_1+kq_1)\ldots\Gamma(\gamma_r+kq_r)}{\prod_{j=1}^{m}\Gamma(k\alpha_{1j}+\beta_{1j})\cdots\prod_{j=1}^{m}\Gamma(k\alpha_{rj}+\beta_{rj})}\right.$$
$$\left.\times\frac{\Gamma(\rho+\mu kr+1)}{\Gamma(\sigma+\rho+\mu kr+1)(\Gamma(k-1))^{r-1}}\,\frac{\Gamma(\alpha+\mu kr)}{\Gamma(\alpha+\beta+\mu kr)}\,\frac{(\xi(x-a)^{\mu})^{kr}}{k!}\right\},$$

now interpreting in view of (1.4), we have the required result (5.1). □

Theorem 6. *Let* $m, r \in \mathbb{N}$, $\alpha_{ij}, \beta_{ij}, \gamma_i, q_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r;\ j = 1, \ldots, m)$ *such that* $\sum_{j=1}^{m} \Re(\alpha_{ij}) > \max\{0; \Re(q_i) - 1\}$; $\Re(q_i) > 0$ *and* $\Re(\sigma) > 0$. *For* $b > x$, *the following integral formula*

$$B\left\{\left(I_{b-}^{\sigma}(b-t)^{\rho}\prod_{i=1}^{r}E_{(\alpha_{ij},\beta_{ij})_m}^{\gamma_i,q_i}\left[\xi(z(b-t))^{\mu}\right]\right)(x):\alpha,\beta\right\} = \frac{\Gamma(\beta)(b-x)^{\sigma+\rho}}{\Gamma(\gamma_1)\ldots\Gamma(\gamma_r)}\,{}_{r+2}\Psi_{(m+1)r+1}\left[\begin{array}{c|c} A_1 & \\ B_1 & (\xi(b-x)^{\mu})^r \end{array}\right]$$

holds, where A_1 *and* B_1 *are defined in equation* (5.2).

Theorem 7. *Let* $m, r \in \mathbb{N}$, $\alpha_{ij}, \beta_{ij}, \gamma_i, q_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r;\ j = 1, \ldots, m)$ *such that* $\sum_{j=1}^{m} \Re(\alpha_{ij}) > \max\{0; \Re(q_i) - 1\}$; $\Re(q_i) > 0$ *and* $\Re(\sigma) \geq 0$. *For* $x > a$, *the following integral formula*

$$B\left\{\left(D_{a+}^{\sigma}(t-a)^{\rho}\prod_{i=1}^{r}E_{(\alpha_{ij},\beta_{ij})_m}^{\gamma_i,q_i}\left[\xi(z(t-a))^{\mu}\right]\right)(x):\alpha,\beta\right\} = \frac{\Gamma(\beta)(x-a)^{\sigma+\rho}}{\Gamma(\gamma_1)\ldots\Gamma(\gamma_r)}\,{}_{r+2}\Psi_{(m+1)r+1}\left[\begin{array}{c|c} A_2 & \\ B_2 & (\xi(x-a)^{\mu})^r \end{array}\right],$$

$$\begin{aligned} A_2 &= (\gamma_1, q_1), \ldots, (\gamma_r, q_r), (\rho+1, \mu r), (\alpha, \mu r), \\ B_2 &= (\beta_{1j}, \alpha_{1j})_{1,m}, \ldots, (\beta_{rj}, \alpha_{rj})_{1,m}, (\rho-\sigma+1, \mu r), (\alpha+\beta, \mu r), (-1,1)_{1,r-1} \end{aligned} \tag{5.4}$$

holds.

TABLE 3. Numerical values of the equation (3.7)

x	$\sigma = 0.1$	$\sigma = 0.3$	$\sigma = 0.5$	$\sigma = 0.7$
0.00	8.84	-4.71	-36.63	-82.17
0.50	7.13	-3.17	-26.86	-59.87
1.00	5.60	-1.98	-18.98	-42.04
1.50	4.26	-1.11	-12.80	-28.17
2.00	3.11	-0.51	-8.11	-17.75
2.50	2.14	-0.14	-4.73	-10.29
3.00	1.35	0.05	-2.43	-5.29
3.50	0.75	0.11	-1.03	-2.25
4.00	0.33	0.08	-0.30	-0.68
4.50	0.08	0.03	-0.04	-0.09
5.00	0.00	0.00	0.00	0.00
5.50	0.13	0.14	0.14	0.14
6.00	0.61	0.73	0.86	0.99
6.50	1.52	1.98	2.53	3.18
7.00	2.93	4.04	5.49	7.31
7.50	4.89	7.06	10.02	13.99
8.00	7.44	11.15	16.43	23.80
8.50	10.63	16.43	24.99	37.36
9.00	14.48	23.01	35.97	55.26
9.50	19.05	31.00	49.64	78.12
10.00	24.35	40.49	66.25	106.53

