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APPROXIMATION BY TRIGONOMETRIC POLYNOMIALS IN THE FRAMEWORK OF WEIGHTED FULLY MEASURABLE GRAND LORENTZ SPACES

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Abstract. In this note we present the fundamental Bernstein and Nikol'skii type inequalities in weighted fully measurable grand Lorentz spaces. These inequalities we apply to obtain the direct and inverse approximation theorems in approximable subspaces of aforementioned function spaces.

Let $1 < p < \infty$. By Φ_p we denote the set of positive measurable functions φ defined on $(0, p-1]$ which are nondecreasing, bounded with a condition $\lim\limits_{x\to 0+} \varphi(x) = 0$.

Let $\varphi \in \Phi_p$. The fully measureble weighted grand Lebesgue space $L_w^{p)s,\varphi}(\mathbb{T})$ is defined as a set of all measurable 2π-periodic functions $f : \mathbb{T} \longrightarrow \mathbb{R}^1$, for which the norm

$$\|f\|_{L_w^{p)s,\varphi}(\mathbb{T})} = \sup_{0<\varepsilon<p-1} (\varphi(\varepsilon))^{\frac{1}{p-\varepsilon}} \left(s \int_0^\infty \Big(w\Big(x \in \mathbb{T} : \ |f(x)| > \lambda \Big) \Big)^{s/p-\varepsilon} \lambda^{s-1} d\lambda \right)^{1/s}$$

is finite.

The space $L_w^{p)s,\varphi}(\mathbb{T})$ is a non-reflexive, non-separable Banach function space.

The subspace of $L_w^{p)s,\varphi}(\mathbb{T})$ in which the smooth functions are dense, is characterized by the equality

$$\lim_{\varepsilon\to 0} (\varphi(\varepsilon))^{\frac{1}{p-\varepsilon}} \left(s \int_0^\infty \Big(w\Big(x \in \mathbb{T} : \ |f(x)| > \lambda \Big) \Big)^{s/p-\varepsilon} \lambda^{s-1} d\lambda \right)^{1/s} = 0.$$

Denote this subspace by $\widetilde{L}_w^{p)s,\varphi}(\mathbb{T})$. First, we treat the fundamental inequalities for trigonometric polynomials in $L_w^{p)s,\varphi}(\mathbb{T})$.

In the sequel we assume that the weights w belong to the well-known Muckenhoupt A_p class.

Let us give the Bernstein type inequality for the Weyl's fractional derivative of trigonometric polynomials.

Theorem 1. *Let $1 < p, s < \infty$, $\varphi \in \Phi_p$ and $w \in A_p$. For an arbitrary trigonometric polynomial T_n and a number $\alpha > 0$ the following inequality*

$$\|T_n^{(\alpha)}\|_{L_w^{p)s,\varphi}} \le cn^\alpha \|T_n\|_{L_w^{p)s,\varphi}}$$

holds, where the constant c is independent of n and T_n.

The next theorem deals with the Nikol'skii type inequality

Theorem 2. *Let $1 < p < q < \infty$, $1 < s < pq/q-p$ and $r = s/\big(1-\big(\frac{1}{p}-\frac{1}{q}\big)s\big)$. Assume that $\varphi(x) = x^\theta$, $\theta > 0$ and $w \in A_{1+\frac{q}{p'}}$, $p' = \frac{p}{p-1}$.*

Then the inequality

$$\|T_n n^{\frac{1}{p}-\frac{1}{q}}\|_{L_w^{q)r,\theta q/p}} \le cn^{\frac{1}{p}-\frac{1}{q}} \|T_n\|_{L_w^{p)s,\theta}}$$

holds.

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In the space $\widetilde{L}_w^{p)s}(\mathbb{T})$, we introduce the structural and constructive characteristics of functions: the moduli of smoothness

$$\Omega(f,\delta)_{L_w^{p)s,\varphi}} = \sup_{0<h\le\delta} \left\| \frac{1}{2h} \int_{x-h}^{x+h} f(t)dt - f(x) \right\|_{L_w^{p)s,\varphi}}$$

and the best approximation by trigonometric polynomial

$$E_n(f)_{L_w^{p)s,\varphi}} = \inf \|f - T_k\|_{L_w^{p)s,\varphi}},$$

where the infimum is taken over all trigonometric polynomials T_k of degree $k \le n$.

The following analogy of Jackson's theorem is valid.

Theorem 3. *Let $1 < p < \infty$, $\varphi \in \Phi_p$, $\alpha \ge 0$, and $w \in A_p(\mathbb{T})$.*

Then for some positive constant c and all f, $f^{(\alpha)} \in \widetilde{L}_w^{p),s,\varphi}$ the following inequality

$$E_n(f)_{L_w^{p)s,\varphi}} \le \frac{c}{(n+1)^\alpha} \Omega\left(f^{(\alpha)}, \frac{1}{n}\right)_{L_w^{p)s,\varphi}}$$

holds.

In the next statement we announce the inverse inequality.

Theorem 4. *Let $1 < p < \infty$, $\varphi \in \Phi_p$ and $w \in A_p(\mathbb{T})$.*

Then the following inequality:

$$\Omega\left(f, \frac{1}{n}\right)_{L_w^{p)s,\varphi}} \le \frac{c}{n^2} \sum_{k=0}^{n-1} (k+1) E_k(f)_{L_w^{p)s,\varphi}},$$

for $f \in \widetilde{L}w^{p)s,\varphi}$ holds, where the constant c is independent of f and n.

Applying the Nikol'skii type inequality, we prove the following statement.

Theorem 5. *Let the conditions of Theorem 2 be satisfied. Assume that for $f \in \widetilde{L}_w^{p)s,\varphi}$*

$$\sum_{k=1}^{\infty} k^{1/p-1/q-1} E_k(f)_{L_w^{p)s,\varphi}} < \infty.$$

Then $f \in \widetilde{L}_w^{q)r,\theta q/p}$ and

$$E_n(f \cdot w^{1/p-1/q})_{L_w^{q)r,\theta q/p}} \le c\left\{ n^{1/p-1/q} E_n(f)_{L_w^{p)s,\theta}} + \sum_{k=n+1}^{\infty} k^{1/p-1/q-1} E_k(f)_{L_w^{p)s,\varphi}} \right\}.$$

The proof of thr above-mentioned theorems are based essentially on the results obtained in [1].

For the similar results in weighted grand Lebesgue spaces we refer the readers to paper [2].

The detiled proofs will be published in Georgian Mathematikal journal.

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