Transactions of A. Razmadze
Mathematical Institute
Vol. **173** (2019), issue 3, 147–157

# BI–LAPLACE–BELTRAMI EQUATION ON A HYPERSURFACE

MEDEA TSAAVA

**Abstract.** We investigate the boundary value problems for the bi–Laplace–Beltrami equation on a smooth bounded surface $\mathscr{C}$ with a smooth boundary in non-classical setting in the Bessel potential space $\mathbb{H}^s_p(\mathscr{C})$ for $s > \dfrac{1}{p}$, $1 < p < \infty$. To the initial BVP we apply a quasi-localization and obtain a model BVP for the bi-Laplacian. The model BVP on the half-plane is investigated by the potential method and is reduced to an equivalent system in Sobolev–Slobodečkii space. Boundary integral equations are investigated in both Bessel potential and Sobolev–Slobodečkii spaces. The property of the obtained system in the non-classical setting is derived, as well.

## Introduction

Let $\mathscr{S} \subset \mathbb{R}^3$ be some smooth closed orientable surface, bordering a compact inner $\Omega^+$ and an outer $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega^+}$ domain. By $\mathscr{C}$ we denote a subsurface of $\mathscr{S}$, which has two faces $\mathscr{C}^-$ and $\mathscr{C}^+$ and inherits the orientation from $\mathscr{S}$: $\mathscr{C}^+$ borders the inner domain $\Omega^+$ and $\mathscr{C}^-$ borders the outer domain $\Omega^-$. $\mathscr{C}$ has the smooth boundary $\Gamma := \partial\mathscr{C}$.

Let $\boldsymbol{\nu}(\omega) = \big(\nu_1(\omega), \nu_2(\omega), \nu_3(\omega)\big)^\top$, $\omega \in \overline{\mathscr{C}}$ be the unit normal vector field on the surface $\mathscr{C}$ and $\partial_{\boldsymbol{\nu}} = \sum\limits_{j=1}^{3} \nu_j \partial_j$ be the normal derivative. Let us consider the bi–Laplace–Beltrami operator in $\mathscr{C}$ written in terms of the Günter's tangent derivatives (see [7, 9, 10] for more details)

$$\Delta^2_{\mathscr{C}} := \sum_{j,k=1}^{3} \mathscr{D}_j^2 \mathscr{D}_k^2, \quad \mathscr{D}_j := \partial_j - \nu_j \partial_{\boldsymbol{\nu}}, \quad j = 1, 2, 3. \tag{0.1}$$

Let $\boldsymbol{\nu}_\Gamma(t) = (\nu_{\Gamma,1}(t), \nu_{\Gamma,2}(t), \nu_{\Gamma,3}(t))^\top$, $t \in \Gamma$, be the unit normal vector field on the boundary $\Gamma$, which is tangential to the surface $\mathscr{C}$ and directed outside of the surface. Let, finally, $\partial_{\boldsymbol{\nu}_\Gamma} := \sum\limits_{j=1}^{3} \nu_{\Gamma,j} \mathscr{D}_j$ denote the corresponding normal derivative on the boundary $\Gamma$.

We study the following boundary value problem for the bi–Laplace–Beltrami equation

$$\begin{cases} \Delta^2_{\mathscr{C}} u(t) = f(t), & t \in \mathscr{C}, \\ u^+(s) = g(s), & \text{on} \quad \Gamma, \\ (\partial_{\boldsymbol{\nu}_\Gamma} u)^+(s) = h(s), & \text{on} \quad \Gamma, \end{cases} \tag{0.2}$$

where $u^+$ and $(\partial_{\boldsymbol{\nu}_\Gamma} u)^+$ denote the traces on the boundary.

We need the Bessel potential $\mathbb{H}^s_p(\mathscr{S})$, $\mathbb{H}^s_p(\mathscr{C})$, $\widetilde{\mathbb{H}}^s_p(\mathscr{C})$ and Sobolev–Slobodečkii $\mathbb{W}^s_p(\mathscr{S})$, $\mathbb{W}^s_p(\mathscr{C})$, $\widetilde{\mathbb{W}}^s_p(\mathscr{C})$ spaces, where $\mathscr{S}$ is a closed smooth surface (without boundary), which contains $\mathscr{C}$ as a subsurface, $1 < p < \infty$, $s \in \mathbb{R}$. Let us commence with the definition of the Bessel potential space on the Euclidean space $\mathbb{H}^s_p(\mathbb{R}^n)$, defined as a subset of the space of Schwartz distributions $\mathbb{S}'(\mathbb{R}^n)$ endowed with the norm (see [14])

$$\big\| u \big| \mathbb{H}^s_p(\mathbb{R}^n) \big\| := \big\| \langle D \rangle^s u \big| L_p(\mathbb{R}^n) \big\|,$$

---

2010 *Mathematics Subject Classification.* 35J40, 35M12, 35J58, 45E05.
*Key words and phrases.* Bi–Laplace–Beltrami equation; Günter's tangential derivatives; Boundary value problems; Boundary condition; Bessel potential spaces.

where $\langle D\rangle^s := \mathscr{F}^{-1}(1+|\xi|^2)^{\frac{s}{2}}\mathscr{F}$ is the Bessel potential and $\mathscr{F}$, $\mathscr{F}^{-1}$ are the Fourier transformations. For the definition of the Sobolev–Slobodečkii space $\mathbb{W}^s_p(\mathbb{R}^n) = \mathbb{B}^s_{p,p}(\mathbb{R}^n)$ (see [14]).

The spaces $\mathbb{H}^s_p(\mathscr{S})$ and $\mathbb{W}^s_p(\mathscr{S})$ are defined, in general, by a partition of the unity $\{\psi_j\}_{j=1}^{\ell}$ subordinated to some covering $\{Y_j\}_{j=1}^{\ell}$ of $\mathscr{S}$ and local coordinate diffeomorphisms (see [12,14] for details)

$$\varkappa_j : X_j \to Y_j, \quad X_j \subset \mathbb{R}^2, \quad j = 1, \dots, \ell.$$

The space $\mathbb{W}^s_p(\mathscr{S})$ coincides with the trace space of $\mathbb{H}_p^{s+\frac{1}{p}}(\mathbb{R}^3)$ on $\mathscr{S}$ and it is known that $\mathbb{W}^s(\mathscr{S}) = \mathbb{H}^s(\mathscr{S})$ for $s \geq 0$, $1 < p < \infty$ (see [14]).

We use, as common, the notation $\mathbb{H}^s(\mathscr{S})$ and $\mathbb{W}^s(\mathscr{S})$ for the spaces $\mathbb{H}^s_2(\mathscr{S})$ and $\mathbb{W}^s_2(\mathscr{S})$ (the case $p = 2$).

The space $\widetilde{\mathbb{H}}^s_p(\mathscr{C})$ is defined as the subspace of $\mathbb{H}^s_p(\mathscr{S})$ of those functions $\varphi \in \mathbb{H}^s_p(\mathscr{S})$, which are supported in the closed sub-surface $\operatorname{supp}\varphi \subset \overline{\mathscr{C}}$, whereas $\mathbb{H}^s_p(\mathscr{C})$ denotes the quotient space $\mathbb{H}^s_p(\mathscr{C}) := \mathbb{H}^s_p(\mathscr{S})\big/\widetilde{\mathbb{H}}^s_p(\mathscr{C}^c)$, and $\mathscr{C}^c := \mathscr{S} \setminus \overline{\mathscr{C}}$ is the complemented sub-surface. For $s > 1/p - 1$ the space $\mathbb{H}^s_p(\mathscr{C})$ can be identified with the space of those distributions $\varphi$ on $\mathscr{C}$ which admit extensions $\ell\varphi \in \mathbb{H}^s_p(\mathscr{S})$, while $\mathbb{H}^s_p(\mathscr{C})$ is identified with the space $r_{\mathscr{C}}\mathbb{H}^s_p(\mathscr{S})$, where $r_{\mathscr{C}}$ is the restriction to the sub-surface $\mathscr{C}$ of $\mathscr{S}$.

