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THE ASYMPTOTIC BEHAVIOR OF PRECISE LOWER ESTIMATE OF RECONSTRUCTION OF A LINEAR ORDER ON A FINITE SET

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Abstract. In the present paper we consider reconstructions of a linear order on a finite set and give the extremal lower estimate of those reconstructions. The asymptotic behavior of such estimate is studied.

The study of discrete point-systems is one of the most important directions in modern mathematics and the methodology of studies of such point-systems is quite diverse. In particular, it uses the methods and principles of combinatorial set theory, mathematical analysis, algebra, graph theory, etc.

The present article is devoted to a concrete topic of discrete mathematics and describes some extremal cases connected with finite linearly ordered point sets. Discrete linearly ordered point-systems can be met in various fields of pure and applied mathematics. One can indicate several such directions in contemporary mathematics, for instance, discrete and computational geometry, classical number theory, combinatorics (finite or infinite), the theory of convex sets, discrete optimization, etc. The investigation of the combinatorial structure of various discrete and finite point-systems in Euclidean spaces is a rather attractive and important topic.

Properties of various discrete point systems are considered in many works (see, t.g., [2–11].)

Throughout this article, we use the following standard notation:

$\mathbf{N}$ is the set of all natural numbers;

$\mathbf{R}$ is the set of all real numbers;

$\mathbf{R}^m$ is the m-dimensional Euclidean space, where $m \geq 1$;

$(X, \leq)$ is a linearly ordered set with $\mathrm{card}(X) = n$, where n is a natural number.

For our further purpose, we need to formulate one important result, which is a particular case of the so-called Master's Theorem. The mentioned universal theorem plays an important role in investigation of various combinatorial problems and questions. Let us formulate a weak version of the Master's Theorem.

Lemma 1. *Let $f : \mathbf{N} \to \mathbf{N}$ be an increasing (in general, not strongly increasing) function such that the inequality*

$$f(2n) \leq 2f(n) + an + b$$

holds true for two fixed real numbers $a \geq 0$ and $b \geq 0$ and for all $n \in \mathbf{N}$. Then there is an upper estimate of f in the form

$$f(n) = O(\log_2(n)).$$

In other words, there exists a real constant $d > 0$ such that

$$f(n) \leq d \cdot \log_2(n)$$

for all natural numbers $n > 1$.

The proof of Lemma 1 can be found in many works, textbooks and monographs (see, e.g., [2, 5, 7]).

Suppose that a nonempty finite linearly ordered set $(X, \preceq)$ is given with

$$\mathrm{card}(X) = n.$$

Take any two-element subset $\{x, y\}$ of X, where $x \neq y$, and compare x and y with respect to "$\preceq$". In out further considerations, such a comparing will be called an elementary operation. Since $\preceq$

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trivially induces the linear ordering on $\{x, y\}$, we have the disjunction $x \preceq y \vee y \preceq x$. Moreover, since $x \neq y$, we can write

$$x \prec y \vee y \prec x.$$

Suppose now that for every two-element subset $\{x, y\}$ of X, we are able to specify, by using exactly one elementary operation, which of the two relations $x \prec y$ and $y \prec x$ is valid. Briefly speaking, we are in the situation where full information on the induced orderings

$$\preceq_{\{x;y\}} \quad (x \in X, y \in X, x \neq y)$$

is available. For our future purpose, several simple auxiliary propositions will be helpful.

Recall that any pair of the form (V, E), where V is a set and E is some subset of the family of all two-element parts of V, is called a graph (see, e.g., [6, 11]).

Lemma 2. *If a finite graph (V, E) is such that*

$$\mathrm{card}(V) = n, \qquad \mathrm{card}(E) < n - 1,$$

then this graph is not connected.