TABLE 4. Numerical values of the equation (3.12)

x	$\sigma = 0.1$	$\sigma = 0.3$	$\sigma = 0.5$	$\sigma = 0.7$
0.00	24.35	40.49	66.25	106.53
0.50	19.05	31.00	49.64	78.12
1.00	14.48	23.01	35.97	55.26
1.50	10.63	16.43	24.99	37.36
2.00	7.44	11.15	16.43	23.80
2.50	4.89	7.06	10.02	13.99
3.00	2.93	4.04	5.49	7.31
3.50	1.52	1.98	2.53	3.18
4.00	0.61	0.73	0.86	0.99
4.50	0.13	0.14	0.14	0.14
5.00	0.00	0.00	0.00	0.00
5.50	0.08	0.03	-0.04	-0.09
6.00	0.33	0.08	-0.30	-0.68
6.50	0.75	0.11	-1.03	-2.25
7.00	1.35	0.05	-2.43	-5.29
7.50	2.14	-0.14	-4.73	-10.29
8.00	3.11	-0.51	-8.11	-17.75
8.50	4.26	-1.11	-12.80	-28.17
9.00	5.60	-1.98	-18.98	-42.04
9.50	7.13	-3.17	-26.86	-59.87
10.00	8.84	-4.71	-36.63	-82.17

Theorem 8. *Let* $m, r \in \mathbb{N}$, $\alpha_{ij}, \beta_{ij}, \gamma_i, q_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \dots, r; j = 1, \dots, m)$ *such that* $\sum_{j=1}^{m} \Re(\alpha_{ij}) > \max\{0; \Re(q_i) - 1\}$; $\Re(q_i) > 0$ *and* $\Re(\sigma) \geq 0$. *For* $b > x$, *the following integral formula*

$$B\left\{ \left(D_{b-}^{\sigma} (b-t)^{\rho} \prod_{i=1}^{r} E_{(\alpha_{ij}, \beta_{ij})_m}^{\gamma_i, q_i} \left[\xi (z(b-t))^{\mu} \right] \right)(x) : \alpha, \beta \right\}$$
$$= \frac{\Gamma(\beta)(b-x)^{\sigma+\rho}}{\Gamma(\gamma_1)\dots\Gamma(\gamma_r)} \, {}_{r+2}\Psi_{(m+1)r+1} \left[\begin{array}{c} A_2 \\ B_2 \end{array} \middle| (\xi(b-x)^{\mu})^r \right]$$

holds, where A_2 *and* B_2 *are defined in equation* (5.4).

Proof. The proof of Theorems 6, 7 and 8 is the same as that of Theorem 5, therefore we omit the details. □

5.2. **Laplace transform.** The Laplace transform of $f(z)$ is defined as [30]

$$L\{f(z)\} = \int_{0}^{\infty} e^{-sz} f(z) dz$$

Theorem 9. *Let $m, r \in \mathbb{N}$, $\alpha_{ij}, \beta_{ij}, \gamma_i, q_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r;\ j = 1, \ldots, m)$ such that $\sum_{j=1}^{m} \Re(\alpha_{ij}) > \max\{0; \Re(q_i) - 1\}$; $\Re(q_i) > 0$ and $\Re(\sigma) > 0$. For $x > a$, the following integral formula*

$$L\left\{ z^{l-1}\left(I_{a+}^{\sigma}(t-a)^{\rho} \prod_{i=1}^{r} E_{(\alpha_{ij},\beta_{ij})_m}^{\gamma_i, q_i}\left[\xi(z(t-a))^{\mu}\right]\right)(x)\right\}$$
$$= \frac{(x-a)^{\sigma+\rho}}{s^l \Gamma(\gamma_1)\ldots\Gamma(\gamma_r)}\, {}_{r+2}\Psi_{mr+r}\left[\begin{array}{c} C_1 \\ D_1 \end{array} \middle| \left(\xi\left(\frac{x-a}{s}\right)^{\mu}\right)^{r}\right], \tag{5.5}$$

$$C_1 = (\gamma_1, q_1), \ldots, (\gamma_r, q_r), (\rho+1, \mu r), (l, \mu r),$$
$$D_1 = (\beta_{1j}, \alpha_{1j})_{1,m}, \ldots, (\beta_{rj}, \alpha_{rj})_{1,m}, (\sigma+\rho+1, \mu r), (-1, 1)_{1,r-1}$$

holds.

Proof. For convenience, we denote the left-hand side of the result (5.5) by $\mathcal{L}$, then using the definition of Laplace transform, we have:

$$\mathcal{L} = \int_0^{\infty} e^{-sz} z^{l-1}\left(I_{a+}^{\sigma}(t-a)^{\rho} \prod_{i=1}^{r} E_{(\alpha_{ij},\beta_{ij})_m}^{\gamma_i, q_i}\left[\xi(z(t-a))^{\mu}\right]\right)(x)dz, \tag{5.6}$$

further applying the result from equation (3.5) to the above equation (5.6), interchanging the order of integration and summation, we have

$$\mathcal{L} = \frac{(x-a)^{\sigma+\rho}}{\Gamma(\gamma_1)\ldots\Gamma(\gamma_r)}\left\{\sum_{k=0}^{\infty} \frac{\Gamma(\gamma_1+kq_1)\ldots\Gamma(\gamma_r+kq_r)}{\prod_{j=1}^{m}\Gamma(k\alpha_{1j}+\beta_{1j})\cdots\prod_{j=1}^{m}\Gamma(k\alpha_{rj}+\beta_{rj})}\right.$$
$$\left.\times \frac{\Gamma(\rho+\mu kr+1)}{\Gamma(\sigma+\rho+\mu kr+1)(\Gamma(k-1))^{r-1}} \frac{(\xi(x-a)^{\mu})^{kr}}{k!}\right\} \int_0^{\infty} e^{-sz} z^{l+\mu kr-1} dz,$$