For $s < 0$, the space is defined by duality, e.g., $\mathbb{H}^s_p(\mathscr{C}) = \big(\widetilde{\mathbb{H}}^{-s}_q(\mathscr{C})\big)'$, where $\dfrac{1}{p} + \dfrac{1}{q} = 1$. The spaces $\widetilde{\mathbb{W}}^s_p(\mathscr{C})$ and $\mathbb{W}^s_p(\mathscr{C})$ are defined similarly.

The Bessel potential $\mathbb{H}^s_p(\Gamma)$, $\mathbb{H}^s_p(\Gamma_0)$, $\widetilde{\mathbb{H}}^s_p(\Gamma_0)$ and Sobolev–Slobodečkii $\mathbb{W}^s_p(\Gamma)$, $\mathbb{W}^s_p(\Gamma_0)$, $\widetilde{\mathbb{W}}^s_p(\Gamma_0)$ spaces on a closed contour $\Gamma$ and an open arc $\Gamma_0$ are defined also similarly.

It is worth noting that for an integer $m = 1, 2, \dots$ the Bessel potential $\mathbb{H}^m_p(\mathscr{S})$ and Sobolev $\mathbb{W}^m_p(\mathscr{S})$ spaces coincide and the equivalent norm in both spaces is defined with the help of the Günter's derivatives (see [6,7,9] and cf. (0.1) for the Günter's derivatives $\mathscr{D}_1, \mathscr{D}_2, \mathscr{D}_3$):

$$\big\|u \,\big|\, \mathbb{W}^m_p(\mathscr{S})\big\| := \Big[\sum_{|\alpha|\leqslant m} \big\|\mathscr{D}^\alpha u \,\big|\, L_p(\mathscr{S})\big\|^p\Big]^{\frac{1}{p}}, \quad \text{where} \quad \mathscr{D}^\alpha := \mathscr{D}_1^{\alpha_1}\mathscr{D}_2^{\alpha_2}\mathscr{D}_3^{\alpha_3}.$$

Let us also consider $\widetilde{\mathbb{H}}^{-2}_0(\mathscr{C})$, a subspace of $\widetilde{\mathbb{H}}^{-2}(\mathscr{C})$, orthogonal to

$$\widetilde{\mathbb{H}}^{-2}_\Gamma(\mathscr{C}) := \Big\{f \in \widetilde{\mathbb{H}}^{-2}(\Omega) \;\big|\; (f,\varphi)_{\mathbb{L}^2(\Omega)} = 0, \quad \varphi \in \mathbb{C}^\infty_0(\Omega)\Big\}.$$

$\widetilde{\mathbb{H}}^{-2}_\Gamma(\mathscr{C})$ consists of those distributions from $\widetilde{\mathbb{H}}^{-2}(\mathscr{C})$ which are supported on $\Gamma$ and $\widetilde{\mathbb{H}}^{-2}(\mathscr{C})$ decomposes into the following direct sum of the subspaces:

$$\widetilde{\mathbb{H}}^{-2}(\mathscr{C}) = \widetilde{\mathbb{H}}^{-2}_\Gamma(\mathscr{C}) \oplus \widetilde{\mathbb{H}}^{-2}_0(\mathscr{C}).$$

The space $\widetilde{\mathbb{H}}^{-2}_\Gamma(\mathscr{C})$ is nontrivial (see [12, §5.1]) and if the right-hand side $f$ is chosen from the orthogonal subspace, the space $\widetilde{\mathbb{H}}^{-2}_0(\mathscr{C})$ guarantees the unique solvability of BVPs (cf. [12] and the next Theorem 0.1).

The Lax–Milgram Lemma applied to the BVP (0.2) gives the following result. Similar proofs see in [15].

**Theorem 0.1.** *The BVP* (0.2) *has a unique solution in the classical weak setting:*

$$u \in \mathbb{H}^2(\mathscr{C}), \quad f \in \widetilde{\mathbb{H}}^{-2}_\Gamma(\mathscr{C}), \quad g \in \mathbb{H}^{3/2}(\Gamma), \quad h \in \mathbb{H}^{1/2}(\Gamma). \tag{0.3}$$

From Theorem 0.1 we cannot even conclude that a solution is continuous. If we succeed in proving that a solution $u$ belongs to the space $\mathbb{H}^2_p(\mathscr{C})$ for some $2 < p < \infty$, we can enjoy even a Hölder continuity of $u$. It is very important to know maximal smoothness of a solution as, for example, in designing approximation methods. To this end, we investigate the solvability properties of the BVP (0.2) in the following non-classical setting:

$$\begin{gathered} u \in \mathbb{H}^s_p(\mathscr{C}), \quad f \in \widetilde{\mathbb{H}}^{s-4}_p(\mathscr{C}) \cap \widetilde{\mathbb{H}}^{-2}_0(\mathscr{C}), \quad g \in \mathbb{H}^{s-1/p}_p(\Gamma), \\ h \in \mathbb{H}^{s-1-1/p}_p(\Gamma), \quad 1 < p < \infty, \quad s > \frac{1}{p} \end{gathered} \tag{0.4}$$

and find necessary and sufficient conditions of solvability.

To formulate the main theorem of the present work we need the following definition.

**Definition 0.2.** The BVP (0.2), (0.4) is Fredholm one if the homogeneous problem $f = g = h = 0$ has a finite number of linearly independent solutions and only a finite number of orthogonality conditions on the data $f, g, h$ ensure the solvability of the BVP.

**Theorem 0.3.** *Let conditions* (0.4) *hold:*

a) *Then a solution to the BVP* (0.2) *is represented by the formula*

$$\begin{aligned} u(x) =& \boldsymbol{N}_{\mathscr{C}} f(x) + \boldsymbol{W}_{(0,\Gamma)} g(x) - \boldsymbol{W}_{(-1,\Gamma)} h(x) + \boldsymbol{W}_{(-2,\Gamma)} \varphi(x) \\ &- \boldsymbol{W}_{(-3,\Gamma)} \psi(x), \quad u \in \mathbb{H}^2_{\#}(\mathscr{C}), \quad x \in \mathscr{C}. \end{aligned} \tag{0.5}$$

*Here* $\boldsymbol{N}_{\mathscr{C}}$, $\boldsymbol{W}_{(j,\Gamma)}$, $\quad j = \overline{-3, 1}$ *are the Newton's and layer potentials, defined below* (*see* (1.5)) *and* $\varphi$, $\psi$ *in* (0.5) *are solutions to the following system of boundary pseudodifferential equations*

$$\begin{cases} \boldsymbol{V}^0_{(-2,\Gamma)} \varphi - \boldsymbol{V}^0_{(-3,\Gamma)} \psi = G & on \quad \Gamma, \\ \boldsymbol{V}^1_{(-1,\Gamma)} \varphi - \boldsymbol{V}^1_{(-2,\Gamma)} \psi = H & on \quad \Gamma, \end{cases} \tag{0.6}$$

$$\varphi \in \widetilde{\mathbb{H}}^r_p(\Gamma), \quad \psi \in \widetilde{\mathbb{H}}^{r-1}_p(\Gamma), \quad G \in \mathbb{H}^r_p(\Gamma), \quad H \in \mathbb{H}^{r-1}_p(\Gamma), \tag{0.7}$$

*where* $r = s - 1/p$, $G$ *and* $H$ *are the functions given in terms of* $f$, $g$, *and* $h$ *in* (1.11) *in* §1 *below.*

b) *Vice versa: if* $u$ *is a solution to the BVP* (0.2) *in the setting* (0.4), *then* $\varphi := u^+$, $\psi := (\partial_{\boldsymbol{\nu}} u)^+$ *are solutions to the system* (0.6).

c) *The system of equations* (0.6) *has a unique pair of solutions* $\varphi \in \widetilde{\mathbb{W}}^{3/2}(\Gamma)$ *and* $\psi \in \widetilde{\mathbb{W}}^{1/2}(\Gamma)$ *in the classical setting for* $p = 2$, $s = 2$.

