Transactions of A. Razmadze
Mathematical Institute
Vol. **173** (2019), 65–88

THE BEHAVIOR OF SOLUTIONS OF THE DIRICHLET, NEUMANN AND MIXED DIRICHLET NEUMANN PROBLEMS IN THE VICINITY OF SHARP EDGES OF A PIECEWISE SMOOTH BOUNDARY

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Abstract. An alternative expression of harmonic function in a three-dimensional case in spherical coordinates is proposed. We consider the behavior of solutions of the Dirichlet, Neumann and mixed Dirichlet-Neumann problems in the vicinity of sharp edges of a piecewise smooth boundary. The conditions of geometry admit only analytical solutions in the vicinity of sharp edges of special type. Some effects of an ideal fluid model in the vicinity of sharp edges of this type are discussed.

1. Introduction

In developing a numerical technique the question of proper modelling of corners in a body has always been a challenge. One can possibly argue, on physical grounds, that a sharp corner in a body is essentially a mathematical artifact. On the other hand, it is well known that variables of interest such as flux may change very rapidly around a rounded corner with a small radius of curvature. Hence for computational efficiency it may be quite advantageous to model a corner as sharp, across which there is a jump in the unit normal and tangential vectors to the boundary of the body. Another type of problem involving mixed boundary values is often encountered in real life. This class of problems involves specifications of incompatible boundary conditions on adjacent segments of the boundary of a body. Such situations may arise irrespective of the local geometry of the boundary of the body being investigated. Hence an ability to model corners effectively and efficiently is very important for numerical techniques for many applications of the Boundary element method (BEM).

A large and growing body of literature exists in this important subject area. It is a difficult task to acknowledge all the contributions in this field in a research article, especially with several excellent reviews published earlier. From a large number of research publications devoted to BEM, very few papers, however, are relevant to the subject of this research. Only publications relevant to the topic are those where singularity of a derivative of solutions is discussed. The reader is referred here to the article by Maz'ya for introduction [16].

The subject has theoretical and practical aspects. It should be pointed out that the majority of researchers consider two-dimensional problems only. Very few manuscripts are devoted to treatments of singularity of a derivative of the solutions in numerical algorithms in three dimensions. Mainly the three-dimensional problems are discussed in a way of a numerical experiment. Let us consider only one representative manuscript written from the sition of numerical algorithms creators [14]; there are numerous sources for additional reading.

The approach to theoretical basis can be gained from the work of Kondrat'ev [7] where the achieved result is the form of a solution of an elliptic equation in the vicinity of irregular points of the boundary (angular or conical points) which consists of a regular function and asymptotic series of solutions of model problems at zero boundary conditions.

This subject receives significant attention since the problem has numerous scientific and technical applications. Most of the cited references and literature on BEM application carry out problem solving without defining a form of the density function of a simple or double layer potential of solutions of the model problems at zero boundary conditions and treating it as an "unknown" function. There is a

2010 *Mathematics Subject Classification.* 76M15, 76B03.
Key words and phrases. Potential theory ideal fluid model.

definite need to build the mathematical basis for the applications in order to clarify the methodology and aid for computational problem solving.

The conclusions of Kondrat'ev's work [7] serve as the foundation for the following research publications that mainly use functional analysis [4–20] (for detailed bibliography see [10], [21]). Georgian school of applied mathematics and its founder Kupradze are especially noteworthy [4], [24].

This research formulates and uses somewhat different approach to investigation of derivative's singularity of solutions of problems under consideration. Most authors of the research cited use functional analysis as a research method, while our research is concentrated on the consideration of a harmonic function in comparison with two equivalent forms: the integral form based on the Method of potential and the form with inclusion of trigonometrical functions. The integral form is convenient for proving of the existence and uniqueness of solutions of the considered problems. The trigonometrical functions allow us to use symmetry productively.

The result of this research is formulation of the basis for numerical algorithm as expressions of density functions of a simple or double layer potential for approximation of the terms of Kondrat'ev series having singularity of a derivative in the vicinity of angular, conical points and points of boundary condition's type change in the Dirichlet, Neumann and mixed Dirichlet-Neumann problems. The expression of a harmonic function in a two-dimensional case in the vicinity of these points had been known, but the density function in the expression of the harmonic function by potentials of simple or double layer was unknown.

In a three-dimensional case, the unknown expressions of harmonic function have been obtained, these can be used as an alternative of Legendre functions. The expressions of these functions by potentials of simple and double layer in the vicinity of these points have been proposed.

As any harmonic function satisfying the condition of radiation can be presented by a sum of simple and double layer potentials, the expressions are sufficient for all solutions of problems under consideration.

The results of this research in theoretical aspect are the alternative expressions of harmonic function in a three-dimensional case (3), (4), which have simpler form, than the Legendre functions. Another result is the relation between solutions of the problems with a smooth and piecewise smooth boundary which has been discovered. This relation can be obtained by the conformal mapping by a power function in a two-dimensional case and by two proposed mappings in a three-dimensional case.

Harmonic functions, having singularity of the derivative $r^{\lambda-1}$, $0 < \lambda < 1$, have in two-dimensional (1), (2) and three-dimensional (3), (4) cases the following expressions:

$$A\, r^{\lambda} \sin(\lambda(\theta + l)), \tag{1}$$

$$B\, r^{\lambda} \cos(\lambda(\theta + l)), \tag{2}$$

$$A\, r^{\lambda} \sin(\lambda\theta) \cos(\beta), \tag{3}$$

$$B\, r^{\lambda} \cos(\lambda\theta), \tag{4}$$

where A, B, λ, l are the constants, r, θ, β are variables. The angles θ, β in (3) are located in two perpendicular planes: if the angle θ is measured from the Ox_1-axis of local orthogonal system of coordinates, the angle β is measured in the plane Ox_2x_3. [1]

Mapping of (1), (2), (3), (4), modificatory the value of the variable λ, maps a harmonic function into a harmonic function. In a two-dimensional case this mapping is conformal by the power function. [2] Below, under the term "conformal mapping" we will mean this case of conformal mappings. In a three-dimensional case we name it "β - mapping". We define two these mappings.

Definition 1. The β_0-mapping: the mapping of a half-space into an infinite wedge with the boundary composed of two half-planes, having intersection in one line. In each plane, perpendicular to the line of intersection, the β_0-mapping corresponds to the conformal mapping in the plane by a power function with center at the point of crossing this plane with this line.

[1]Expressions (1), (2) exist for all values of λ. The proof of the existence of expressions (3), (4) for all values of λ is in Section 3.