applying the definition of Laplace transform, after little simplification, we have

$$\mathcal{L} = \frac{(x-a)^{\sigma+\rho}}{s^l\Gamma(\gamma_1)\ldots\Gamma(\gamma_r)}\left\{\sum_{k=0}^{\infty} \frac{\Gamma(\gamma_1+kq_1)\ldots\Gamma(\gamma_r+kq_r)}{\prod_{j=1}^{m}\Gamma(k\alpha_{1j}+\beta_{1j})\cdots\prod_{j=1}^{m}\Gamma(k\alpha_{rj}+\beta_{rj})}\right.$$
$$\left.\times \frac{\Gamma(\rho+\mu kr+1)\Gamma(l+\mu kr)}{\Gamma(\sigma+\rho+\mu kr+1)(\Gamma(k-1))^{r-1}} \frac{1}{k!}\left(\xi\left(\frac{x-a}{s}\right)^{\mu}\right)^{kr}\right\} \tag{5.7}$$

interpret the above equation (5.7) in the view of (1.4), we can easily arrive at the required result (5.5). □

Theorem 10. *Let $m, r \in \mathbb{N}$, $\alpha_{ij}, \beta_{ij}, \gamma_i, q_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r;\ j = 1, \ldots, m)$ such that $\sum_{j=1}^{m} \Re(\alpha_{ij}) > \max\{0; \Re(q_i) - 1\}$; $\Re(q_i) > 0$ and $\Re(\sigma) > 0$. For $b > x$, the following integral formula*

$$L\left\{ z^{l-1}\left(I_{b-}^{\sigma}(b-t)^{\rho} \prod_{i=1}^{r} E_{(\alpha_{ij},\beta_{ij})_m}^{\gamma_i, q_i}\left[\xi(z(b-t))^{\mu}\right]\right)(x)\right\} = \frac{(b-x)^{\sigma+\rho}}{s^l \Gamma(\gamma_1)\ldots\Gamma(\gamma_r)}\, {}_{r+2}\Psi_{mr+r}$$
$$\times\left[\begin{array}{c} (\gamma_1, q_1), \ldots, (\gamma_r, q_r), (\rho+1, \mu r), (l, \mu r) \\ (\beta_{1j}, \alpha_{1j})_{1,m}, \ldots, (\beta_{rj}, \alpha_{rj})_{1,m}, (\sigma+\rho+1, \mu r), (-1, 1)_{1,r-1} \end{array} \middle| \left(\xi\left(\frac{b-x}{s}\right)^{\mu}\right)^{r}\right]$$

holds.

Theorem 11. *Let $m, r \in \mathbb{N}$, $\alpha_{ij}, \beta_{ij}, \gamma_i, q_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r;\ j = 1, \ldots, m)$ such that $\sum_{j=1}^{m} \Re(\alpha_{ij}) > \max\{0; \Re(q_i) - 1\}$; $\Re(q_i) > 0$ and $\Re(\sigma) \geq 0$. For $x > a$, the following integral formula*

$$L\left\{ z^{l-1} \left(D^{\sigma}_{a+} (t-a)^{\rho} \prod_{i=1}^{r} E^{\gamma_i, q_i}_{(\alpha_{ij}, \beta_{ij})_m} \left[\xi (z(t-a))^{\mu} \right] \right)(x) \right\} = \frac{(x-a)^{\sigma+\rho}}{s^l \Gamma(\gamma_1) \ldots \Gamma(\gamma_r)} {}_{r+2}\Psi_{mr+r}$$
$$\times \left[\begin{array}{c} (\gamma_1, q_1), \ldots, (\gamma_r, q_r), (\rho+1, \mu r), (l, \mu r) \\ (\beta_{1j}, \alpha_{1j})_{1,m}, \ldots, (\beta_{rj}, \alpha_{rj})_{1,m}, (\rho - \sigma + 1, \mu r), (-1, 1)_{1, r-1} \end{array} \middle| \left(\xi \left(\frac{x-a}{s} \right)^{\mu} \right)^{r} \right]$$

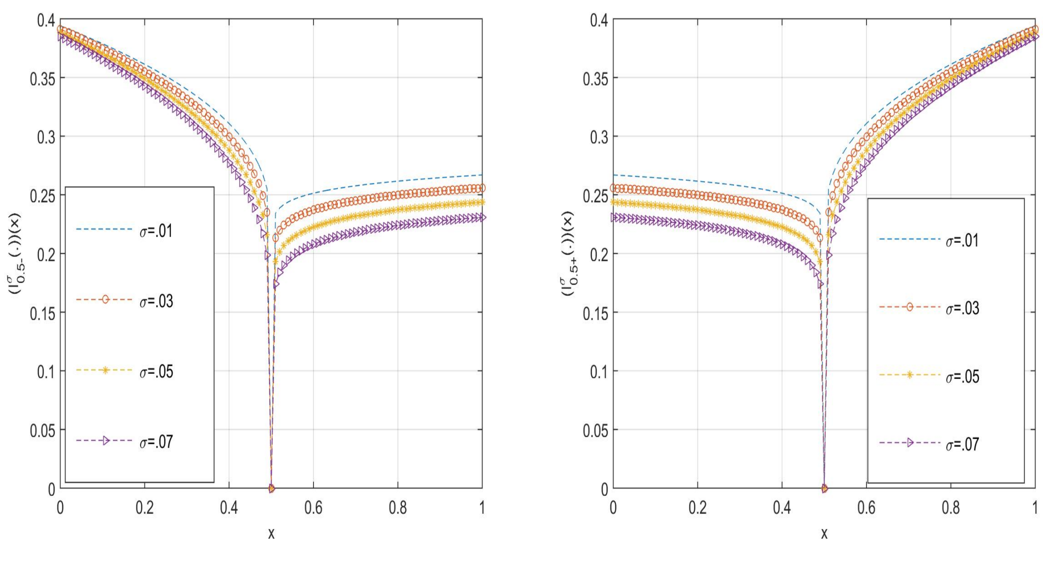
holds.