The proof of Theorem 0.3 is exposed in §1.

The system of boundary pseudodifferential equations (0.6) we will consider also in the Sobolev–Slobodečkii space setting

$$\varphi \in \widetilde{\mathbb{W}}^r_p(\Gamma), \quad \psi \in \widetilde{\mathbb{W}}^{r-1}_p(\Gamma), \quad G \in \mathbb{W}^r_p(\Gamma), \quad H \in \mathbb{W}^{r-1}_p(\Gamma). \tag{0.8}$$

To formulate the theorem, consider the following model system of singular integral equations (SIEs) in two settings:

$$\begin{cases} i S_{\mathbb{R}} \psi_0(t) = G_0(t), \\ i S_{\mathbb{R}} \varphi_0(t) = H_0(t), & t \in \mathbb{R} \end{cases} \tag{0.9}$$

in the Sobolev–Slobodečkii

$$\varphi_0, \psi_0 \in \widetilde{\mathbb{W}}^{r-1}_p(\mathbb{R}), \quad G_0, H_0 \in \mathbb{W}^{r-1}_p(\mathbb{R}) \tag{0.10a}$$

and the Bessel potential space

$$\varphi_0, \psi_0 \in \widetilde{\mathbb{H}}^{r-1}_p(\mathbb{R}), \quad G_0, H_0 \in \mathbb{H}^{r-1}_p(\mathbb{R}) \tag{0.10b}$$

settings. Here

$$S_{\mathbb{R}} v(t) := \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{v(\tau) d\tau}{\tau - t}, \quad v \in \mathbb{L}_p(\mathbb{R}) \tag{0.11}$$

is understood in the sense of Cauchy's principal value.

**Theorem 0.4.** *Let* $1 < p < \infty$, $r = s - \dfrac{1}{p} > -1$. *The system of boundary pseudodifferential equations* (0.6) *is Fredholm one in the Sobolev–Slobodečkii* (0.7) *and Bessel potential* (0.8) *space settings if the system of boundary integral equations* (0.9) *is locally invertible at* 0 *in the settings* (0.10a) *and* (0.10b), *respectively.*

**Remark 0.5.** Theorem 0.4 is proved at the end of §1. For the proof we apply a quasi-localization of the BVP (0.2) with the corresponding model BVP on the half-space (see Lemma 1.5). The constraint $r > -1$ is then natural since we deal with the boundary value problem.

In a forthcoming paper the problem will be treated by a direct application of the local quasi-equivalence to the equation (0.6).

A quasi-localization means "freezing coefficients" and "rectifying" underling contours and surfaces. For details of a quasi-localization we refer the reader to papers [13] and [1], where the quasi-localization is well described for singular integral operators and for BVPs, respectively. We also refer to [8, §3], where a short introduction to quasi-localization is exposed.

In the present case under consideration we get 2 different model problems by localizing the mixed BVP (0.2) to:

1 inner points of $\mathscr{C}$;
2 inner points on the boundary $\Gamma$.

The model BVPs obtained by a quasi-localization, are well investigated in the first case and such model problems have unique solutions without additional constraints. In the second case we get a mixed BVP on the half-plane for the bi-Laplace equation (cf. (1.13) below). System (0.9) is related to this model problem (1.13) just as the BVP (0.2) is related to system (0.6).

## 1. Potential Operators and Boundary Integral Equations

Let $\mathscr{S}$ be a closed, sufficiently smooth orientable surface in $\mathbb{R}^n$. We use the notation $\mathbb{X}^s_p(\mathscr{S})$ for either the Bessel potential $\mathbb{H}^s_p(\mathscr{S})$ or the Sobolev–Slobodečkii $\mathbb{W}^s_p(\mathscr{S})$ spaces for $\mathscr{S}$ closed or open and a similar notation $\widetilde{\mathbb{X}}^s_p(\mathscr{S})$ for $\mathscr{S}$ open.

Consider the space

$$\mathbb{X}^s_{p,\#}(\mathscr{S}) := \left\{\varphi \in \mathbb{X}^s_p(\mathscr{S}) \,:\, (\varphi, 1) = 0\right\}, \tag{1.1}$$

where $(\cdot,\cdot)$ denotes the duality pairing between the adjoint spaces. It is obvious that $\mathbb{X}^s_{p,\#}(\mathscr{S})$ does not contain nonzero constants: if $c_0 = \text{const} \in \mathbb{X}^s_{p,\#}(\mathscr{S})$ then

$$0 = (c_0, 1) = c_0(1,1) = c_0 \text{mes}\, \mathscr{S}$$

and $c_0 = 0$. Moreover, $\mathbb{X}^s_p(\mathscr{S})$ decomposes into the direct sum

$$\mathbb{X}^s_p(\mathscr{S}) = \mathbb{X}^s_{p,\#}(\mathscr{S}) + \{\text{const}\} \tag{1.2}$$

and the dual (adjoint) space is

$$\left(\mathbb{X}^s_{p,\#}(\mathscr{S})\right)^* = \mathbb{X}^{-s}_{p',\#}(\mathscr{S}), \quad p' := \frac{p}{p-1}. \tag{1.3}$$

The following is a part of Theorem 10 proved in [10].

**Theorem 1.1.** *Let $\mathscr{S}$ be $\ell$-smooth, $\ell = 1, 2, \ldots$, $1 < p < \infty$, and $|s| \leqslant \ell$. Let $\mathbb{X}^s_{p,\#}(\mathscr{S})$ be the same as in* (1.1)–(1.3)*. The bi–Laplace–Beltrami operator $\Delta^2_{\mathscr{S}} := \Delta_{\mathscr{S}}\Delta_{\mathscr{S}}$ is invertible between the spaces with detached constants*

$$\Delta^2_{\mathscr{S}} \,:\, \mathbb{X}^{s+1}_{p,\#}(\mathscr{S}) \to \mathbb{X}^{s-1}_{p,\#}(\mathscr{S}), \tag{1.4}$$

*i.e., has the fundamental solution $\mathscr{K}_{\mathscr{S}}$ in the setting* (1.4)*.*

Let $\mathscr{C} \subset \mathscr{S}$ be a subsurface with a smooth boundary $\Gamma := \partial\mathscr{C}$. With the fundamental solution $\mathscr{K}_{\mathscr{S}}$ of the bi–Laplace–Beltrami operator at hand we can consider on the surface $\mathscr{C}$ the standard layer potentials:

$$\begin{aligned}
\boldsymbol{N}_{\mathscr{C}} v(x) &:= \int\limits_{\mathscr{C}} \mathscr{K}_{\mathscr{S}}(x,y)v(y)\,d\sigma \\
\boldsymbol{W}_{(0,\Gamma)} v(x) &:= \int\limits_{\Gamma} (\partial_{\boldsymbol{\nu}_\Gamma}\Delta\mathscr{K}_{\mathscr{S}})(x,\tau)v(\tau)d\tau, \\
\boldsymbol{W}_{(-1,\Gamma)} v(x) &:= \int\limits_{\Gamma} (\Delta\mathscr{K}_{\mathscr{S}})(x,\tau)v(\tau)d\tau, \quad x\in\mathscr{C}, \\
\boldsymbol{W}_{(-2,\Gamma)} v(x) &:= \int\limits_{\Gamma} (\partial_{\boldsymbol{\nu}_\Gamma(\tau)}\mathscr{K}_{\mathscr{S}})(x,\tau)v(\tau)d\tau, \quad x\in\mathscr{C}, \\
\boldsymbol{W}_{(-3,\Gamma)} v(x) &:= \int\limits_{\Gamma} \mathscr{K}_{\mathscr{S}}(x,\tau)v(\tau)d\tau, \quad x\in\mathscr{C}.
\end{aligned} \tag{1.5}$$

