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BICRITICAL POINTS IN PROBLEM ON THE STABILITY OF HEAT-CONDUCTING FLOWS BETWEEN HORIZONTAL POROUS CYLINDERS

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Abstract. The stability of heat-conducting flow between horizontal porous rotating cylinders with a constant azimuthal pressure gradient is studied. It is assumed that the flow is subjected to the action of a radial flow through the cylinder walls and a radial temperature gradient. The aim of this paper is to find the intersection points of neutral curves that correspond to flow instability and appearance of complex regimes.

1. Formulation of the Problem

We consider the steady heat-conducting flow between horizontal porous rotating cylinders with a constant azimuthal pressure gradient maintained by a pumping of a fluid around the annulus at cylinders. It is assumed that the cylinders heated up to different temperatures and the flow is subjected to the action of a radial converging and diverging fluid through the permeable cylinder walls and radial temperature gradient. External mass forces absent, the fluid inflow through the wall of one cylinder is equal to the fluid outflow through the other one.

We denote the radii, angular velocities and temperature of the inner and outer cylinders by R_1, Ω_1, T_1, and R_2, Ω_2, T_2, respectively. Assume that on the surface of the cylinders the following boundary conditions

$$\begin{aligned} &v_r' = U_0, \quad v_\varphi' = \Omega_1 R_1, \quad v_z' = 0, \quad T' = T_1 \ \ (r = R_1),\\ &v_r' = \frac{U_0}{R}\,, \quad v_\varphi' = \Omega_2 R_2, \quad v_z' = 0, \quad T' = T_2 \ \ (r = R_2) \end{aligned} \tag{1.1}$$

are fulfilled, where $R = \frac{R_2}{R_1}$, $V'(v_r', v_\theta', v_z')$ is the velocity vector, U_0 is the radial velocity through the wall of the inner cylinder.

Under the above assumption, using the Navier–Stokes system, heat transfer, continuity equations and an equation of state [5] in terms of cylindrical coordinates r, θ, z with z-axis coinciding with that of cylinders we obtain the following exact solution for the velocity V_0, temperature T_0, pressure Π_0:

$$\begin{gathered} V_0 = \{u_0(r), v_0(r), 0\}, \quad T_0 = c_1 + c_2 r^{\varkappa P_r},\\ u_0(r) = \frac{R_1 U_0}{r}\,, \quad v_0(r) = \begin{cases} \dfrac{K}{\varkappa}\Big(a r^{\varkappa+1} + \dfrac{b}{r} - r\Big) + A r^{\varkappa+1} + \dfrac{B}{r}, & \varkappa \neq -2,\\ \dfrac{K}{2}\Big(\dfrac{a_1 \ln r + b_1}{r}\Big) + \dfrac{A_1 \ln r + B_1}{r}, & \varkappa = -2, \end{cases}\\ \frac{\partial \Pi_0}{\partial r} = \frac{\rho(u_0^2 + v_0^2)}{r}\,, \end{gathered} \tag{1.2}$$

where

$$K = \frac{1}{2\rho\nu}\Big(\frac{\partial \Pi_0}{\partial \theta}\Big)_0 = \text{const}\,, \quad a = \frac{R^2 - 1}{(R^{\varkappa+2} - 1)R_1^{\varkappa}}\,, \quad a_1 = \frac{R_1^2(R^2 - 1)}{\ln R}\,,$$

$$b = \frac{R_2^2(R^{\varkappa} - 1)}{R^{\varkappa+2} - 1}, \quad b_1 = -\frac{R_1^2 \ln R_2 - R_2^2 \ln R_1}{\ln R}\,,$$