Theorem 12. *Let $m, r \in \mathbb{N}$, $\alpha_{ij}, \beta_{ij}, \gamma_i, q_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r;\ j = 1, \ldots, m)$ such that $\sum_{j=1}^{m} \Re(\alpha_{ij}) > \max\{0; \Re(q_i) - 1\}$; $\Re(q_i) > 0$ and $\Re(\sigma) \geq 0$. For $b > x$, the following integral formula*

$$L\left\{ z^{l-1} \left(D^{\sigma}_{b+} (b-t)^{\rho} \prod_{i=1}^{r} E^{\gamma_i, q_i}_{(\alpha_{ij}, \beta_{ij})_m} \left[\xi (z(b-t))^{\mu} \right] \right)(x) \right\} = \frac{(b-x)^{\sigma+\rho}}{s^l \Gamma(\gamma_1) \ldots \Gamma(\gamma_r)} {}_{r+2}\Psi_{mr+r}$$
$$\times \left[\begin{array}{c} (\gamma_1, q_1), \ldots, (\gamma_r, q_r), (\rho+1, \mu r), (l, \mu r) \\ (\beta_{1j}, \alpha_{1j})_{1,m}, \ldots, (\beta_{rj}, \alpha_{rj})_{1,m}, (\rho - \sigma + 1, \mu r), (-1, 1)_{1, r-1} \end{array} \middle| \left(\xi \left(\frac{b-x}{s} \right)^{\mu} \right)^{r} \right]$$

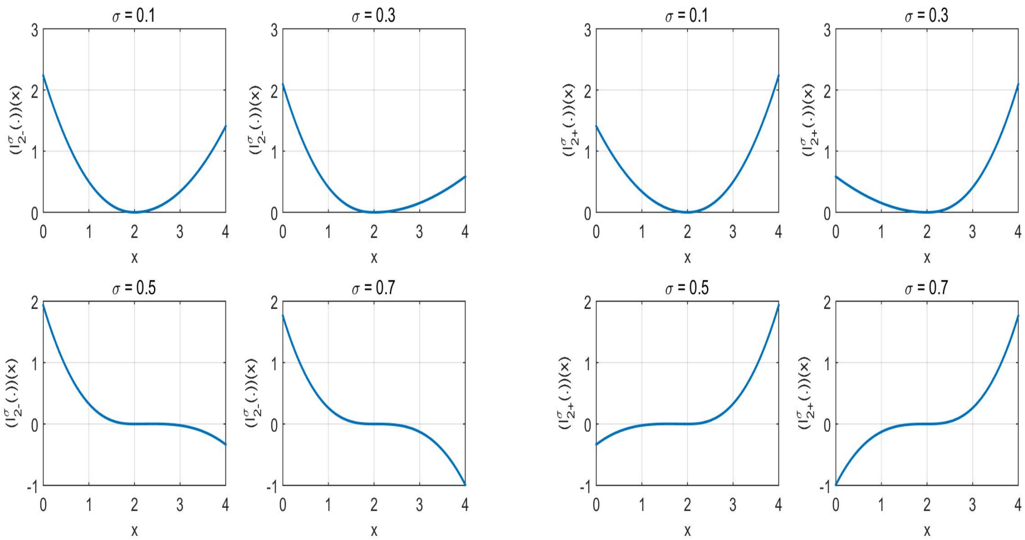
holds.

Proof. The proof of Theorems 10, 11 and 12 is the same as that of Theorem 9, therefore we omit the details. □



(a) Plot of Equation (3.6) (b) Plot of Equation (3.1)

FIGURE 1. Graph of fractional integral formulae for $\rho = .01$; $a = b = .5$



(a) Plot of Equation (3.6) (b) Plot of Equation (3.1)