The potential operators, defined above, have standard boundedness properties

$$\begin{aligned}
\boldsymbol{N}_{\mathscr{C}} &: \mathbb{H}^s_{p,\#}(\mathscr{C}) \longrightarrow \mathbb{H}^{s+4}_{p,\#}(\mathscr{C})\,, \\
\boldsymbol{W}_{(0,\Gamma)} &: \mathbb{H}^s_{p,\#}(\Gamma) \longrightarrow \mathbb{H}^{s+3+\frac{1}{p}}_{p,\#}(\mathscr{C})\,, \\
\boldsymbol{W}_{(-1,\Gamma)} &: \mathbb{H}^s_{p,\#}(\Gamma) \longrightarrow \mathbb{H}^{s+2+\frac{1}{p}}_{p,\#}(\mathscr{C}), \\
\boldsymbol{W}_{(-2,\Gamma)} &: \mathbb{H}^s_{p,\#}(\Gamma) \longrightarrow \mathbb{H}^{s+1+\frac{1}{p}}_{p,\#}(\mathscr{C}), \\
\boldsymbol{W}_{(-3,\Gamma)} &: \mathbb{H}^s_{p,\#}(\Gamma) \longrightarrow \mathbb{H}^{s+\frac{1}{p}}_{p,\#}(\mathscr{C})
\end{aligned}$$

and any solution to the mixed BVP (0.2) in the space $\mathbb{H}^2_{\#}(\mathscr{C}) := \mathbb{H}^2_{2,\#}(\mathscr{C})$ is represented as follows:

$$\begin{aligned}
u(x) =& \boldsymbol{N}_{\mathscr{C}} f(x) + \boldsymbol{W}_{(0,\Gamma)} u^+(x) - \boldsymbol{W}_{(-1,\Gamma)}(\partial_{\boldsymbol{\nu}_\Gamma} u)^+(x) + \boldsymbol{W}_{(-2,\Gamma)}(\Delta u)^+(x) \\
&- \boldsymbol{W}_{(-3,\Gamma)}(\partial_{\boldsymbol{\nu}_\Gamma}\Delta u)^+(x), \quad u\in\mathbb{H}^2_{\#}(\mathscr{C}), \quad x\in\mathscr{C}.
\end{aligned} \tag{1.6}$$

Since $\mathbb{X}^s_p = \mathbb{X}^s_{p,\#} + \{\mathrm{const}\}$, we can extend layer potentials to the entire space as follows:

$$\begin{aligned}
&\text{for} \quad \varphi = \varphi_0 + c, \quad \varphi_0 \in \mathbb{X}^s_{p,\#}, \quad c = \text{const}, \\
&\text{we set} \quad \boldsymbol{W}_{(j,\Gamma)}\varphi = \boldsymbol{W}_{(j,\Gamma)}\varphi_0 + c, \quad \boldsymbol{N}_{\mathscr{C}} f = \boldsymbol{N}_{\mathscr{C}} f_0 + c, \quad j = \overline{-3,0}
\end{aligned} \tag{1.7}$$

i.e., by setting $\boldsymbol{W}_{(j,\Gamma)} c = \boldsymbol{N}_{\mathscr{C}} c = c$.

**Lemma 1.2.** *The representation formula* (1.6) *remains valid for a solution in the space* $\mathbb{H}^2(\mathscr{C})$*, provided the potentials are extended as in* (1.7).

*Proof.* Indeed, since $u = u_0 + c$, $u_0 \in \mathbb{H}^s_{p,\#}(\mathscr{C})$, $u \in \mathbb{H}^s_p(\mathscr{C})$, we apply the extension formulae (1.7), the representation formula (1.6) for a solution in the space $\mathbb{H}^2_{\#}(\mathscr{C})$ and get the representation formula (1.6) for a solution in the space $\mathbb{H}^2(\mathscr{C})$:

$$\begin{aligned}
u(x) &= u_0(x) + c = \boldsymbol{N}_{\mathscr{C}} f_0(x) + \boldsymbol{W}_{(0,\Gamma)} u_0^+(x) - \boldsymbol{W}_{(-1,\Gamma)}(\partial_{\boldsymbol{\nu}_\Gamma} u_0)^+(x) \\
&\quad + \boldsymbol{W}_{(-2,\Gamma)}(\Delta u_0)^+(x) - \boldsymbol{W}_{(-3,\Gamma)}(\partial_{\boldsymbol{\nu}_\Gamma}\Delta u_0)^+(x) + c \\
&= \boldsymbol{N}_{\mathscr{C}}(f(x)-c) + \boldsymbol{W}_{(0,\Gamma)}(u-c)^+(x) - \boldsymbol{W}_{(-1,\Gamma)}(\partial_{\boldsymbol{\nu}_\Gamma}(u-c))^+(x) \\
&\quad + \boldsymbol{W}_{(-2,\Gamma)}(\Delta(u-c))^+(x) - \boldsymbol{W}_{(-3,\Gamma)}(\partial_{\boldsymbol{\nu}_\Gamma}\Delta(u-c))^+(x) + c \\
&= \boldsymbol{N}_{\mathscr{C}} f(x) + \boldsymbol{W}_{(0,\Gamma)} u^+(x) - \boldsymbol{W}_{(-1,\Gamma)}(\partial_{\boldsymbol{\nu}_\Gamma} u)^+(x) \\
&\quad + \boldsymbol{W}_{(-2,\Gamma)}(\Delta u)^+(x) - \boldsymbol{W}_{(-3,\Gamma)}(\partial_{\boldsymbol{\nu}_\Gamma}\Delta u)^+(x), \quad u\in\mathbb{H}^1(\mathscr{C}), \quad x\in\mathscr{C}.
\end{aligned} \tag{1.8}$$

The lemma is proof. □

*Proof of Theorem* 0.3. Let us recall the Plemelji formulae

$$
\begin{aligned}
&(\boldsymbol{W}_{(0,\Gamma)}v)^{\pm}(t)=\pm\frac{1}{2}v(t)+\boldsymbol{W}_{(0,\Gamma)}v(t), \quad (\partial_{\boldsymbol{\nu}_\Gamma}\boldsymbol{V}^0_{(0,\Gamma)}\psi)^{\pm}(t)=\boldsymbol{V}^1_{(+1,\Gamma)}v(t),\\
&(\boldsymbol{W}_{(-1,\Gamma)}v)^{\pm}(t)=\boldsymbol{V}^0_{(-1,\Gamma)}v(t), \quad (\partial_{\boldsymbol{\nu}_\Gamma}\boldsymbol{W}_{(-1,\Gamma)})^{\pm}(t)=\mp\frac{1}{2}v(t)+\boldsymbol{V}^1_{(0,\Gamma)}v(t),\\
&(\boldsymbol{W}_{(-2,\Gamma)}v)^{\pm}(t)=\boldsymbol{V}^0_{(-2,\Gamma)}v(t), \quad (\partial_{\boldsymbol{\nu}_\Gamma}\boldsymbol{W}_{(-2,\Gamma)})^{\pm}(t)=\boldsymbol{V}^1_{(-1,\Gamma)}v(t),\\
&(\boldsymbol{W}_{(-3,\Gamma)}v)^{\pm}(t)=\boldsymbol{V}^0_{(-3,\Gamma)}v(t), \quad (\partial_{\boldsymbol{\nu}_\Gamma}\boldsymbol{W}_{(-3,\Gamma)})^{\pm}(t)=\boldsymbol{V}^1_{(-2,\Gamma)}v(t),
\end{aligned}
\tag{1.9}
$$