[2]The conformal mapping by power function $w = z^k$ is the mapping of a half plane in the angular domain [9].

Definition 2. The β_1-mapping: the mapping of a half-space in the interiority or exteriority of an infinite cone, which is the space with exception of the infinite cone. In each plane, having the axis of the cone, the β_1-mapping corresponds to the conformal mapping in a plane by the power function with center at the cone apex. If the index is not marked below, the β-mapping is the substitution of β_0-mapping or β_1-mapping.

In the domain Θ with a closed smooth boundary $S \in C_1$ the harmonic function $u \in C_2(\Theta) \cap C_1(\overline{\Theta})$ under the condition of radiation:

$$|u| < \frac{c}{\sqrt{r}} \quad \text{as} \quad r \to \infty \quad \text{in a two-dimensional case,} \quad |u| < \frac{c}{r} \quad \text{as} \quad r \to \infty \quad \text{in a three-dimensional case,}$$

where c is a constant, r is a distance from S, can be expressed as a sum of potentials of simple and double layers

$$\delta u(p) = -W_S(p, u) + V_S\left(p, \frac{\partial u}{\partial n}\right), \tag{5}$$

where $\delta = 2$, if $p \in \Theta \backslash S$, and $\delta = 1$, if $p \in S$.

If Θ in (5) is finitesimal (internal problem), it is simply connected. If it is not finitesimal (external problem), Θ is a complement of some simply connected domain with respect to the plane in a two-dimensional case and to the space in a three-dimensional case. The condition of radiation is necessary for external problem only.

The functions V and W, called as a potential of simple and double layer in a two-dimensional case, are defined as

$$V_S(p, \varphi) = -\frac{1}{\pi} \int\limits_S \ln(\overline{r}(p, q))\varphi(q) dS_q, \quad W_S(p, \varphi) = -\frac{1}{\pi} \int\limits_S \frac{\partial}{\partial n_q}\left(\ln(\overline{r}(p, q)\right)\varphi(q) dS_q;$$

the potentials of simple $\overline{V}$ and double $\overline{W}$ layer in a three-dimensional case are defined as [3]

$$\overline{V}_S(p, \varphi) = \frac{1}{2\pi} \int\limits_S \frac{\varphi(q)}{\overline{r}(p, q)} dS_q,$$

$$\overline{W}_S(p, \varphi) = \frac{1}{2\pi} \int\limits_S \frac{\partial}{\partial n_q} \left(\frac{1}{\overline{r}(p, q)} \right) \varphi(q) dS_q,$$

where $\overline{r}$ is a distance from the point q to the point p, n is the normal vector to S, external to Θ, φ is a function of density. If the index is not marked, we mean the potential in S. The notation n_q describes the normal vector n at the point q. In the text below, we omit the first argument of $V, W, \overline{V}, \overline{W}$ if the point of observation p has been specified in the text.

We can find in [26, p. 91] formula (5) for $p \in \Theta \backslash S, \delta = 2$, which was obtained for a smooth in Lyapunov's sense boundary. It is still true for $S \in C_1$. Let us show that the limit of this expression at approach p to $S \in C_1$ for $u \in C_2(\Theta) \cap C_1(\overline{\Theta})$ exists and is equal to (5) for $\delta = 1$.

There are the limiting values for the double layer potential W with differentiable density $\varphi_2 \in C_1(S)$ and potential of simple layer V with continuous density $\varphi_1 \in C_0(S)$ on $S \in C_1$ in two– and in three-dimensional cases:

$$\Big[W(\varphi_2)\Big]^{\pm} = \mp\varphi_2 + W(\varphi_2), \tag{6}$$

$$\Big[V(\varphi_1)\Big]^{\pm} = V(\varphi_1), \tag{7}$$

[3] The underlining is added for distinguishing between the two-dimensional and three-dimensional cases.

where the upper index corresponds to the approach from Θ, and the lower index corresponds to that from the region outside of Θ, which complements Θ with respect to the plane or three-dimensional space. [4]

We can find (5), $p \in S$, taking into account (6), (7), as the limiting value of (5), $p \in \Theta \backslash S$, as $p \to S$ from inside of Θ. [5]

For $\varphi_2 \in C_1(S)$, there are the limiting values of normal derivative for $p \in S(C_1)$ in the two– and in three-dimensional cases:

$$\left[\frac{\partial W(\varphi_2)}{\partial n}\right]^{\pm} = Q(\varphi_2 - \varphi_2(p)), \tag{9}$$

$$\left[\frac{\partial V(\varphi_2)}{\partial n}\right]^{\pm} = \pm\varphi_2(p) + \Gamma(\varphi_2), \tag{10}$$

where

$$Q(\varphi_2 - \varphi_2(p)) = \frac{1}{2\pi}\int_S \Pi(n_p, n_q)(\varphi_2(q) - \varphi_2(p))dS_q,$$

$$\Pi(m_{p_1}, m_{q_1}) = \frac{\partial}{\partial m_{p_1}}\left(\frac{\partial \Upsilon(p_1, q_1)}{\partial m_{q_1}}\right),$$

$$\Gamma(\varphi) = \frac{1}{2\pi}\int_S \frac{\partial \Upsilon(p, q)}{\partial n_p}\varphi_2(q)dS_q,$$

$\Upsilon(p,q) = \dfrac{2}{\ln(\overline{r}(p,q))}$ in the two-dimensional case, $\Upsilon(p,q) = \dfrac{1}{\overline{r}(p,q)}$ in the three-dimensional case; m_{p_1}, m_{q_1} are some unit vectors at the points p_1, q_1.

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[4]If we use Gauss's theorem (8) we can obtain (6) because of continuity of the integral $W(\varphi_2 - \varphi_2(p_1))$ when point p cross the boundary $S \in C_1$ at the point $p_1 \in S$.

$$\int_S \overline{\Upsilon}(p,q)dS_q = -2, \;\; p \in \Theta\backslash S, \qquad \int_S \overline{\Upsilon}(p,q)dS_q = 0, \;\; p \notin \Theta \cup S, \qquad \int_S \overline{\Upsilon}(p,q)dS_q = -1, \;\; p \in S, \tag{8}$$

where

$$\overline{\Upsilon}(p,q) = \frac{1}{\pi}\frac{\partial}{\partial n_q}\left(\ln\left(\frac{1}{r(p,q)}\right)\right) \text{ in two-dimensional case,}$$

$$\overline{\Upsilon}(p,q) = \frac{1}{2\pi}\frac{\partial}{\partial n_q}\left(\frac{1}{r(p,q)}\right) \text{ in three-dimensional case.}$$

The expression (7) follows from continuity of the integral with weak singularity V.