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$$A = \frac{\Omega_1(\Omega R^2 - 1)}{(R^{\varkappa+2} - 1)R_1^{\varkappa}}, \quad A_1 = \frac{\Omega_1 R_1^2(\Omega R^2 - 1)}{\ln R},$$
$$B = \frac{\Omega_1 R_2^2(R^{\varkappa} - \Omega)}{R^{\varkappa+2} - 1}, \quad B_1 = -\frac{\Omega_1 R_1^2(\ln R_2 - \Omega R^2 \ln R_1)}{\ln R},$$
$$c_1 = \frac{T_1 R^{\mathrm{Pr}\,\varkappa} - T_2}{R^{\varkappa\,\mathrm{Pr}} - 1}, \quad c_2 = \frac{T_2 - T_1}{R_1^{\varkappa\,\mathrm{Pr}}(R^{\varkappa\,\mathrm{Pr}} - 1)},$$

$\varkappa = \frac{U_0 R_1}{\nu}$ is the radial Reynolds number, $Pr = \frac{\nu}{\chi}$ is the Prandtl number, ρ is the fluid density, ν and χ are, respectively, the coefficients of kinematic viscosity and thermal diffusion. The radial flow is inward for $\varkappa < 0$ (converging flow) and outward for $\varkappa > 0$ (diverging flow).

The flow with the velocity vector V_0, temperature T_0 and pressure Π_0 is called the main stationary flow. This flow is a superposition of the heat-conducting flow in the transverse direction(maintained by a pumping fluid round the cylinders) and a distribution of angular velocities (maintained by the rotation of the two cylinders). Our aim is to find the intersection points of neutral curves which correspond to flow instability and appearance of complex regimes.

2. Neutral Curves

Let the perturbed state be taken as

$$V' = V_0 + V(v_r, v_\theta, v_z), \quad T' = T_0 + \tau, \quad \Pi' = \Pi_0 + \Pi. \tag{2.1}$$

Taking into account that the main stationary flow consists a rotating shear flow, we denote rotation shear S by $\frac{V_m}{d}$, where V_m is an average velocity in the azimuthal direction, $d = R_2 - R_1$ is a gap width between cylinders. Introducing dimensionless variables for time, velocity, temperature and pressure by S, R_2, SR_2, $T_2 - T_1$, $\nu\rho' S$ in the system of Navier-Stokes equations, for the vector-functions $F = \{v_r, v_\theta, v_z, \tau\}$ and $F_1 = \{u_r, u_\theta, u_z, T_1\}$, we obtain the following nonlinear problem of finding perturbations V, τ and Π:

$$\begin{gathered} \frac{\partial F}{\partial t} + NF - \frac{1}{\mathrm{Ta}} MF + \frac{1}{\mathrm{Ta}} \nabla_1 \Pi = -\mathcal{L}(F, F_1), \\ (\nabla_1, rF) = 0, \quad F\big|_{r=1,R} = 0, \end{gathered} \tag{2.2}$$

where

$$MF = \Big\{\Delta_1 v_r - \frac{1-\varkappa}{r^2} v_r - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}, \Delta_1 v_\theta - \frac{1+\varkappa}{r^2} v_\theta + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}, \Delta_1 v_z, \frac{1}{\mathrm{Pr}}\Delta_1 \tau\Big\},$$
$$NF = \omega_1 \frac{\partial F}{\partial \theta} + \Big\{ \mathrm{Ra}\,\omega_2 \tau - 2\,\mathrm{Ta}\,\omega_1 v_\theta, -g_1 v_r, 0, \frac{g_2}{\mathrm{Pr}} v_r, \Big\},$$
$$\mathcal{L}(F, F_1) = \Big\{(F, \nabla_1)u_r - \frac{v_\theta u_\theta}{r},\ (F, \nabla_1)u_\theta + \frac{v_r u_\theta}{r}, (F, \nabla_1)u_z, (F, \nabla_1)T_1\Big\},$$
$$\Delta_1 = \frac{\partial^2}{\partial r^2} + \frac{1-\varkappa}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}, \quad \nabla_1 = \Big\{\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}, \frac{\partial}{\partial z}, 0\Big\},$$
$$\mathrm{Ta} = \frac{\Omega_1 R_2^2}{2} \text{ is Taylor number},$$
$$\mathrm{Ra} = \frac{\mu}{\lambda}, \quad \mu = \frac{\beta(T_2 - T_1)}{2}, \ \beta \text{ is coefficient of thermal expansion},$$
$$\lambda = \frac{V_m}{\Omega_1 R_2} \text{ is ratio of the average velocities of pumping liquid and rotation},$$
$$V_m = K\,\frac{R_1 R^2}{R-1} D(R), \quad D(R) = \frac{R^{\varkappa} - 1}{R^{\varkappa+2} - 1}\ln R - \frac{\varkappa(R^2 - 1)}{2R^2(\varkappa + 2)},$$
$$\omega_1 = \frac{v_0(r)}{r} = \lambda g(r) + g_0(r), \quad \omega_2 = \omega_1^2 r,$$
$$g(r) = \frac{d}{R_2}\,\frac{D1(R) r^{\varkappa+2} + D2(R) - r^2}{rD(R)}, \quad g_0(r) = D3(R) r^{\varkappa+1} + \frac{D4(R)}{r},$$