where $t\in\partial\Omega_\alpha$ and

$$
\begin{aligned}
\boldsymbol{V}^0_{(-3,\Gamma)}v(t)&:=\int_\Gamma \mathscr{K}_{\mathscr{S}}(t,\tau)v(\tau)d\tau, \qquad t\in\Gamma,\\
\boldsymbol{V}^0_{(-2,\Gamma)}v(t)&:=\int_\Gamma (\partial_{\boldsymbol{\nu}_{\Gamma(\tau)}}\mathscr{K}_{\mathscr{S}})(t,\tau)v(\tau)d\tau,\\
\boldsymbol{V}^1_{(-2,\Gamma)}v(t)&:=\int_\Gamma (\partial_{\boldsymbol{\nu}_{\Gamma(t)}}\mathscr{K}_{\mathscr{S}})(t,\tau)v(\tau)d\tau,\\
\boldsymbol{V}^1_{(-1,\Gamma)}v(t)&:=\int_\Gamma (\partial_{\boldsymbol{\nu}_{\Gamma(t)}}\partial_{\boldsymbol{\nu}_{\Gamma(\tau)}}\mathscr{K}_{\mathscr{S}})(t,\tau)v(\tau)d\tau,
\end{aligned}
\tag{1.10}
$$

are pseudodifferential operators on $\Gamma$, have orders $-3$, $-2$, $-2$ and $-1$, respectively, and represent the direct values of the corresponding potentials $\boldsymbol{W}_{-3,\Gamma}$, $\boldsymbol{W}_{-2,\Gamma}$, $\partial_{\boldsymbol{\nu}_\Gamma}\boldsymbol{W}_{-3,\Gamma}$ and $\partial_{\boldsymbol{\nu}_\Gamma}\boldsymbol{W}_{-2,\Gamma}$.

By applying the Plemelji formulae (1.9) to (1.6), we get

$$
\begin{cases}
u^+(t)=g(t)=(\boldsymbol{N}_{\mathscr{C}}f)^+ +\dfrac{1}{2}g(t)+\boldsymbol{V}^0_{(0,\Gamma)}g(t)-\boldsymbol{V}^0_{(-1,\Gamma)}h(t)\\
\quad +\boldsymbol{V}^0_{(-2,\Gamma)}\varphi(t)-\boldsymbol{V}^0_{(-3,\Gamma)}\psi(t),\\
(\partial_{\boldsymbol{\nu}_\Gamma}u)^+(t)=h(t)=(\partial_{\boldsymbol{\nu}_\Gamma}\boldsymbol{N}_{\mathscr{C}}f)^+ +\boldsymbol{V}^1_{(+1,\Gamma)}g(t)+\dfrac{1}{2}h(t)-\boldsymbol{V}^1_{(0,\Gamma)}h(t)\\
\quad +\boldsymbol{V}^1_{(-1,\Gamma)}\varphi(t)-\boldsymbol{V}^1_{(-2,\Gamma)}\psi(t), \quad t\in\Gamma.
\end{cases}
$$

We obtain system (0.6), where

$$
\begin{aligned}
G&:=\left[\frac{1}{2}g-(\boldsymbol{N}_{\mathscr{C}}f)^+-\boldsymbol{V}^0_{(0,\Gamma)}g+\boldsymbol{V}^0_{(-1,\Gamma)}h\right]\in\mathbb{H}^{s-1/p}_p(\Gamma),\\
H&:=\left[\left(\frac{1}{2}h-\partial_{\boldsymbol{\nu}_\Gamma}\boldsymbol{N}_{\mathscr{C}}f\right))^+-\boldsymbol{V}^1_{(0,\Gamma)}g+\boldsymbol{V}^1_{(-1,\Gamma)}h\right]\in\mathbb{H}^{s-1-1/p}_p(\Gamma).
\end{aligned}
\tag{1.11}
$$

Thus, we have proved the inverse assertion of Theorem 0.3: if $u$ is a solution to the BVP (0.2), the functions $\varphi$ and $\psi$ are solutions to system (0.6).

The direct assertion is even easier to prove:

- the function in (1.8) represented by the potentials, satisfies the equation (0.2);
- if $\varphi$ and $\psi$ are solutions to system (0.6), using Plemelji formulae (1.9), it can easily be verified that $u$ in (1.8) satisfies the boundary conditions in (0.2).

The existence and uniqueness of a solution to the BVP (0.2) in the classical setting (0.3) is stated in Theorem 0.1, while for system (0.6) it follows from the equivalence with the BVP (0.2). $\square$

The remainder of the paper is devoted to the proof of solvability properties of the system (0.6) in the non-classical setting (0.4).

On the 2-dimensional Euclidean space we consider the following equation:

$$
\Delta^2 u=f^0 \quad \text{on} \quad \mathbb{R}^2, \quad u\in\mathbb{H}^s_p(\mathbb{R}^2), \quad f^0\in\mathbb{H}^{s-4}_p(\mathbb{R}^2),
\tag{1.12}
$$

also the model

$$\begin{cases}\Delta^2 u(x) = f_1(x), & x \in \mathbb{R}^2_+,\\ u^+(t) = g_1(t), & t \in \mathbb{R},\\ -(\partial_2 u)^+(t) = h_1(t), & t \in \mathbb{R}.\end{cases} \tag{1.13}$$

boundary value problems for the Laplace equation on the upper half plane $\mathbb{R}^2_+ := \mathbb{R} \times \mathbb{R}^+$, where $\partial_{\boldsymbol{\nu}_\Gamma} = -\partial_2$ is the normal derivative on the boundary of $\mathbb{R}^2_+$.

The BVP (1.13) will be treated in the non-classical setting:

$$f_1 \in \widetilde{\mathbb{H}}^{s-4}_p(\mathbb{R}^2_+) \cap \widetilde{\mathbb{H}}^{-2}_0(\mathbb{R}^2_+), \quad g_1 \in \mathbb{H}^{s-1/p}_p(\mathbb{R}), \quad h_1 \in \mathbb{H}^{s-1-1/p}_p(\mathbb{R}), \tag{1.14}$$
$$1 < p < \infty, \quad s > \frac{1}{p}.$$

**Proposition 1.3.** *The bi-Laplace equation* (1.12) *has a unique solution, as well.*

*Proof.* The assertion is a well-known classical result available in many textbooks on partial differential equations (see e.g. [12]). □

As a particular case of Theorem 0.1 (can easily be proved with the Lax–Milgram Lemma), we have the following

**Proposition 1.4.** *The BVP* (1.13) *has a unique solution $u$ in the classical weak setting*

$$u \in \mathbb{H}^2(\mathbb{R}^2_+), \quad f_1 \in \widetilde{\mathbb{H}}^{-2}_0(\mathbb{R}^2_+), \quad g_1 \in \mathbb{H}^{3/2}(\mathbb{R}), \quad h_1 \in \mathbb{H}^{1/2}(\mathbb{R}),$$

**Lemma 1.5.** *The BVP* (0.2) *is Fredholm one in the non-classical setting* (0.4) *if the model mixed BVP* (1.13) *is locally Fredholm (i.e., is locally invertible) at* 0 *in the non-classical setting* (1.14).