[5]The second Green's formula [26, p. 90], where one of the functions is Newtonian or logarithmic source and the second is harmonic function $u \in C_2(\Theta) \cap C_1(\overline{\Theta})$, at the point $p_1 \in \Theta\backslash S$ in the vicinity of $p \in S$, $S \in C_1$, by the Gauss's theorem (8) can be represented in the form

$$2\big(u(p_1) - u(p)\big) = -W\big(p_1, u - u(p)\big) + V\left(p_1, \frac{\partial u}{\partial n}\right).$$

When $p_1 \to p$, the left hand-side tends to zero, the right-hand side is continuous for $p \in S$,

$$= -W\big(p, u - u(p)\big) + V\left(p, \frac{\partial u}{\partial n}\right), p_1 = p, p \in S.$$

The equivalent of this equation is

$$u(p) = -W(p, u) + V\left(p, \frac{\partial u}{\partial n}\right), p \in S.$$

$Q(\varphi_2 - \varphi_2(p))$ is the operator of the function $\varphi_2 - \varphi_2(p)$, $\varphi_2 \in C_1(S)$, integral in the operator exists in $p \in S$, $S \in C_1$, in the sense of principal value. [6] Γ is singular integral. [7]

If $u \in C_m(\Theta) \cap C_k(\overline{\Theta})$, $m \geq 2, k \geq 1$, the reasoning is also true right up to $m = \infty, k = \infty$, when u is analytical in some region, which includes Θ.

In the part below we consider the behavior of the potentials on a piecewise smooth boundary next to the angular or conical point and to the point of change of type of the boundary conditions in the function space $u \in L_2^{(1)}(\Theta)$. [8] [9]

Definition 3. If

$$\int\limits_{\Theta} \left(|u|^2 + |\text{grad}u|^2\right) d\Theta < \infty, \text{ we say that } u \in L_2^{(1)}(\Theta).$$

We make generalization of (5) for piecewise smooth boundary by using conformal mapping and β - mappings and prove expressions 3), (4).

Comparing the expressions (1), (2) and (3), (4) with (5), in the vicinity of these points the following result can be obtained: geometry of sharp edges of special type allows only analytical solutions in the Dirichlet, Neumann and mixed Dirichlet-Neumann problems. This result has important consequences, and as it will be discussed later, can be used in an ideal fluid model in applications, where this model corresponds to the real physical processes with required degree of accuracy, because the velocity of potential flow in this model is a solution of the Neumann problem.

2. The Potentials $V_L(r^\lambda)$, $W_L(r^\lambda)$ on the Ray L.

Let us consider the integral $W_L(\varphi)$, $\varphi = r^\lambda$, in the two-dimensional case on the ray located along the Ox_1-axis of the left orthogonal coordinate system for positive values of x_1 (L coincides with the right half of the Ox_1)-axis, r is the distance from the point of the ray emergence at the origin of

[6]Because of the Gauss's theorem, in the two-dimensional and in three-dimensional cases (8) for $p_1 \in \Theta \backslash S$ in the vicinity of $p \in S$ the expression

$$\frac{\partial W(p_1, \varphi_2 - \varphi_2(p))}{\partial n_p} = \frac{\partial W(p_1, \varphi_2)}{\partial n_p} \tag{11}$$

is true. Let us split S in two parts: $S = S_R \cup S_a$, where S_R is a part of S inside of the circle in two-dimensional case or sphere in the three-dimensional case with small radius R and center at p , S_a is the remaining part of S. According to the condition $\varphi_2 \in C_1(S)$, we can present φ_2 in p by two terms of Taylor series and addition. When $p_1 \to p$, because of the subtracting in (11) the first term has no influence on the integral sum. If $S \in C_1$, when R is small, we can replace S_R by a segment of tangent line in the two-dimensional case and by a circle in a tangent plane in the three-dimensional case, and the corresponding integral on S_R of the second term is equal to zero, it exists as singular integral. The integral of the addition on S_R converges as integral with a weak singularity. Therefore in the limiting expression of (11) as $p_1 \to p$ the integral exists in the sense of the principal value because there is the limit of the integral sum on S_a when the radius of the circle or sphere tends to zero. It is also true at approaching $p_1 \to p$, if $p_1 \notin \Theta$. Finally, we get (9), where integral in the right-hand side exists in the sense of the principal value for $\varphi_2 \in C_1(S)$, $S \in C_1$.

[7]When R, is small we can replace S_R by a segment of tangent line in the two-dimensional case and by a circle in a tangent plane in the three-dimensional case. The integral Γ on S_R of the second term of Taylor series in p of φ_2 at approach $p_1 \to p$, $p_1 \in \Theta \backslash S$ converges as the integral with a weak singularity. Integral Γ of addition on S_R and integral Γ of φ_2 on S_a converges. We have to consider the integral Γ of the first term on S_R at approach $p_1 \to p$. The first term is a constant, consequently, the integral Γ from it coincides on S_R with the integral in Gauss's theorem (8) with a different sign. Therefore the gap in $p \in S$ of limiting values of the integral from different sides of S is defined by the integral in the Gauss's theorem. Finally, we get (10).

The integral Γ on S_R of φ_2 is equal to zero because the numerator of Γ is equal to zero when the points p and q belong to one line with normal vector n_p in the two-dimensional case or one plane with a normal vector n_p in the three-dimensional case. Consequently, the condition [26, p. 58, (3.20)] of the existence of singular integral is satisfied.

[8]If the following expression of the first Green's formula, where both of the functions are equal to u, exists:

$$\int\limits_S u \frac{\partial u}{\partial n} dS = \int\limits_{\Theta} \sum_{k=1}^{m} \left(\frac{\partial u}{\partial x_k}\right)^2 d\Theta$$

($m = 2$ in the two-dimensional case, $m = 3$ in the three-dimensional case), the function $u \in L_2^{(1)}(\Theta)$. The quadratic form in the right part corresponds to the expression of energy. The condition $u \in L_2^{(1)}(\Theta)$ is equivalency of finiteness of value of energy in applications.