The above assertion immediately follows from the fact that any nonempty finite connected graph (V, E) contains a subtree (V, E') such that

$$\mathrm{card}(E') = \mathrm{card}(V) - 1.$$

Lemma 3. *Let $(L, \leq)$ be a linearly ordered set and let X and Y be any two nonempty disjoint finite subsets of L such that*

$$X = \{x_1, x_2, \dots, x_m\}, \quad Y = \{y_1, y_2, \dots, y_n\},$$
$$x_1 < x_2 < x_3 < \dots < x_m, \quad y_1 < y_2 < y_3 < \dots < y_n.$$

Consider the set $Z = X \cup Y$. Then $m + n - 1$ elementary operations are sufficient for describing the ordering on Z induced by $\leq$.

The proof of this lemma can be done by induction on the sum $m + n$.

Lemma 4. *Let $(X, \preceq)$ be a linearly ordered set with* $\mathrm{card}(X) = n$, *where $n \geq 2$. Then no test of $n - 2$ (or less) pairs of elements from X can give full description of the ordering $\preceq$.*

Proof. Suppose otherwise. Then there are $n - 2$ (or less) elementary operations which allow us to reconstruct the given linear ordering $\preceq$.

Consider the graph (V, E) where $V = X$ and E consists of all those two-element parts of V which were used in the process of making the above-mentioned elementary operations. Our assumption means that $\mathrm{card}(E) \leq n - 2$.

Then Lemma 2 says that the graph (V, E) is not connected. i.e., we have a representation

$$X = V = V_1 \cup V_2,$$

where V_1 and V_2 are nonempty disjoint sets and no edge of (V, E) has one vertex in V_1 and the order vertex in V_2.

Let us denote

$$\preceq' = (\preceq \cap (V_1 \times V_1)) \cup (\preceq \cap (V_2 \times V_2)).$$

Obviously $\preceq'$ is a partial ordering on X. Taking into account this circumstance, we can readily define two distinct linear orderings on X which both satisfy the list of the carried out elementary operations.

Namely, the first linear ordering is such that it extends $\preceq'$ and all elements from V_1 are strictly less than all those from V_2, and the second linear ordering is such that it also extends $\preceq'$, but all elements from V_2 are strictly less than all elements from V_1. In other words, we see that the test suggested by the made $n - 2$ elementary operations could not reconstruct the given linear ordering $\preceq$. □

In connection with the last lemma, there naturally arise the questions: how many two-element subsets of X should be taken for total reconstruction of the linear ordering $\preceq$ on X?

Equivalently, one may ask: how many elementary operations are sufficient for the total description of the linear orderings $\preceq$ on X?

By using Lemma 1 and Lemma 3, one can deduce the next well-known statement.

Theorem 1. *The minimal number of those two-element subsets of a nonempty finite linearly ordered set* $(L, \leq)$ *with* $\text{card}(L) = n$ *that suffices to reconstruct the given ordering* $\leq$ *is estimated from the above by* $O(n \cdot \log_2(n))$.

Equivalently, $O(n \cdot \log_2(n))$ *elementary operations are enough to reconstruct the linear ordering* $\leq$ *on* L.

Example. Let $(L, \preceq)$ be an arbitrary linearly ordered set. Recall that for every set $\{x, y\} \subset L$, where $x \neq y$, the elementary operation corresponding to $\{x, y\}$ allows one to recognize the induced ordering on $\{x, y\}$ or, in other words, provides information which of the two relations $x \prec y$ and $y \prec x$ is valid.

Let now $X_1, X_2, X_3, \ldots, X_m$ be some subsets of a linearly ordered set $(L, \preceq)$. Consider the Cartesian product $X_1 \times X_2 \times X_3 \times \times \times X_m$ and equip it with the so-called lexicographical ordering $\leq$. In particular, if $m = 2$, then we have

$$(x_1, y_1) < (x_2, y_2) \Leftrightarrow ((x_1 \prec x_2 \vee (x_1 = x_2 \& y_1 \prec y_2)),$$

where (x_1, y_1) and (x_2, y_2) are any two distinct pairs from $X_1 \times X_2$.