*Proof.* We apply quasi-localization of the boundary value problem (0.2) in the more general non-classical setting (0.4), which includes the classical setting (0.3) as a particular case (see [1, 3] for details of quasi-localization of boundary value problems and also [2, 11, 13] for general results on localization and quasi-localization).

Prior proceeding with the quasi-localization let us explain shortly why the quasi-localization can be performed in the Bessel potential (Besov) spaces.

Localizing classes consist of multiplication operators by smooth functions and since localization is performed in quotient spaces modulo compact operators it suffices to note that smooth functions commute with the Bessel potential in quotient algebra (see [3]) and, therefore, their norms coincide with the norm in $\mathbb{L}_p$-space, i.e., with the supremum-norm. This makes localization ("freezing coefficients") easy.

Concerning the "rectification": since the difference of "pull-back" of the original operator and its local representative is locally compact in $\mathbb{L}_p$ and is bounded in the Bessel potential spaces $\mathbb{H}^s_p$, it is locally compact in all $\mathbb{H}^r_p$-spaces for $r < s$ (Krasnoselskij theorem).

By quasi-localization at the point $\omega \in \overline{\mathscr{C}}$ we first localize to the tangential plane $\mathbb{R}^2(\omega)$ (tangential half- plane $\mathbb{R}^2_+(\omega)$) to $\mathscr{C}$ at $\omega \in \mathscr{C}$ (at $\omega \in \Gamma = \partial\mathscr{C}$, respectively). The differential operators remain the same

$$\begin{gathered}\Delta^2_{\mathbb{R}^2} := \sum_{j,k=1}^{3} \mathscr{D}_j^2 \mathscr{D}_k^2, \quad \mathscr{D}_j = \partial_j - \nu_j \partial_{\boldsymbol{\nu}},\\ \partial_{\boldsymbol{\nu}} = \sum_{j=1}^{3} \nu_j \partial_j, \quad \partial_{\boldsymbol{\nu}_\Gamma} = \sum_{j=1}^{3} \nu_{\Gamma,j} \mathscr{D}_j,\end{gathered} \tag{1.15}$$

but the normal vector $\boldsymbol{\nu}(\omega)$ to the tangent plane $\mathbb{R}^2$ and the normal vector $\boldsymbol{\nu}_\Gamma(\omega)$ to the boundary of the tangent plane $\mathbb{R}(\omega) = \partial\mathbb{R}^2_+(\omega)$ are now constant. Next, we rotate the tangent planes $\mathbb{R}^2(\omega)$ and $\mathbb{R}^2_+(\omega)$ to match them to the planes $\mathbb{R}^2$ and $\mathbb{R}^2_+$. The normal vector fields will transform into $\boldsymbol{\nu} = (0, 0, 1)$ and $\boldsymbol{\nu}_\Gamma = (0, -1, 0)$. The rotation is an isomorphism of the spaces $\mathbb{W}^r_p(\mathbb{R}^2(\omega)) \to \mathbb{W}^r_p(\mathbb{R}^2)$,

$\mathbb{W}^r_p(\mathbb{R}^2_+(\omega)) \to \mathbb{W}^r_p(\mathbb{R}^2_+)$, $\widetilde{\mathbb{W}}^r_p(\mathbb{R}^2_+(\omega)) \to \widetilde{\mathbb{W}}^r_p(\mathbb{R}^2_+)$ etc., and transforms the operators in (1.15) into the operators

$$\Delta^2_{\mathbb{R}^2(\omega)} \ \to\ \Delta^2 := \sum_{j,k=1}^{2} \partial_j^2\partial_k^2, \quad \mathscr{D}_j \ \to\ \partial_j, \quad j,k=1,2, \quad \mathscr{D}_3 \ \to\ 0,$$
$$\partial_{\boldsymbol{\nu}(\omega)} \ \to\ \partial_3, \quad \partial_{\boldsymbol{\nu}_\Gamma(\omega)} \ \to\ -\partial_2,$$

and we get (1.12), (1.13) as a local representatives of BVP (0.2).

For the BVP (0.2) in the non-classical setting (0.4) we get the following local quasi-equivalent equations and BVPs at different points of the surface $\omega\in\overline{\mathscr{C}}$:

i. the equation (1.12) at 0 if $\omega\in\mathscr{C}$ is an inner points of the surface;

iv. the mixed BVP (1.13) in the non-classical setting (1.14) at 0 if $\omega\in\Gamma$.

The main conclusion of the present theorem on Fredholm properties of BVPs (0.2) and (1.13) follows from Proposition 1.3 and the general theorem on quasi-localizaion (see [1–3,11,13]): *The BVP* (0.2), (0.4) *is Fredholm one if all local representatives* (1.12) *and* (1.13) *in non-classical settings are locally Fredholm (i.e., are locally invertible).* □

Now we concentrate on the model mixed BVP (1.13). To this end, let us recall that the function

$$\mathscr{K}^2_\Delta(x) := \frac{1}{8\pi}|x-y|^2\ln|x-y|$$

is the fundamental solution to the bi-Laplace's equation in two variables

$$\begin{aligned} &\Delta^2\mathscr{K}^2_\Delta(x)=\delta(x), \qquad x\in\mathbb{R}^2,\\ &\Delta^2=(\partial_1^2+\partial_2^2)^2=(\partial_{\boldsymbol{\nu}}^2+\partial_\ell^2)^2. \end{aligned} \tag{1.16}$$

Standard Newton and layer potential operators (cf. (1.5)) acquire the following forms:

$$\boldsymbol{N}_{\mathbb{R}^2_+}v(x) := \frac{1}{8\pi}\int\limits_{\mathbb{R}^2_+}|x-y|^2\ln|x-y|v(y)\,dy,$$
$$\boldsymbol{W}_{(0,\mathbb{R})}v(x) := -\frac{1}{8\pi}\int\limits_{\mathbb{R}}\partial_{y_2}(\partial^2_{y_1}+\partial^2_{y_2})\big|(x_1,x_2)-(\tau,y_2)\big|^2\ln\big|(x_1,x_2)-(\tau,y_2)\big|\Big|_{y_2=0}v(\tau)\,d\tau,$$
$$\boldsymbol{W}_{(-1,\mathbb{R})}v(x) = \frac{1}{8\pi}\int\limits_{\mathbb{R}}(\partial^2_{y_1}+\partial^2_{y_2})\big|(x_1,x_2)-(\tau,y_2)\big|^2\ln\big|(x_1,x_2)-(\tau,y_2)\big|\Big|_{y_2=0}v(\tau)\,d\tau,$$
$$\boldsymbol{W}_{(-2,\mathbb{R})}v(x) := -\frac{1}{8\pi}\int\limits_{\mathbb{R}}\partial_{y_2}\big|(x_1,x_2)-(\tau,y_2)\big|^2\ln\big|(x_1,x_2)-(\tau,y_2)\big|\Big|_{y_2=0}v(\tau)\,d\tau,$$
$$\boldsymbol{W}_{(-3,\mathbb{R})}v(x) := \frac{1}{8\pi}\int\limits_{\mathbb{R}}\big|x-(\tau,0)\big|^2\ln\big|x-(\tau,0)\big|v(\tau)\,d\tau.$$