[9]We define the function space as it has been done in [15, pp. 122–130], this is used for applications in Section 5.

coordinates, $0 < \lambda < 1$. Let us show that the limiting values of $W_L(r^\lambda)$ at approach p from region of negative values of x_2 (upper sign) and from the region of positive values of x_2 (lower sign) are equal to (12) when we mean the normal vector $\overrightarrow{n} = (0, 1)$.

$$\left[W_L(r^\lambda)\right]^{\pm} = \mp r^\lambda. \tag{12}$$

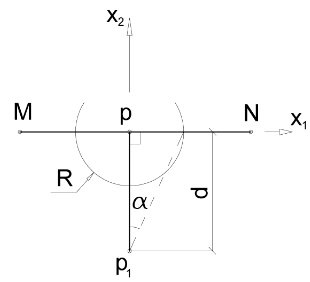

FIGURE 1. Approach point p_1 to the point $p \in]MN[$.

♢ If we consider potential $W_{[MN]}(\varphi)$, $\varphi \in C_n, n \geq 1$, on the segment of straight line $[MN]$ at the approach point p_1 to the point $p \in]MN[$ in the local system of coordinates (Figure 1), we find: only thefirst term of Taylor series of presentation of φ at p corresponds to the integral having different limiting values in the sides of $[MN]$. Integrals of other terms are continuous when the point p_1 crosses the $[MN]$, they are equal to zero for $p \in]MN[$ when p_1 coincides with p. Let $[MN]$ consist of two parts: $[MN] = S_R \cup S_a$, where S_R is a part of $[MN]$ inside of the circle with small radius R and center at p. The integral of the first term on S_a creates continuous in p function, which is equal to zero at $p \in]MN[$ when p_1 coincides with p. The integral $W_{[MN]}(\varphi) = 0$ for $p \in]MN[$ when p_1 coincides with p because the numerator of $W_{[MN]}$ is equal to zero, it is still true, if we replace $[MN]$ by the ray with the vertex in M. Consequently, we have to consider the integral of the first term on S_R

$$\frac{\varphi(p)}{\pi} \int\limits_{S_R} \frac{d}{x_1^2 + d^2} dx_1 = \mathrm{sign}(d) \frac{\varphi(p)}{\pi} \left(\arctan \frac{x_1}{|d|}\right) \Big|_{-R}^{R},$$

when the point of observation p_1 approaches p (Figure 1). The arctangent $\arctan \frac{R}{|d|})$ in the last expression is equal to the value of the angle α (Figure 1) which approaches $\pi/2$ when the point p_1 approaches the point p.

When $\varphi(p) = r^\lambda$, $0 < \lambda < 1$, $p \in [MN]$, r is the distance from the point M, the limiting values of the last expression at the approach point p_1 to the point p is equivalent of (12) if we replace $[MN]$ by the ray with the vertex in M. We can do this because the corresponding integral on the infinite part of the ray converges at p_1 as $p_1 \to p$ and this integral is equal to zero when p_1 coincides with p; in the vertex of the ray this function $\varphi(p) = r^\lambda$ is equal to zero. ♢

Let us consider the function

$$A(r, \Omega) = -r^\lambda \frac{\sin(\lambda(\Omega - \pi))}{\sin(\lambda\pi)} - W_L(r^\lambda)$$

and its derivative

$$\frac{\partial A}{\partial x_1} \tag{13}$$

in the polar coordinate system:

$$x_1 = r\cos(\Omega), x_2 = r\sin(\Omega), \tag{14}$$

the angle Ω is measured anticlockwise from Ox_1, the ray L coincides with the right half of the Ox_1-axis. The harmonic function (13) in the domain of plane with a slit in the half of the $Ox_1, x_1 \geq 0$-axis, has zero values in the boundary: in the half of the $Ox_1, x_1 \geq 0$-axis, and in the infinite boundary. Consequently, (13) is zero function. Therefore the function A is a constant. Since A is equal to zero at the origin of coordinates, this constant is equal to zero. Thus we have found: the harmonic function of the potential $W_L(r^\lambda)$, $0<\lambda<1$ which is defined on the ray L, has the expression

$$W_L(r^\lambda) = -r^\lambda \frac{\sin(\lambda(\Omega-\pi))}{\sin(\lambda\pi)}, \quad 0 \le \Omega \le 2\pi. \tag{15}$$

If we consider a derivative of $W_L(r^\lambda)$ by the angle Ω in the polar coordinate system in continuation of ray L at $\Omega=\pi$, we find that it is equal to [10]

$$-\frac{\lambda r^\lambda}{\sin(\lambda\pi)}. \tag{16}$$

This result justifies (15).

Let us consider simple layer V with density r^λ which is defined on the ray $V_L(r^\lambda)$ in the polar system of coordinates (14). There are the relations between values of derivatives of some function T when $x_2=0$ for $\Omega=0$, $\Omega=2\pi$, $\Omega=\pi$

$$\left.\frac{\partial T}{\partial x_2}\right|_{\Omega=0} = \left.\frac{\partial T}{r\partial\Omega}\right|_{\Omega=0}, \quad \left.\frac{\partial T}{\partial x_2}\right|_{\Omega=2\pi} = \left.\frac{\partial T}{r\partial\Omega}\right|_{\Omega=2\pi}, \quad \left.\frac{\partial T}{\partial x_2}\right|_{\Omega=\pi} = -\left.\frac{\partial T}{r\partial\Omega}\right|_{\Omega=\pi}.$$

Owing to the last expressions, (15) and the relation

$$\Gamma_L(\varphi) = -W_L(\varphi), \tag{17}$$

the function $V_L(r^\lambda)$ has to satisfy the conditions [11]

$$\left.\frac{\partial V_L(r^\lambda)}{r\partial\Omega}\right|_{\Omega=0} = -r^\lambda, \quad \left.\frac{\partial V_L(r^\lambda)}{r\partial\Omega}\right|_{\Omega=2\pi} = r^\lambda, \quad \left.\frac{\partial V_L(r^\lambda)}{r\partial\Omega}\right|_{\Omega=\pi} = 0.$$

Only one two-dimensional harmonic function (2) satisfies the conditions, thus:

$$V_L(r^\lambda) = \frac{1}{\lambda+1}\frac{r^{\lambda+1}}{\sin(\lambda\pi)}\cos((\lambda+1)(\Omega-\pi)) + C, \tag{18}$$

$0<\lambda<1$, $0\le\Omega\le 2\pi$, C is a constant.