FIGURE 2. Graph of fractional integral formulae for $\rho = 1.8$; $a = b = 2$

5.3. **Whittaker transform.**

Theorem 13. *Let* $m, r \in \mathbb{N}$, $\alpha_{ij}, \beta_{ij}, \gamma_i, q_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r;\ j = 1, \ldots, m)$ *such that* $\sum_{j=1}^{m} \Re(\alpha_{ij}) > \max\{0; \Re(q_i) - 1\}; \Re(q_i) > 0$ *and* $\Re(\sigma) > 0$. *For* $x > a$, *the following integral formula*

$$\int_0^\infty z^{\zeta-1} e^{-\eta z/2} W_{\tau,\omega}(\eta z) \left\{ \left(I^\sigma_{a+} (t-a)^\rho \prod_{i=1}^{r} E^{\gamma_i, q_i}_{(\alpha_{ij}, \beta_{ij})_m} \left[\xi (z(t-a))^\mu \right] \right)(x) \right\}$$

$$= \frac{(x-a)^{\sigma+\rho}}{\eta^\zeta \Gamma(\gamma_1) \ldots \Gamma(\gamma_r)} \, {}_{r+3}\Psi_{(m+1)r+1} \left[\begin{array}{c} A \\ B \end{array} \middle| \left(\xi \left(\frac{x-a}{\eta} \right)^\mu \right)^r \right] \tag{5.8}$$

holds, where

$$\begin{aligned} A &= (\gamma_1, q_1), \ldots, (\gamma_r, q_r), (\rho+1, \mu r), (1/2 + \omega + \zeta, \mu r), (1/2 - \omega + \zeta, \mu r) \\ B &= (\beta_{1j}, \alpha_{1j})_{1,m}, \ldots, (\beta_{rj}, \alpha_{rj})_{1,m}, (\sigma + \rho + 1, \mu r), (1/2 - \tau + \zeta, \mu r), (-1, 1)_{1,r-1}. \end{aligned} \tag{5.9}$$

Proof. For convenience, we denote the left-hand side of the result (5.8) by $\mathcal{W}$, further applying the result from equation (3.5) to the above equation, interchanging the order of integration and summation, we have

$$\mathcal{W} = \frac{(x-a)^{\sigma+\rho}}{\Gamma(\gamma_1) \ldots \Gamma(\gamma_r)} \left\{ \sum_{k=0}^{\infty} \frac{\Gamma(\gamma_1 + kq_1) \ldots \Gamma(\gamma_r + kq_r)}{\prod_{j=1}^{m} \Gamma(k\alpha_{1j} + \beta_{1j}) \cdots \prod_{j=1}^{m} \Gamma(k\alpha_{rj} + \beta_{rj})} \right.$$

$$\left. \times \frac{\Gamma(\rho + \mu k r + 1)}{\Gamma(\sigma + \rho + \mu k r + 1)(\Gamma(k-1))^{r-1}} \frac{(\xi (x-a)^\mu)^{kr}}{k!} \right\} \int_0^\infty z^{\zeta + \mu k r - 1} e^{-\eta z/2} W_{\tau,\omega}(\eta z) dz,$$

by substituting $\eta z = t$, we have

$$\mathcal{W} = \frac{(x-a)^{\sigma+\rho}}{\Gamma(\gamma_1)\dots\Gamma(\gamma_r)}\left\{\sum_{k=0}^{\infty}\frac{\Gamma(\gamma_1+kq_1)\dots\Gamma(\gamma_r+kq_r)}{\prod_{j=1}^{m}\Gamma(k\alpha_{1j}+\beta_{1j})\cdots\prod_{j=1}^{m}\Gamma(k\alpha_{rj}+\beta_{rj})}\right.$$
$$\left.\times\frac{\Gamma(\rho+\mu kr+1)}{\Gamma(\sigma+\rho+\mu kr+1)(\Gamma(k-1))^{r-1}}\frac{(\xi(x-a)^{\mu})^{kr}}{k!}\right\}\frac{1}{\eta^{\zeta+\mu kr}}\int_0^{\infty}t^{\zeta+\mu kr-1}e^{-t/2}W_{\tau,\omega}(t)dt.$$

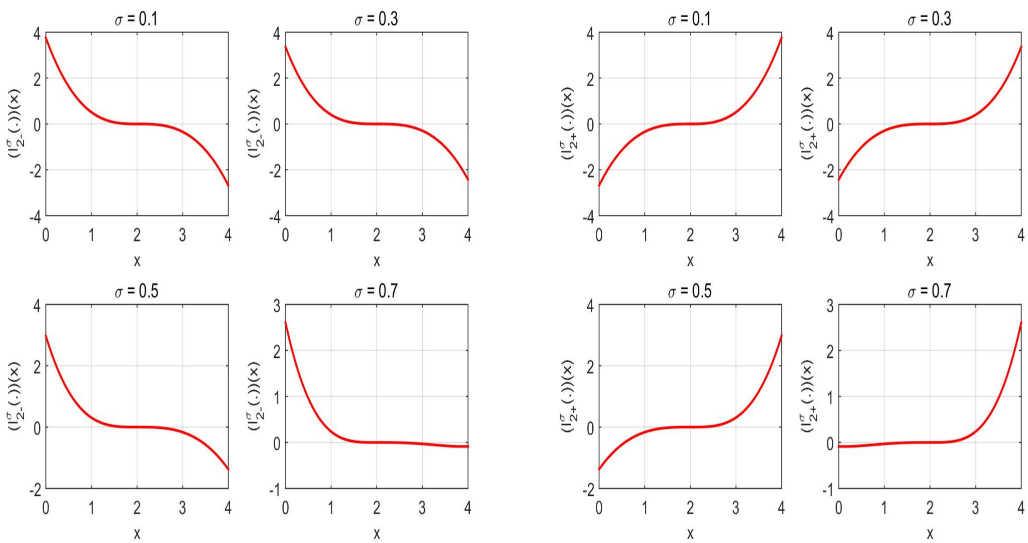
Now using the integral formula for Whittaker function