The pseudodifferential operators on $\boldsymbol{V}^0_{-3,\mathbb{R}}$, $\boldsymbol{V}^0_{-2,\mathbb{R}}$, $\boldsymbol{V}^1_{-2,\mathbb{R}}$ and $\boldsymbol{V}^1_{-1,\mathbb{R}}$ associated with the layer potentials (see (1.10)), acquire the form

$$\boldsymbol{V}^0_{(-3,\mathbb{R})}v(x) := \frac{1}{8\pi}\int\limits_{\mathbb{R}}\big((x_1-\tau)^2+x_2^2\big)\ln\big((x_1-\tau)^2+x_2^2\big)^{1/2}v(\tau)d\tau, \quad t\in\mathbb{R},$$
$$\begin{aligned}\boldsymbol{V}^0_{(-2,\mathbb{R})}v(x) &:= -\frac{1}{8\pi}\int\limits_{\mathbb{R}}\partial_{y_2}\big((x_1-\tau)^2+(x_2-y_2)^2\big)\ln\big((x_1-\tau)^2+(x_2-y_2)^2\big)^{1/2}\Big|_{y_2=0}v(\tau)d\tau\\ &:= \frac{(x_2-y_2)}{8\pi}\int\limits_{\mathbb{R}}\big(2\ln\big((x_1-\tau)^2+(x_2-y_2)^2\big)^{1/2}+1\big)\Big|_{y_2=0}v(\tau)d\tau\end{aligned}$$

$$
\begin{aligned}
&:= \frac{x_2}{8\pi}\int\limits_{\mathbb{R}} \big(2\ln\big((x_1-\tau)^2+x_2^2\big)^{1/2}+1\big)v(\tau)d\tau,\\
\boldsymbol{V}^1_{(-2,\mathbb{R})}v(x) &:= -\frac{1}{8\pi}\int\limits_{\mathbb{R}} \partial_{x_2}\big((x_1-\tau)^2+(x_2-y_2)^2\big)\ln\big((x_1-\tau)^2+(x_2-y_2)^2\big)^{1/2}\Big|_{y_2=0}v(\tau)d\tau\\
&:= \frac{(y_2-x_2)}{8\pi}\int\limits_{\mathbb{R}} \big(2\ln\big((x_1-\tau)^2+(x_2-y_2)^2\big)^{1/2}+1\big)\Big|_{y_2=0}v(\tau)d\tau\\
&:= -\frac{x_2}{8\pi}\int\limits_{\mathbb{R}} \big(2\ln\big((x_1-\tau)^2+x_2^2\big)^{1/2}+1\big)v(\tau)d\tau,\\
\boldsymbol{V}^1_{(-1,\mathbb{R})}v(x) &:= -\frac{1}{8\pi}\int\limits_{\mathbb{R}} \partial^2_{x_2}\big((x_1-\tau)^2+(x_2-y_2)^2\big)\ln\big((x_1-\tau)^2+(x_2-y_2)^2\big)^{1/2}\Big|_{y_2=0}v(\tau)d\tau\\
&:= -\frac{1}{4\pi}\int\limits_{\mathbb{R}} \Big(\ln\big((x_1-\tau)^2+(x_2-y_2)^2\big)^{1/2}+\frac{(y_2-x_2)^2}{(x_1-\tau)^2+(x_2-y_2)^2}+\frac{1}{2}\Big)\Big|_{y_2=0}v(\tau)d\tau\\
&:= -\frac{1}{4\pi}\int\limits_{\mathbb{R}} \Big(\ln\big((x_1-\tau)^2+x_2^2\big)^{1/2}+\frac{x_2^2}{(x_1-\tau)^2+x_2^2}+\frac{1}{2}\Big)v(\tau)d\tau,
\end{aligned}
$$

and when $x_2\to 0$, we get

$$
\begin{aligned}
&\boldsymbol{V}_{(-3,\mathbb{R})}v(t) := \lim_{x_2\to 0}\boldsymbol{V}^0_{(-3,\mathbb{R})}v(x) = \frac{1}{8\pi}\int\limits_{\mathbb{R}}(t-\tau)^2\ln|t-\tau|v(\tau)d\tau,\\
&\boldsymbol{V}_{(-2,\mathbb{R})}v(t) := \lim_{x_2\to 0}\boldsymbol{V}^0_{(-2,\mathbb{R})}v(x) = 0, \quad \boldsymbol{V}^*_{(-2,\mathbb{R})}v(t) := \lim_{x_2\to 0}\boldsymbol{V}^1_{(-2,\mathbb{R})}v(x) = 0, \qquad (1.17)\\
&\boldsymbol{V}_{(-1,\mathbb{R})}v(t) := \lim_{x_2\to 0}\boldsymbol{V}^1_{(-1,\mathbb{R})}v(x) = -\frac{1}{4\pi}\int\limits_{\mathbb{R}}\Big(\ln|t-\tau|+\frac{1}{2}\Big)v(\tau)d\tau.
\end{aligned}
$$

Now we prove the following

**Lemma 1.6.** *Let $1<p<\infty$, $s>\dfrac{1}{p}$. Let $g_1\in\mathbb{H}^{s-1/p}_p(\mathbb{R})$ and $h_1\in\mathbb{H}^{s-1-1/p}_p(\mathbb{R})$ (non-classical formulation (1.14)). A solution to the BVP (1.13) is represented by the formula*

$$
\begin{aligned}
u(x) = \boldsymbol{N}_{\mathbb{R}^2_+}f(x)+\boldsymbol{W}_{(0,\mathbb{R})}g_1(x)-\boldsymbol{W}_{(-1,\mathbb{R})}h_1(x)+\boldsymbol{W}_{(-2,\mathbb{R})}\varphi_0(x)&\\
-\boldsymbol{W}_{(-3,\mathbb{R})}\psi_0(x), \quad x\in\mathbb{R}^2& \qquad (1.18)
\end{aligned}
$$

*and $\varphi^0$ and $\psi^0$ are the solutions to the system of pseudodifferential equations*

$$
\begin{cases}
\boldsymbol{V}^0_{(-2,\mathbb{R})}\varphi_0-\boldsymbol{V}^0_{(-3,\mathbb{R})}\psi_0 = G_0 & \text{on} \quad \mathbb{R},\\
\boldsymbol{V}^1_{(-1,\mathbb{R})}\varphi_0-\boldsymbol{V}^1_{(-2,\mathbb{R})}\psi_0 = H_0 & \text{on} \quad \mathbb{R},
\end{cases} \qquad (1.19)
$$

$$
\begin{aligned}
&\varphi_0\in\widetilde{\mathbb{H}}^{s-1/p}_p(\mathbb{R}), \quad \psi_0\in\widetilde{\mathbb{H}}^{s-1-1/p}_p(\mathbb{R}),\\
&G_0\in\mathbb{H}^{s-1/p}_p(\mathbb{R}), \quad H_0\in\mathbb{H}^{s-1-1/p}_p(\mathbb{R}), \qquad (1.20)
\end{aligned}
$$

*where*

$$
\begin{aligned}
G_0 &:= \Big[\frac{1}{2}g_1-(\boldsymbol{N}_{\mathbb{R}^2_+}f)^+-\boldsymbol{V}^0_{(0,\mathbb{R})}g_1+\boldsymbol{V}^0_{(-1,\mathbb{R})}h_1\Big]\in\mathbb{H}^{s-1/p}_p(\mathbb{R}),\\
H_0 &:= \Big[\Big(\frac{1}{2}h_1-\partial_{\boldsymbol{\nu}_\Gamma}\boldsymbol{N}_{\mathbb{R}^2_+}f\Big)^+-\boldsymbol{V}^1_{(0,\mathbb{R})}g_1+\boldsymbol{V}^1_{(-1,\mathbb{R})}h_1\Big]\in\mathbb{H}^{s-1-1/p}_p(\Gamma).
\end{aligned}
$$

*The system of boundary pseudodifferential equations (1.19) has a unique pair of solutions $\varphi_0$ and $\psi_0$ in the classical setting $p=2$, $s=1$.*

*Proof.* By repeating word by word the proof of Theorem 0.3, we prove the equivalence via the representation formulae (1.18) of the BVP (1.13) in the non-classical setting (1.14) and of the system (1.19).