Let the potential W be defined on two rays with one vertex at the origin of coordinates located under the angles: plus α and minus α from the Ox_1-axis, and have density with an absolute value r^λ on each ray and symmetric normal vectors on the rays which coincide with external normal vectors of the wedge with aperture angle 2α . This potential creates the functions u_w^+ and u_w^-, when the density functions on the rays have symmetric positive values or antisymmetric values: positive values on the one ray and negative values on the other ray, accordingly.

$$u_w^+(r,\Omega) = -\Big(r^\lambda \sin(\lambda(\Omega-\alpha-\pi)) + r^\lambda \sin(\lambda(-\Omega-\alpha+\pi))\Big)/\sin(\lambda\pi),$$
$$u_w^-(r,\Omega) = -\Big(r^\lambda \sin(\lambda(\Omega-\alpha-\pi)) - r^\lambda \sin(\lambda(-\Omega-\alpha+\pi))\Big)/\sin(\lambda\pi).$$

After transformations and substitution $\theta=\Omega-\pi$, we get

$$u_w^+(r,\theta) = -\frac{2\sin(-\lambda\alpha)}{\sin(\lambda\pi)} r^\lambda \cos(\lambda\theta), \tag{19}$$

$$u_w^-(r,\theta) = -\frac{2\cos(-\lambda\alpha)}{\sin(\lambda\pi)} r^\lambda \sin(\lambda\theta), \tag{20}$$

where$-\pi+\alpha\le\theta\le\pi-\alpha$.

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[10]After substitution $\Omega=\pi$ the value of the derivative by Ω has the expression

$$-\frac{1}{\pi}\int\limits_0^\infty \frac{rb^\lambda}{(r+b)^2}db = -\frac{\lambda r^\lambda}{\sin(\lambda\pi)}.$$

This integral can be calculated by "Wolfram Mathematica 9" (www.wolfram.com):

FullSimplify[Integrate[(r*b^L)/(r + b)^2, {b, 0, Infinity}]]

[11]Despite the fact that the functions $V_L(r^\lambda)$, $W_L(r^\lambda)$ are defined for $0\le\Omega\le2\pi$, the two-dimensional harmonic functions (1) , (2) exist for all values of Ω, thus the derivatives for $\Omega=0, \Omega=2\pi$ can be considered, the values of derivatives of $V_L(r^\lambda)$ for $\Omega=0,\Omega=2\pi$ are the limits as aspiration $\Omega\to0$ and $\Omega\to2\pi$.

If we consider the potential V on the two rays with same combination of density functions and normal vector's directions, we obtain

$$u_v^+(r,\theta) = \frac{1}{\lambda+1}\frac{2\cos\big((\lambda+1)\alpha\big)}{\sin(\lambda\pi)} r^{\lambda+1}\cos\big(\theta(\lambda+1)\big) + C_1, \tag{21}$$

$$u_v^-(r,\theta) = \frac{-1}{\lambda+1}\frac{2\sin\big((\lambda+1)\alpha\big)}{\sin(\lambda\pi)} r^{\lambda+1}\sin\big(\theta(\lambda+1)\big) + C_2, \tag{22}$$

where $-\pi+\alpha \le \theta \le \pi-\alpha$, C_1, C_2 are the constants. [12]

Let us introduce the notation: $\psi_1(r,\theta) = C_\psi u_w^+(R,\theta)$, $\psi_2(r,\theta) = C_\psi u_w^-(r,\theta)$, $\psi_3(r,\theta) = C_\psi u_v^+(r,\theta)$, $\psi_4(r,\theta) = C_\psi u_v^-(r,\theta)$, where $C_\psi = \dfrac{\lambda+1}{4}$.

Expressions (19), (20), (21), (22) were obtained for $0<\lambda<1$. For these values of λ, the integrals $W_L(r^\lambda)$, $V_L(r^\lambda)$ converge at a point for which $r<\infty$.

There are the values $\lambda \ge 1$ for which expressions (19), (20), (21), (22) are equal to zero when the point of observation of the potentials W, V of the two rays belongs to one of the rays. In this case the values of the potential W, V of the two rays are finite for all points where $r<\infty$ despite the potentials $W_L(r^\lambda)$, $V_L(r^\lambda)$ of one ray do not converge at these points, the rays "counterpoise" each other. We can consider expressions (19), (20), (21), (22) having zero values at the points of rays for $\lambda \ge 1$ as a result of conformal mapping by power function with center in the common vertex of the rays from the same expressions for $0<\lambda<1$. The mapping do not change the zero boundary conditions therefore the integrals of the potentials converge for $\lambda \ge 1$ at the points of the rays after the mapping as they have zero values there.

The integrals converge in the described sense only. We do not need to calculate these integrals numerically, since expressions (19), (20), (21), (22) have already been obtained, but we have to take into account the possibility of changing the multiplier function in different ranges of λ. The functions (19), (20), (21), (22) can be written in the form

$$f_w(\lambda)u_w^+(\lambda,r,\theta) = B_w(\lambda,\alpha)r^\lambda\cos(\lambda\theta), \tag{23}$$

$$f_w(\lambda)u_w^-(\lambda,r,\theta) = A_w(\lambda,\alpha)r^\lambda\sin(\lambda\theta). \tag{24}$$

$$f_v(\lambda)u_v^+(\lambda,r,\theta) = B_v(\lambda,\alpha)r^{\lambda+1}\cos(\theta(\lambda+1)), \tag{25}$$

$$f_v(\lambda)u_v^-(\lambda,r,\theta) = A_v(\lambda,\alpha)r^{\lambda+1}\sin(\theta(\lambda+1)), \tag{26}$$

where $f_w(\lambda)=1$, $f_v(\lambda)=\lambda+1$ at $0\le\lambda\le1$. If λ and α are such that (23), (24), (25), (25) are equal to zero on the rays, the functions A_w, B_w, A_v, B_v depend on λ only, as the values of α are determined by the values of λ. Since A_w, B_w, A_v, B_v are periodic, the range $0\le\lambda\le1$ defines these functions for all values of λ. The limiting values for $\lambda\to0$ and $\lambda\to1$ of (23), (24), (25), (26) under the described conditions exist and are equal to zero. Since $u_w^+, u_w^-, u_v^+, u_v^-$ are equal to zero on the rays for $0\le\lambda\le1$ and conformal mapping by power function of the functions created by $u_w^+, u_w^-, u_v^+, u_v^-$ for $0\le\lambda\le1$, have zero values on the rays for all values of λ, the integrals, corresponding to $u_w^+, u_w^-, u_v^+, u_v^-$, converge for all values of λ at the points of the rays. Therefore they converge at other points of the domain at $r<\infty$, $\lambda\ge0$ and at $r\ne0$, $\lambda<0$ for all values of λ. The right– and left-hand sides of expressions (23), (24), (25), (26) are harmonic functions of the forms (1), (2). Consequently, the result of conformal mapping by a power function of the both parts of equalities (23), (24), (25), (26) are harmonic functions of the forms (1), (2), despite the factors $f_w(\lambda), f_v(\lambda)$ are not known for all values of λ.