$$\int_0^{\infty}t^{\alpha-1}e^{-t/2}W_{\tau,\omega}(t)dt = \frac{\Gamma\left(1/2+\omega+\alpha\right)\Gamma\left(1/2-\omega+\alpha\right)}{\Gamma\left(1/2-\tau+\alpha\right)}\quad\left(\Re(\alpha\pm\omega) > \frac{-1}{2}\right)$$

and after little simplification, we have

$$\mathcal{W} = \frac{(x-a)^{\sigma+\rho}}{\eta^{\zeta}\Gamma(\gamma_1)\dots\Gamma(\gamma_r)}\left\{\sum_{k=0}^{\infty}\frac{\Gamma(\gamma_1+kq_1)\dots\Gamma(\gamma_r+kq_r)}{\prod_{j=1}^{m}\Gamma(k\alpha_{1j}+\beta_{1j})\cdots\prod_{j=1}^{m}\Gamma(k\alpha_{rj}+\beta_{rj})}\right.$$
$$\left.\times\frac{\Gamma(\rho+\mu kr+1)\Gamma\left(1/2+\omega+\zeta+\mu kr\right)\Gamma\left(1/2-\omega+\zeta+\mu kr\right)}{\Gamma(\sigma+\rho+\mu kr+1)\Gamma\left(1/2-\tau+\zeta+\mu kr\right)(\Gamma(k-1))^{r-1}}\frac{1}{k!}\left(\xi\left(\frac{x-a}{\eta}\right)^{\mu}\right)^{kr}\right\}\tag{5.10}$$

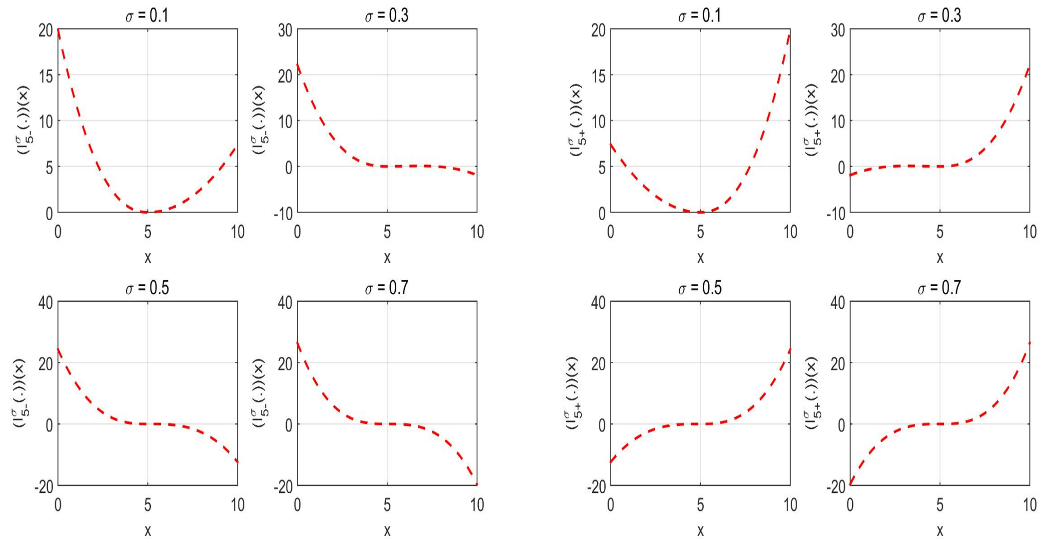
interpret the above equation (5.10) in the view of (1.4), we can easily arrive at the required result (5.5). □



(a) Plot of Equation (3.6)

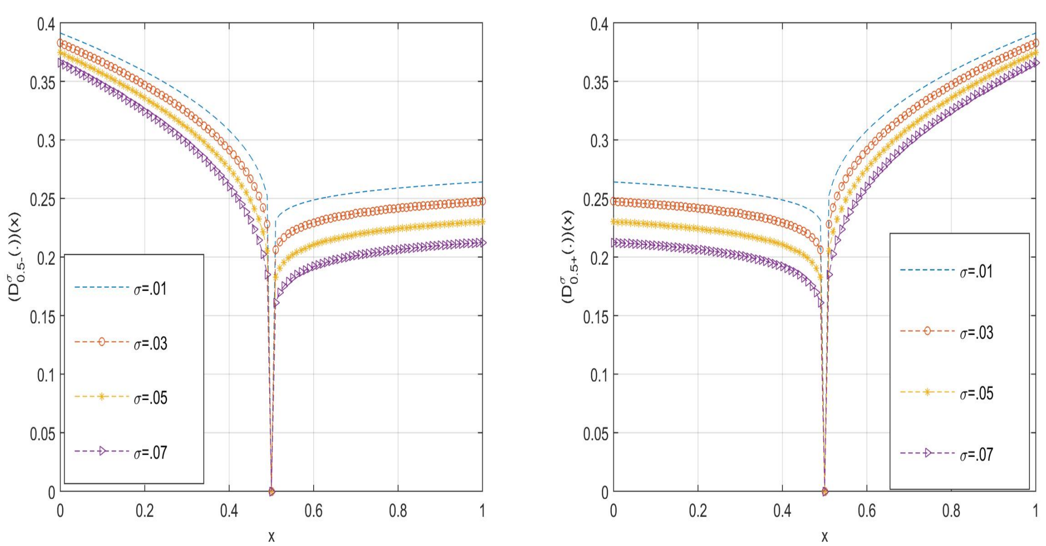
(b) Plot of Equation (3.1)