Let n be a nonzero natural number and Z be the subsets of the Cartesian product $X_1 \times X_2 \times X_3 \times \times \times X_m$ with $\text{card}(Z) = n$. Using Theorem 1, one can demonstrate that $O(n \cdot \log_2(n))$ elementary operations, each of which is applied either to a two-element subset of X_1, or to a two-element subset of $X_2, \ldots$, or to a two-element subset of X_m, are sufficient to reconstruct the lexicographical ordering on Z .

For more detailed information about Theorem 1, its generalizations and applications in the discrete and combinatorial geometry, see [3–5, 8, 9, 13].

Theorem 2. *Let* $(X, \leq)$ *be a nonempty finite linearly ordered set with* $\text{card}(X) = n$. *The probability that exactly* $n - 1$ *elementary operations suffice to reconstruct the given ordered* $\leq$, *is equal to*

$$p = \frac{1}{\dbinom{\binom{n}{2}}{n-1}}.$$

Proof. First of all, let us find a number of all elementary operations on $(X, \leq)$. In fact, we wish to find a number of all two-element subsets of the given linearly ordered set $(X, \leq)$. It is well known that, the number of all two-element subsets of a finite set with $\text{card}(X) = n$ is $\binom{n}{2}$.

At the second step we calculate the number of all possible $(n-1)$-subsets of $\binom{n}{2}$, which is obviously equal to

$$\binom{\binom{n}{2}}{n-1}.$$

Let us prove that exactly one set of two-element subsets enables us to reconstruct the given ordering.

Let us consider two-element subsets $\{x_i, x_j\}$, which allow us to reconstruct the given linear ordering

$$x_1 < x_2 < \cdots < x_n.$$

Notice that to reconstruct given ordering, at least one pair must exist which contains x_1, otherwise reconstruction will be not uniquely determined.

Let us look at those two-element subsets that contain the element x_1. Enumerate all elements of X in the pairs with x_1 as follows:

$$x_{i_1}, x_{i_2}, \ldots, x_{i_k}$$

and consider the set of all such two-element subsets which contain x_1

$$\{x_1, x_{i_1}\}, \{x_1, x_{i_2}\}, \dots, \{x_1, x_{i_k}\}.$$

We fixed the ordering

$$x_2 < x_3 < \dots < x_n$$

and by induction $n-2$ elementary operations are enough to reconstruct the above mentioned ordering.

Suppose that $k \geq 2$. Then $n-k-1$ elementary operations are needed to reconstruct the ordering of the remaining elements $x_2, x_3, \dots, x_n$.

It is clear that if $k \geq 2$, then

$$n-k-1 < n-2.$$

But this contradicts the inductive assumption.

Therefore, $k = 1$ and x_1 is in the pair with just x_{i_1}.

Now let us prove that

$$x_{i_1} = x_2.$$

Suppose that $x_{i_1} \neq x_2$ and $x_{i_1} = x_k$ $(k \neq 2)$. In such a case we get another ordering of the given set $(X, \leq)$.

For example,

$$x_2 < x_3 < \dots < x_{k-1} < x_1 < x_k < \dots < x_n$$

or

$$x_2 < x_3 < \dots < x_1 < x_{k-1} < x_k < \dots < x_n.$$

It is clear that such an ordering is not the given one. Consequently, x_{i_1} should be x_2 in order for a linear ordering to be uniquely determined. $\square$

We thus obtain

$$p = \frac{1}{\binom{\binom{n}{2}}{n-1}}.$$

By using the well-known combinatorial formulas, we get

$$\binom{\binom{n}{2}}{n-1} = \frac{\left(\frac{n(n-1)}{2}\right)!}{(n-1)!\left(\frac{(n-1)(n-2)}{2}\right)!}.$$

A precise estimate of $n!$ that is of importance both for numerical calculations and for theoretical analysis is Stirlinng's formula (see, e.g., [1]):

$$n! \sim n^n e^{-n} \sqrt{2\pi n}.$$

This formula implies that

$$\lim_{n\to\infty} \frac{n!}{n^n e^{-n}\sqrt{2\pi n}} = 1.$$

For more details about this formula and its applications see, e.g., [12].