Any harmonic function has maximal and minimal values in the boundary of the domain where the function is defined. The potentials are equal to zero in the rays and expressions (19), (20), (21), (22)

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[12] Expression (21) with the value of the angle $\alpha = \dfrac{\pi}{2(\lambda+1)}$ is equal to constant C_1. Conformal mapping from constant function by power function of the domain $-\pi+\alpha\le\theta\le\pi-\alpha$ to a domain with different value of α is the same constant function. Consequently, in all expressions (21), equal to the constant, this constant has the same magnitude; the value of C_1 does not depend on λ and α. Below, in the footnote 18it is shown that this is possible if $C_1=0$ only. As $C_1=2C$, $C=0$ in (18), $C_2=0$ in (22).

have infinite values in the infinite boundary $r = \infty$, consequently, the potentials have finite values at the points of the domain where $r < \infty$. If we consider expressions (19), (20), (21), (22) having zero values at the points of the rays for $\lambda \leq 0$ as a result of conformal mapping by a power function with center in the common vertex of the rays from the same expressions for $0 < \lambda < 1$ we find that the integrals of the potentials converge at all points of the domain with the exception of the point of singularity: $r = 0$. [13]

3. The Potentials $\overline{V}_{\widehat{S}}(r^\lambda)$, $\overline{W}_{\widehat{S}}(r^\lambda)$ on the Sector $\widehat{S}$.

A solution of three-dimensional Laplace equation in spherical coordinate system:

$$\triangle u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin(\Omega)}\frac{\partial}{\partial \Omega}\left(\sin(\Omega)\frac{\partial u}{\partial \Omega}\right) + \frac{1}{r^2 \sin(\Omega)^2}\frac{\partial^2 u}{\partial \beta^2} = 0 \tag{27}$$

can be considered in the form

$$u = r^\lambda U(\Omega)\cos(\kappa\beta), \tag{28}$$

where κ is integer, $\kappa \geq 0$ [26, p. 319]. Equation (27) has singular points for $r = 0$, $\Omega = 0$, $\Omega = \pi$.

Let us suppose that expression (28), $0 < \lambda < 1$, is an axi-symmetric solution of equation (27), which is equal outside of a cone to the potential $\overline{V}$ located on the cone surface. Because of the axial symmetry, solution (28) has to have the expression ($\kappa = 0$)

$$u = r^\lambda U_v(\Omega) = \int_0^{2\pi} \overline{V}_{\widehat{S}}(\varphi_v) d\beta, \tag{29}$$

where the angle β is measured in the plane, perpendicular to the axis of the cone; the angle Ω is measured from axis of the cone, Ω is equal to a half of aperture angle at the points of cone surface; $\overline{V}_{\widehat{S}}(\varphi_v)$ is the potential $\overline{V}$ located on the sector $\widehat{S}$ of the cone with an infinitely small angle $d\beta$ which corresponds to the sector with an infinitely small angle $d\eta$ in the local coordinate system (Figure 2). In the limiting case, when the cone surface transforms by the β_1-mapping in a plane, perpendicular to the axis of the cone, the angle η coincides with β. [14]

When we consider an integral of some function $A(p, q)$ over a square of sector $\widehat{S}$ in the local system of coordinates (Figure 2):

$$\int_{\widehat{S}} A(p, q) dS_q = \int_r A(p, q) r dr d\eta,$$

we mean: a sector is located on the plane Ox_1x_3, the angle η is measured from Ox_1, the bisector of the sector $d\eta$ is located iin the Ox_1-axis. Element of the surface integral over the square of the sector: $d\widehat{S}_q = r dr d\eta$, where q is the point on the bisector in the Ox_1-axis, r is the distance from the vertex (Figure 2). Consequently, at the point p of the bisector integral $\overline{V}_{\widehat{S}}(\varphi_v)$, $\varphi_v \in C_0(\widehat{S})$, is the integral with a weak singularity.

The first and second derivatives of (29) in direction of perpendicular to a plane, having axis of the cone, are equal to zero in the plane because of axial symmetry of the problem. Therefore $r^\lambda U_v(\Omega)$ is a two-dimensional harmonic function in any plane having the axis of the cone, it has the axis of symmetry in the axis of the cone. Consequently, the β_1-mapping of the cone transforms two-dimensional harmonic function $r^\lambda U_v(\Omega)$ into another two-dimensional harmonic function with same the form of expression (2), but different value of λ in the expression.

Because the conformal mapping by a power function with center at the point $r = 0$ can be done for any expression (2) and can transform the expression in other expression with the same form, but different value of λ, there are special case of the β_1-mapping which transforms the cone surface into a

[13] The right-hand sides of expressions (19), (20), (21), (22) for zero values on the rays include the solution of the Dirichlet or Neumann problem for a wedge under zero boundary conditions [26, pp. 305–310].

[14] The solution of the Dirichlet problem with singularity of derivative $r^{\lambda-1}, 0 < \lambda < 1$, outside of the cone in the vicinity of its vertex: $r = 0$, has to be under the zero boundary conditions. This result is related to the term "Noetherian" [7] and can be illustrated by the conformal mapping as is shown in the text below. Here we do not use this restriction, therefore (29) may be not equal to zero on the cone surface.

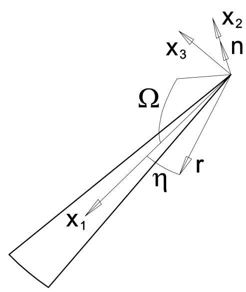

FIGURE 2. Sector $\widehat{S}$ with infinitely small angle $d\eta$.

ray and the corresponding expression (29). In this case, the Newtonian potential $\overline{V}_{\widehat{S}}(\varphi_v)$ on each of the sectors (Figure 2) after transformation will correspond to the ray, and the right-hand side of (29) after the mapping will correspond to 2π such rays. [15] Consequently, the potential of one ray $\overline{V}_{\widehat{S}}(\varphi_v)$ has axial symmetry and expression (28) for $\kappa = 0$ in which the two-dimensional harmonic function $r^\lambda U_v(\Omega)$ is a sum of symmetric functions having expression (2) and a constant. [16]

Let us consider the function

$$\Gamma_{\widehat{S}}(\varphi_v) = -\overline{W}_{\widehat{S}}(\varphi_v), \tag{30}$$

where $\overline{W}_{\widehat{S}}$ is the potential $\overline{W}$ on the sector (2). The function (30) has to be a harmonic function having expression (28) for $\kappa = 1$, because of one plane of symmetry Ox_1x_2. Therefore the potential $\overline{W}_{\widehat{S}}(\varphi_v)$ has the expression

$$fpw\overline{W}_{\widehat{S}}(\varphi_v) = r^\lambda U_w(\Omega)\cos(\beta), \tag{31}$$

where the angle β is measured in plane Ox_2x_3 from the Ox_2-axis (2). The function $r^\lambda U_w(\Omega)$ in the plane Ox_1x_2 is two-dimensional harmonic function because it is the derivative of two-dimensional harmonic function in this plane (30).