$$g_1(r) = \frac{dv_0}{dr} + \frac{v_0}{r} = \frac{d}{R_2}\,\frac{D1(R)(\varkappa+2)r^{\varkappa} - 2}{D(R)} + (\varkappa+2)r^{\varkappa} D3(R),$$
$$D1(R) = \frac{(R^2-1)R^{\varkappa}}{R^{\varkappa+2}-1}, \quad D2(R) = 1 - D1(R),$$
$$D3(R) = \frac{(\Omega R^2-1)R^{\varkappa}}{R^{\varkappa+2}-1}, \quad D4(R) = \frac{R^{\varkappa}-\Omega}{R^{\varkappa+2}-1},$$
$$g_2(r) = \frac{\varkappa \Pr^2 R^{\varkappa \Pr}}{R^{\varkappa \Pr}-1}\, r^{\varkappa \Pr - 1}.$$

Problem (2.2) is written in terms of the Boussinesq approximation, which is based on the assumption that the thermal expansion coefficient is small [1]. In the sequel it will always be assumed that the velocity, temperature and pressure components are periodic with respect to z and θ with the known periods $2\pi/\alpha$ and $2\pi/m$, respectively.

The theoretical and experimental studies have shown that after the loss of stability of main flow between rotating cylinders there occured secondary modes either axisymmetric or nonaxisymmetric disturbances as vortices and oscillatory modes in the form of traveling waves.

To study the transition to complex regimes of special attention are the points of intersection of neutral curves, corresponding to the two above-mentioned kinds of the secondary flows, since at these points with a high probability may appear various regimes, including the complex one [2, 4].

Let $(\mathrm{Ra}_0, \mathrm{Ta}_0)$ be the point lying on the plane of parameters $(\mathrm{Ra}, \mathrm{Ta})$ and corresponding to the intersection of the neutral curves corresponding to the monotonic ($m = 0$) axisymmetric and oscillatory nonaxisymmetric loss of stability of main flow (1.2). Under the definite values of parameters of the problem, the neutral curves may be nonintersecting that indicates that under the corresponding values of parameters of the problem we cannot expect the appearance of complex regimes.

To construct neutral curves, we assume that the perturbations V, temperature τ and pressure Π are infinitely small. The neutral curves, which corresponds to the bifurcation of vortex and azimuthal waves are found by solving the spectral problems:

$$(M - \mathrm{Ta}\, N)\Phi_0 = \nabla_1 p_0, \quad (\nabla_1, r\Phi_0) = 0, \quad \Phi_0\big|_{r=1,R} = 0, \tag{2.3}$$

and

$$(M - \mathrm{Ta}\, N - ic\,\mathrm{Ta})\Phi_1 = \nabla_1 p_1, \quad (\nabla_1, r\Phi_1) = 0, \quad \Phi_1\big|_{r=1,R} = 0, \tag{2.4}$$

where

$$\Phi_0 = \big\{u_0(r), v_0(r), iw_0(r), \tau_0(r)\big\}e^{i\alpha z}, \quad p_0 = q_0(r)e^{i\alpha z}, \tag{2.5}$$
$$\Phi_1 = \big\{u_1(r), v_1(r), w_1(r), \tau_1(r)\big\}e^{-i(m\theta+\alpha z)}, \quad p_1 = q_1(r)e^{-i(m\theta+\alpha z)}, \tag{2.6}$$

c – unknow frequency of neutral azimuthal waves.