The existence and uniqueness of a solution to the BVP (1.13) in the classical setting (1.14) is stated in Proposition 1.4, while for system (1.19) it follows from the proved equivalence with the BVP (1.13). □

**Lemma 1.7.** *Let* $1 < p < \infty$, $s > \dfrac{1}{p}$. *The system of boundary pseudodifferential equations* (1.19) *is locally invertible at* 0 *if and only if the system* (0.9) *is locally invertible at* 0 *in the non-classical setting* (0.10a) *and the space parameters are related as follows:* $r = s - \dfrac{1}{p} > 0$.

*Proof.* Due to the equalities (1.17) $\boldsymbol{V}^0_{(-2,\mathbb{R})}\varphi_0 = 0$, $\boldsymbol{V}^1_{(-2,\mathbb{R})}\psi_0 = 0$ and the equation in (1.19) acquires the form

$$\begin{cases} -\dfrac{1}{8\pi}\displaystyle\int_{\mathbb{R}} (t-\tau)^2 \ln|t-\tau|\psi_0(\tau)d\tau = G(t), \quad t \in \mathbb{R}, \\ -\dfrac{1}{4\pi}\displaystyle\int_{\mathbb{R}} \big(\ln|t-\tau| + \dfrac{1}{2}\big)\varphi_0(\tau)d\tau = H(t), \quad t \in \mathbb{R}. \end{cases}$$

Multiply both equations by -4, apply to the first equation the differentiation $\partial_t^3$ and to the second equation $\partial_t$. We get

$$\begin{cases} \dfrac{1}{\pi}\displaystyle\int_{\mathbb{R}} \dfrac{\psi_0(\tau)d\tau}{\tau - t} = G(t), \\ \dfrac{1}{\pi}\displaystyle\int_{\mathbb{R}} \dfrac{\varphi_0(\tau)d\tau}{\tau - t} = H(t), \quad t \in \mathbb{R}. \end{cases}$$

The obtained equation coincides with system (0.9).

Invertibility of the singular integral operator follows from the following equality (see [4, 5, 11])

$$\mathscr{F} S_{\mathbb{R}}\varphi(\xi) = -\operatorname{sign}\xi\varphi(\xi),$$

since $S_{\mathbb{R}}\varphi(\xi) = \mathscr{F}^{-1}(-\operatorname{sign}\xi)\mathscr{F}$, we get

$$\begin{aligned} S^2_{\mathbb{R}}\varphi(\xi) &= \mathscr{F}^{-1}(-\operatorname{sign}\xi)\mathscr{F}\mathscr{F}^{-1}(-\operatorname{sign}\xi)\mathscr{F}\varphi(\xi) \\ &= \mathscr{F}^{-1}(-\operatorname{sign}\xi)^2\mathscr{F}\varphi(\xi) = \mathscr{F}^{-1}\mathscr{F}\varphi(\xi) = \varphi(\xi). \end{aligned}$$

Here

$$\mathscr{F}u(\xi) := \int_{\mathbb{R}^n} e^{i\xi x}u(x)dx, \quad \xi \in \mathbb{R}^n,$$

is the Fourier transform and

$$\mathscr{F}^{-1}v(\xi) := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-i\xi x}v(\xi)d\xi, \quad x \in \mathbb{R}^n,$$

is its inverse transform.

To prove the local equivalence at 0 of systems (1.19) and (0.9) we note that the differentiation

$$\partial_t := \frac{d}{dt} : \mathbb{H}^r_p(\mathbb{R}) \to \mathbb{H}^{r-1}_p(\mathbb{R}), \quad \partial_t : \widetilde{\mathbb{H}}^r_p(\mathbb{R}) \to \widetilde{\mathbb{H}}^{r-1}_p(\mathbb{R})$$

is invertible at any finite point $x \in \mathbb{R}$ and the inverse operator is

$$\Big(\frac{d}{dt}\Big)^{-1}\varphi(t) = \int_{-\infty}^{t} \varphi(\tau)d\tau. \qquad □$$

*Proof of Theorem* 0.4. By Theorem 0.3, system (0.6) is Fredholm one in the Bessel potential space setting (0.7) if the BVP (0.2) is Fredholm in the non-classical setting (0.4). On the other hand, by Lemma 1.5 the BVP (0.2) is Fredholm in the non-classical setting (0.4) if the BVP (1.13) is locally invertible at 0 in the non-classical setting (1.14). And, finally, by Lemma 1.6 and Lemma 1.7, the BVP (1.13) is locally invertible in the non-classical setting (1.14) if the system of boundary integral equations (0.9) is locally invertible at 0 in the Bessel potential space setting (0.10b). □

## Acknowledgement

The author was supported by the grant of the Shota Rustaveli Georgian National Science Foundation PhD–F–17–197.

## References

1. L. P. Castro, R. Duduchava, F. O. Speck, Localization and minimal normalization of some basic mixed boundary value problems. *Factorization, singular operators and related problems (Funchal, 2002)*, 73–100, *Kluwer Acad. Publ. Dordrecht*, 2003.
2. V. D. Didenko, B. Silbermann, *Approximation of Additive Convolution-Like Operators. Real $C^*$-Algebra Approach.* Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2008.
3. R. Duduchava, On multidimensional singular integral operators I–II. *J. Operator Theory* **11** (1984), 41–76, 199–214.
4. R. Duduchava, On general singular integral operators of the plane theory of elasticity. *Rend. Sem. Mat. Univ. Politec. Torino* **42** (1984), no. 3, 15–41.
5. R. Duduchava, General singular integral equations and fundamental problems basic theorems of the plane theory of elasticity. (Russian) *Trudi Tbiliss. Mat. Inst. Razmadze Acad. Nauk Gruzin. SSR* **82** (1986), 45–89.
6. R. Duduchava, The Green formula and layer potentials. *Integral Equations Operator Theory* **41** (2001), no. 2, 127–178.
7. R. Duduchava, Partial differential equations on hypersurfaces. *Mem. Differential Equations Math. Phys.* **48** (2009), 19–74.
8. R. Duduchava, Mellin convolution operators in Bessel potential spaces with admissible meromorphic kernels. *Mem. Differ. Equ. Math. Phys.* **60** (2013), 135–177.
9. R. Duduchava, D. Mitrea, M. Marius, Differential operators and boundary value problems on hypersurfaces. *Math. Nachr.* **279** (2006), 996–1023.
10. R. Duduchava, M. Tsaava, T. Tsutsunava, Mixed boundary value problem on hypersurfaces. *Int. J. Differ. Equ.* 2014, Art. ID 245350, 8 pp.
11. I. Gohberg, N. Krupnik, One-Dimensional Linear Singular Integral Equations, I-II, Oper. Theory Adv. Appl. **53-54**, Birkhäuser, Basel, 1979.
12. G. C. Hsiao, W. L. Wendland, *Boundary Integral Equations, Applied Mathematical Sciences.* vol. **164**, Springer, Berlin Heidelberg 2008, ISBN 978-3-540-15284-2.
13. I. B. Simonenko, A new general method of investigating linear operator equations of singular integral equation type. I. (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* **29**, 1965, 567–586.
14. H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators.* Second edition, Johann Ambrosius Barth, Heidelberg, 1995.
15. M. Tsaava, The boundary value problems for the bi-Laplace-Beltrami equation. *Mem. Differ. Equ. Math. Phys.* **77** (2019), 93–103.

(Received 04.09.2019)

I. Javakhishvili Tbilisi State University, Georgia
*E-mail address*: m.caava@yahoo.com