Figure 3. Graph of fractional integral formulae for $\rho = 2.5$; $a = b = 2$



(a) Plot of Equation (3.6) (b) Plot of Equation (3.1)

FIGURE 4. Graph of fractional integral formulae for $\rho = 1.9$; $a = b = 5$



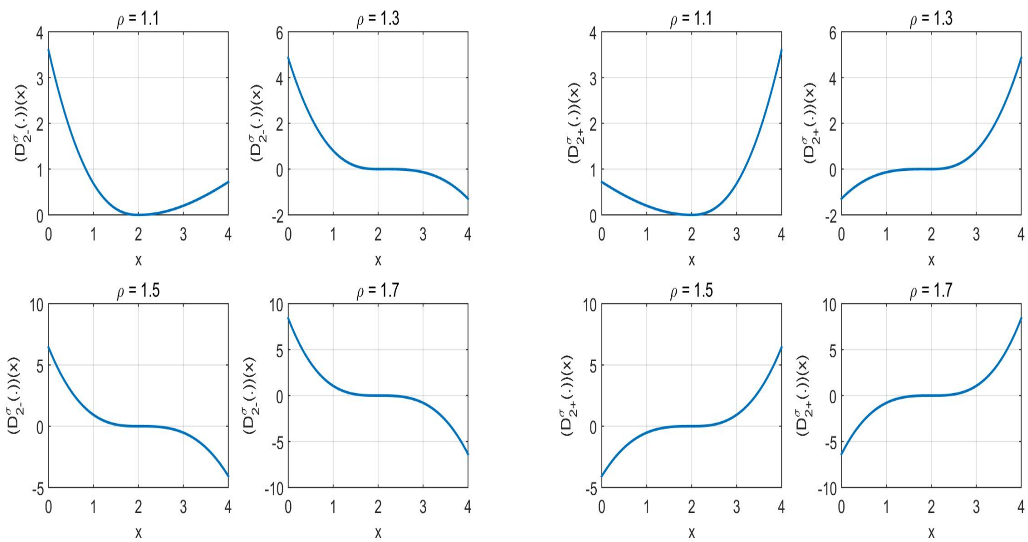
(a) Plot of Equation (3.12) (b) Plot of Equation (3.7)

FIGURE 5. Graph of fractional derivative formulae for $\rho = .01$; $a = b = 0.5$

Theorem 14. *Let* $m, r \in \mathbb{N}$, $\alpha_{ij}, \beta_{ij}, \gamma_i, q_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r;\ j = 1, \ldots, m)$ *such that* $\sum_{j=1}^{m} \Re(\alpha_{ij}) > \max\{0; \Re(q_i) - 1\}$; $\Re(q_i) > 0$ *and* $\Re(\sigma) > 0$. *For* $b > x$, *the following integral formula*

$$\int_0^\infty z^{\zeta-1} e^{-\eta z/2} W_{\tau,\omega}(\eta z) \left\{ \left(I_{b-}^{\sigma} (b-t)^{\rho} \prod_{i=1}^{r} E_{(\alpha_{ij},\beta_{ij})_m}^{\gamma_i, q_i} \left[\xi (z(b-t))^{\mu} \right] \right)(x) \right\}$$
$$= \frac{(b-x)^{\sigma+\rho}}{\eta^{\zeta} \Gamma(\gamma_1) \ldots \Gamma(\gamma_r)} {}_{r+3}\Psi_{(m+1)r+1} \left[\begin{matrix} A \\ B \end{matrix} \middle| \left(\xi \left(\frac{b-x}{\eta} \right)^{\mu} \right)^{r} \right]$$

holds, where A *and* B *are defined in equation* (5.9).



(a) Plot of Equation (3.12) (b) Plot of Equation (3.7)

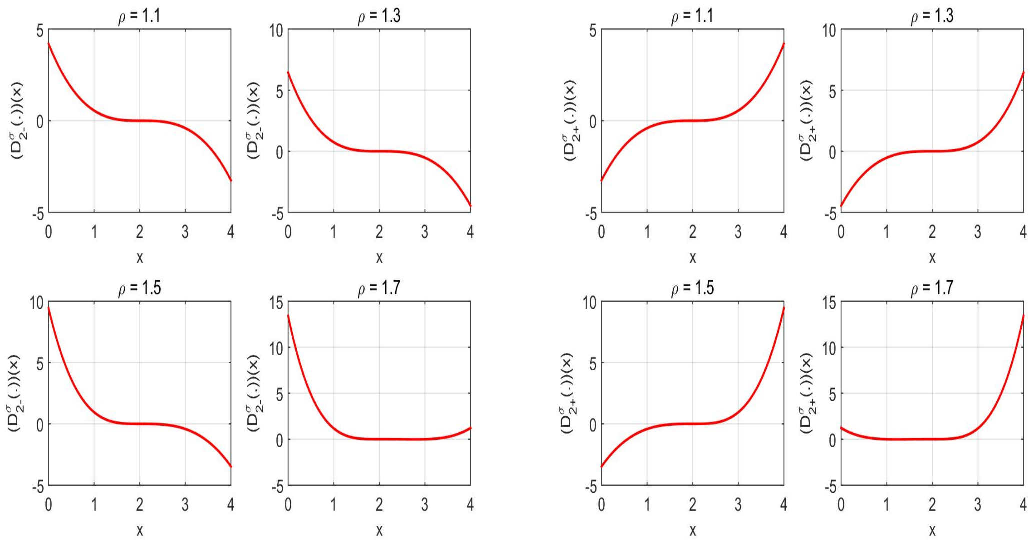
FIGURE 6. Graph of fractional derivative formulae for $\sigma = 1$; $a = b = 2$