Applying Stirling's approximation in our case, we get

$$\frac{\left(\frac{n(n-1)}{2}\right)!}{(n-1)!\left(\frac{(n-1)(n-2)}{2}\right)!} \sim \frac{\left(\frac{n(n-1)}{2e}\right)^{\frac{n(n-1)}{2}} \sqrt{\frac{2\pi n(n-1)}{2}}}{\left(\frac{n-1}{e}\right)^{n-1}\sqrt{2\pi(n-1)}\left(\frac{(n-1)(n-2)}{2e}\right)^{\frac{(n-1)(n-2)}{2}}\sqrt{\pi(n-1)(n-2)}}.$$

Since

$$\frac{e^{-\frac{(n-1)n}{2}}}{e^{-(n-1)}e^{-\frac{(n-2)(n-1)}{2}}} = 1,$$

we have

$$\binom{\binom{n}{2}}{n-1} \sim \frac{\left(\frac{n(n-1)}{2}\right)^{\frac{n(n-1)}{2}}}{(n-1)^{n-1}\left(\frac{(n-1)(n-2)}{2}\right)^{\frac{(n-1)(n-2)}{2}}}\sqrt{\frac{n}{2\pi(n-2)(n-1)}}$$

$$= \left[\frac{\left(\frac{n(n-1)}{2}\right)^{\frac{n}{2}}}{(n-1)\left(\frac{(n-1)(n-2)}{2}\right)^{\frac{(n-2)}{2}}}\right]^{n-1}\sqrt{\frac{n}{2\pi(n-2)(n-1)}}$$

$$= \left[\frac{(n(n-1))^{\frac{n}{2}}2^{-\frac{n}{2}}}{(n-1)((n-1)(n-2))^{\frac{(n-2)}{2}}2^{-\frac{n-2}{2}}}\right]^{n-1}\sqrt{\frac{n}{2\pi(n-2)(n-1)}}$$

$$= \frac{1}{2^{n-1}}\left[\frac{n^{\frac{n}{2}}(n-1)^{\frac{n}{2}}}{(n-1)(n-1)^{\frac{(n-2)}{2}}(n-2)^{\frac{(n-2)}{2}}}\right]^{n-1}\sqrt{\frac{n}{2\pi(n-2)(n-1)}}$$

$$= \frac{1}{2^{n-1}}\left[\frac{n^{\frac{n}{2}}}{(n-2)^{\frac{n}{2}}(n-2)^{-1}}\right]^{n-1}\sqrt{\frac{n}{2\pi(n-2)(n-1)}}$$

$$= \frac{1}{2^{n-1}}\left[\frac{n-2}{\left(\frac{n-2}{n}\right)^{\frac{n}{2}}}\right]^{n-1}\sqrt{\frac{n}{2\pi(n-2)(n-1)}}$$

$$= \frac{1}{2^{n-1}}(n-2)^{n-1}e^{n-1}(n-2)^{-\frac{1}{2}}\sqrt{\frac{n}{2\pi(n-1)}}$$

$$= \frac{1}{2^{n-1}}(n-2)^{n-\frac{3}{2}}e^{n-1}\sqrt{\frac{n}{2\pi(n-1)}} \sim n^n\left(\frac{e}{2}\right)^n\sqrt{\frac{1}{2\pi}}.$$

Remark. The number $n^n(\frac{e}{2})^n\sqrt{\frac{1}{2\pi}}$ is much bigger than n^n for sufficiently large natural numbers n. Since the probability p in Theorem 2 is asymptotically equal to $\frac{1}{n^n(\frac{e}{2})^n\sqrt{\frac{1}{2\pi}}}$, we conclude that even for $n = 15$, this probability is almost zero.

In case $n = 15$, the 15^{15} is an extremely big number. In particular, about 10^{11} many stars are in the Milky Way.

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