Problems of eigenvalues (2.3) and (2.4) have been solved by the shooting method for fixed λ, $\varkappa$, α,R, m, Pr, Ω. Thus, for the fixed values of these parameters we established the dependence of the critical value of the number Ta, Ra and the neutral mode frequency c corresponding to the bifurcation of vortices and azimuthal waves origination on a number Ω. Further, using the Newton method, we minimize the difference between the obtained critical values of Ta_0. This allows us to calculate with sufficient exactness the values Ta_0,Ra_0 and c_0 corresponding to the point of intersection of neutral curves.

The calculations in this paper were performed for the case $R = 2$ (radius of the outer cylinders is two times greater than that of the inner ones), $m = 0, 1$, for various values of axial wave number α, $\Pr = 7$ (the working medium is water) and for small absolute values $\varkappa$ ($-2 < \varkappa < 2$). The results of calculations are presented in Tables 1 and 2.

3. Conclusions

As our calculations show, these intersections of neutral curves take place especially when the liquid pumping is in the direction of the rotation inner cylinder.

When the liquid pumping and the inner cylinder rotate in the same direction we can expect the occurrence of complex modes. In the case where the outer cylinder is rest (Table 1) we find that intersections of neutral curves take place when temperature of the inner cylinder is higher than that of the outer for different axial wave numbers. If the liquid pumping and both cylinders rotate in the same direction, we can expect the occurrence of complex modes when temperature of the outer cylinder is higher than that of the inner one. But for the opposite rotating cylinders when the inner cylinder rotates in the same direction as of the pumping, there arise complex regimes, if temperature of the inner cylinder exceeds that of the outer one for both diverging and converging flows (Table 2).

When the pumping flow is directed to the opposite direction of the rotating inner cylinder, the neutral curves do not intersect and thus it is difficult to expect the occurrence of complex regimes. In this case we find very high frequency of neutral azimuthal waves.

Table 1. The points of intersection of neutral curves $\lambda = 1$, $\Omega = 0$

$\varkappa$	$\alpha = 5$			$\alpha = 8$		
	Ra_0	Ta_0	c_0	Ra_0	τ_0	c_0
−1.9	−2.2118	60.9357	2.5582	−0.5292	48.736	2.6613
−1.5	−0.607	59.501	2.5798	−0.209	49.592	2.6649
−1.1	−0.1559	60.7709	2.5911	−0.0695	51.14	2.665
−0.5	−0.03525	65.1397	2.5924	−0.0184	53.989	2.6618
−0.2	−0.02168	67.8398	2.59219	−0.0117	55.599	2.66
0.5	−0.01567	75.1248	2.5938	−0.0083	59.809	2.657
1	−0.02122	81.1659	2.59819	−0.01068	69.227	2.6567
1.5	−0.03486	87.971	2.60578	−0.01674	67.025	2.6587
2	−0.060456	495.619	2.617139	−0.0283	71.2452	2.6635

Table 2. The points of intersection of neutral curves $\lambda = 1$, $\alpha = 4$

$\varkappa$	$\Omega = 0.1$			$\varkappa$	$\Omega = -0.2$		
	Ra_0	Ta_0	c_0		Ra_0	Re_0	c_0
0.2	0.626	76.221	4.247	2	−0.105	115.42	2.48
0.18	0.6784	71.69	4.20716	1.5	−0.0602	104.8	2.459
0.16	0.7165	68.69	4.17436	1	−0.0035	95.57	2.445
−0.2	0.9096	57.835	4.003	−0.1	−0.0307	79.244	2.426
−0.5	1.1257	55.508	3.906	−1.5	−2.129	74.297	2.355
−0.8	1.4729	54.938	3.818	−1.9	−7.345	78.84	2.317

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