Theorem 15. *Let* $m, r \in \mathbb{N}$, $\alpha_{ij}, \beta_{ij}, \gamma_i, q_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r;\ j = 1, \ldots, m)$ *such that* $\sum_{j=1}^{m} \Re(\alpha_{ij}) > \max\{0; \Re(q_i) - 1\}$; $\Re(q_i) > 0$ *and* $\Re(\sigma) \geq 0$. *For* $x > a$, *the following integral formula*

$$\int_0^\infty z^{\zeta-1} e^{-\eta z/2} W_{\tau,\omega}(\eta z) \left\{ \left(D_{a+}^{\sigma} (t-a)^{\rho} \prod_{i=1}^{r} E_{(\alpha_{ij},\beta_{ij})_m}^{\gamma_i, q_i} \left[\xi (z(t-a))^{\mu} \right] \right)(x) \right\}$$
$$= \frac{(x-a)^{\sigma+\rho}}{\eta^{\zeta} \Gamma(\gamma_1) \ldots \Gamma(\gamma_r)} {}_{r+3}\Psi_{(m+1)r+1} \left[\begin{matrix} C \\ D \end{matrix} \middle| \left(\xi \left(\frac{x-a}{\eta} \right)^{\mu} \right)^{r} \right]$$

holds, where

$$\begin{aligned} C &= (\gamma_1, q_1), \ldots, (\gamma_r, q_r), (\rho+1, \mu r), (1/2+\omega+\zeta, \mu r), (1/2-\omega+\zeta, \mu r) \\ D &= (\beta_{1j}, \alpha_{1j})_{1,m}, \ldots, (\beta_{rj}, \alpha_{rj})_{1,m}, (\rho-\sigma+1, \mu r), (1/2-\tau+\zeta, \mu r), (-1, 1)_{1,r-1} \end{aligned} \tag{5.11}$$



(a) Plot of Equation (3.12) (b) Plot of Equation (3.7)

FIGURE 7. Graph of fractional derivative formulae for $\sigma = 1.5$; $a = b = 2$

Theorem 16. *Let* $m, r \in \mathbb{N}$, $\alpha_{ij}, \beta_{ij}, \gamma_i, q_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r;\ j = 1, \ldots, m)$ *such that* $\sum_{j=1}^{m} \Re(\alpha_{ij}) > \max\{0; \Re(q_i) - 1\}$; $\Re(q_i) > 0$ *and* $\Re(\sigma) \geq 0$. *For* $b > x$, *the following integral formula*

$$\int_0^\infty z^{\zeta-1} e^{-\eta z/2} W_{\tau,\omega}(\eta z) \left\{ \left(D_{b+}^{\sigma} (b-t)^{\rho} \prod_{i=1}^{r} E_{(\alpha_{ij},\beta_{ij})_m}^{\gamma_i,q_i} \left[\xi (z(b-t))^{\mu} \right] \right)(x) \right\}$$

$$= \frac{(b-x)^{\sigma+\rho}}{\eta^{\zeta} \Gamma(\gamma_1) \ldots \Gamma(\gamma_r)} {}_{r+3}\Psi_{(m+1)r+1} \left[\begin{array}{c} C \\ D \end{array} \middle| \left(\xi \left(\frac{b-x}{\eta} \right)^{\mu} \right)^{r} \right]$$

holds, where C *and* D *are defined in equation* (5.11).

Proof. The proof of Theorems 14, 15 and 16 is the same as that of Theorem 13, therefore we omit the details. □

6. Conclusion

In this paper, we have established some image formulas by applying the Riemann–Liouville fractional derivative and integral operators on the product of generalized multi-index Mittag–Leffler function $E_{(\alpha_j,\beta_j)_m}^{\gamma,q}(.)$. Then, some more image formulas are derived by employing the integral transform. To study the nature of these formulae, numerical results and their graphs are plotted for different values of the parameters involved in our main results, which can be simply interpreted and observed. For the numerical results and the graphs, the author has chosen $r = m = 2$. Further, for more investigation of these formulas the reader can choose any value of r and m.

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[1]Department of Mathematics, Baba Farid College, Bathinda-151001 (India)
E-mail address: mehar.jallandhra@gmail.com

[2]Department of Mathematics, Islamic Azad University, Central Tehran Branch, 13185.768, Tehran (Iran)
E-mail address: hamedelectroj@gmail.com

[3]Department of Mathematics, Faculty of Science, Nevsehir Haci Bektas Veli University, 50300 Nevsehir (Turkey)
E-mail address: msenol@nevsehir.edu.tr