Expression (31) has anti-symmetry by the plane Ox_1x_3 and symmetry by the plane Ox_1x_2. Therefore the two-dimensional harmonic function $r^\lambda U_w(\Omega)$ has the expression (1)

$$s400r^\lambda U_w(\Omega) = c_w r^\lambda \sin(\lambda(\Omega - \pi)), \tag{32}$$

where the constant c_w is unknown, $c_w \neq 0$.

Consequently,

$$\left.\frac{\partial \overline{W}_{\widehat{S}}(\varphi_v)}{\partial \Omega}\right|_{\substack{\beta=0\\ \Omega=\pi}} = r^\lambda \left.\frac{\partial U_w(\Omega)}{\partial \Omega}\right|_{\Omega=\pi} = \lambda c_w r^\lambda \cos(\lambda(\Omega-\pi))\Big|_{\Omega=\pi} = \lambda c_w r^\lambda. \tag{33}$$

Let us suppose $\varphi_v = r^\lambda$, $0 < \lambda < 1$ and consider under this guess the following derivative: [17]

$$\left.\frac{\partial \overline{W}_{\widehat{S}}(r^\lambda)}{\partial \Omega}\right|_{\substack{\beta=0\\ \Omega=\pi}} = -\frac{1}{2\pi}\int_0^\infty \frac{r b^{\lambda+1}}{(r+b)^3}\,db = -\frac{1}{4}\lambda(1+\lambda)\frac{r^\lambda}{\sin(\lambda\pi)}. \tag{34}$$

If we compare (34) with (33), we find that our guess is true: $c_w = -\dfrac{1+\lambda}{4\sin(\lambda\pi)}$. Finally, we get

$$\overline{W}_{\widehat{S}}(r^\lambda) = -(\lambda+1)r^\lambda\frac{\sin(\lambda(\Omega-\pi))}{4\sin(\lambda\pi)}\cos(\beta), \quad 0<\lambda<1, \quad 0 \le \Omega \le 2\pi, \quad -\frac{\pi}{2} \le \beta \le \frac{\pi}{2}. \tag{35}$$

[15] For $0 < \lambda < 1$, after the mapping the value of λ in new expression (29) will be in the same range of values.

[16] The angle $d\eta$ is infinitesimal, the Newtonian potential on the sector (Figure 2) behaves as an (integral) sum of Newtonian sources located on the ray of its bisector. Therefore the potential field is axisymmetric, rotation of the sector around its bisector does not change the field. We can consider one sector as we consider two sectors below (41) with the same result: $\kappa = 0$.

[17] The following integral can be calculated analogously (16), footnote 10.

The two-dimensional harmonic functions $r^\lambda U_v(\Omega)$, $r^\lambda U_w(\Omega)$ of sector's $\widehat{S}$ (2) have relation between each other in the plane Ox_1x_2 (30) which is the same relation of the functions $V_L(r^\lambda)$, $W_L(r^\lambda)$ of the ray L (17 Expressions of the functions $r^\lambda U_w(\Omega)$ (32) and $W_L(r^\lambda)$ (15) are identical to the difference only in coefficient value. Consequently, the functions $r^\lambda U_v(\Omega)$, $V_L(r^\lambda)$ are identical with difference only in a value of this coefficient. Thus we get [18]

$$\overline{V}_{\widehat{S}}(r^\lambda) = \frac{r^{\lambda+1}}{4\sin(\lambda\pi)}\cos\big((\lambda+1)(\Omega-\pi)\big), \quad 0<\lambda<1, \quad 0\le\Omega\le 2\pi, \quad -\frac{\pi}{2}\le\beta\le\frac{\pi}{2}. \tag{36}$$

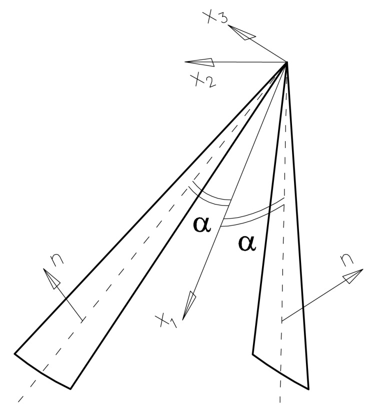

FIGURE 3. Two sectors (see 2) are in the plane Ox_1x_2.

Let us consider two sectors (3) which are the pair of sectors (2). Let us show that the potentials $\overline{V}$, $\overline{W}$, having values of density function in the plane Ox_1x_2 and the direction of normal vectors, same as the density function and the direction of normal vectors on the pair of rays of the expressions (19), (20), (21), (22), define the three-dimensional harmonic functions

$$\overline{u}_w^+(r,\theta,\beta) = -(\lambda+1)\frac{\sin(-\lambda\alpha)}{2\sin(\lambda\pi)}r^\lambda\cos(\lambda\theta), \tag{37}$$

$$\overline{u}_w^-(r,\theta,\beta) = -(\lambda+1)\frac{\cos(-\lambda\alpha)}{2\sin(\lambda\pi)}r^\lambda\sin(\lambda\theta)\cos(\beta), \tag{38}$$

$$\overline{u}_v^+(r,\theta,\beta) = \frac{\cos\big((\lambda+1)\alpha\big)}{2\sin(\lambda\pi)}r^{\lambda+1}\cos\big(\theta(\lambda+1)\big), \tag{39}$$

$$\overline{u}_v^-(r,\theta,\beta) = \frac{-\sin\big((\lambda+1)\alpha\big)}{2\sin(\lambda\pi)}r^{\lambda+1}\sin\big(\theta(\lambda+1)\big)\cos(\beta), \tag{40}$$

where $0<\lambda<1$, $\theta=\Omega-\pi$, $-\pi+\alpha\le\theta\le\pi-\alpha$, $-\pi/2\le\beta<\pi/2$.

Because of the above-described relations of the two-dimensional harmonic functions of the sector $\widehat{S}$ and the ray L in the plane Ox_1x_2 (3), expressions (37), (38), (39), (40) are identical to (19), (20), (21), (22) in this plane with difference only in coefficient value. As each of the functions (37), (38), (39), (40) is a three-dimensional harmonic function having the plane of symmetry Ox_1x_2 that has expression

[18] The potential $\overline{V}_{\widehat{S}}(r^\lambda)$ can be calulated explicitly in the continuation of the sector's bisector $\Omega=\pi$ (Figure 2)

$$\overline{V}_{\widehat{S}}(r^\lambda) = \frac{1}{2\pi}\int_0^\infty \frac{b^{\lambda+1}}{r+b}db = \frac{r^{\lambda+1}}{2\sin(\lambda\pi)}, \quad \text{where} \quad -2<\lambda<-1.$$

Additive constant is equal to zero in this expression therefore additive constant is equal to zero in the source expression (18): $C=0, C_1=0, C_2=0$. See footnotes (12) and the end of Section 2.

The additional conclusion of this result is justification of the supposition at the end of Section 2: the multiplier function in different ranges of λ has different forms as the coefficient of r^λ in the range $-2<\lambda<-1$ is doubled in comparison with the same coefficient in the range $0<\lambda<1$ (36). (Since the cited computer program shows the wrong ranges of λ, the author used [12, p. 360] for integration.)

(28) in which the two-dimensional harmonic function $r^\lambda U(\Omega)$ is already known, we have to find the coefficient κ only. [19] The β_1-mapping of expression (28) transforms the two-dimensional harmonic function $r^\lambda U(\Omega)$ into another two-dimensional harmonic function with same form and different value of λ, the function $\cos(\kappa\beta)$ not changes. Consequently, we can transform a pair of the sectors by the β_1-mapping into one sector in the plane Ox_1x_3 (3). After this mapping we have one sector with a double density function for the potential $\overline{W}$ when values of the density function are anti-symmetrical, consequently, the function $\cos(\kappa\beta)$ in (38) is equal to $\cos(\beta)$ as it is in the expression for one sector (35). By the analogy, the function $\cos(\kappa\beta)$ is equal to 1 in (39) when values of the density function are symmetrical due to (36).

Let us rewrite the functions in the form

$$\begin{aligned}\overline{u}_w^+(r,\theta,\beta) &= \psi_1(r,\theta)\cos(\kappa_1\beta),\\ \overline{u}_w^-(r,\theta,\beta) &= \psi_2(r,\theta)\cos(\kappa_2\beta),\\ \overline{u}_v^+(r,\theta,\beta) &= \psi_3(r,\theta)\cos(\kappa_3\beta),\\ \overline{u}_v^-(r,\theta,\beta) &= \psi_4(r,\theta)\cos(\kappa_4\beta),\end{aligned}$$

where $0<\lambda<1$, $-\pi+\alpha\le\theta\le\pi-\alpha$, $-\pi/2\le\beta<\pi/2$, the functions ψ_i are known, $\kappa_2=1$, $\kappa_3=0$, the coefficients κ_1, κ_4 are unknown. Let us find them.

♢ If the function ξ is created by the integral (β_0 is measured as β)

$$\begin{aligned}\xi(r,\theta,\beta) &= \int_{-\pi/2}^{\pi/2}\overline{u}_w^+(r,\theta,\beta+\beta_0)d\beta_0 = \int_{-\pi/2}^{\pi/2}\psi_1(r,\theta)\cos(\kappa_1(\beta+\beta_0))d\beta_0\\ &= \psi_1(r,\theta)\frac{1}{\kappa_1}\sin(\kappa_1(\beta+\beta_0))\Big|_{-\pi/2}^{\pi/2},\end{aligned} \tag{41}$$

it has to have the axial symmetry by Ox_1. This symmetry exists if $\kappa_1=0$, or $\kappa_1=2n, n\in N$. For $\alpha>\pi/2$, the function $\overline{u}_w^+$ in the ranges of the angles does not change its sign. This is possible for $\kappa_1=0$ only. The β - mapping does not change the value of κ_1. Hence $\kappa_1=0$. Analogously, we can find: $\kappa_3=0$. The functions $\overline{u}_w^-,\overline{u}_v^-$ have one plane of symmetry, therefore: $\kappa_2=1$, $\kappa_4=1$. ♢

Finally, we get (37), (38), (39), (40).

In expression (28) for $\kappa=0$ (37), (39) $\left.\dfrac{U(\Omega)}{\partial\Omega}\right|_{\theta=0}=0$, for $\kappa=1$ (38), (40) $U(\Omega)\Big|_{\theta=0}=0$, consequently, the Laplace equation (27) with a solution in the form (28) does not degenerate in the axis $\theta=0$, $\Omega=\pi$. [20]

The expressions (37), (38), (39), (40) represent the three-dimensional harmonic function for $\lambda\ne n, n\in N$. When $\lambda=n$, $n\in N$, the denominator of the expressions converts to zero, consequently, in this case the expressions exist if they are equal to zero only. We have proved the existence of three-dimensional harmonic functions (3), (4) for $\lambda\ne n$, $n\in N$. [21]

♢ The expression of the three-dimensional harmonic function (28) having only one plane of symmetry is equal in the plane to the two-dimensional harmonic function $r^\lambda U(\Omega)$ of the form (1), (2): $\kappa=1$; $\beta=0$ in the plane of symmetry. The derivative of expression (28) by r, having values of the derivative of (1), (2) by r in the plane of symmetry, does not change it's form when the value of λ changes, therefore the first summand in (27) does not change it's form, the second summand does not change its form because the derivative of the expression (28) by Ω, having values of derivative of (1),

[19] The first and second derivatives of (28) in the direction of the normal vector of the plane Ox_1x_2 are equal to zero because this plane is the plane of symmetry in the considered problems, therefore the function $r^\lambda U(\Omega)$ is two-dimensional harmonic function in expressions (28) of three-dimensional harmonic functions of the solutions of these problems.

[20] The points on the axis $\theta=\pi$ are not in the domain with the range of the angle θ: $-\pi+\alpha\le\theta\le\pi-\alpha$.

[21] The conformal mapping by a power function of two-dimensional harmonic functions involved in (37), (38), (39), (40) is carried out similarly to the two-dimensional case. See the end of Section 2.

(2) by Ω in the plane of symmetry, does not change its form. The third summand in (27) is equal to

$$\frac{1}{r^2\sin(\Omega)^2}\frac{\partial^2 u}{\partial\beta^2} = -\frac{\cos(\beta)}{r^2\sin(\Omega)^2}\left(r^\lambda U(\Omega)\right)$$

and does not change its form, as well. The conformal mapping by a power function with center at the point $r = 0$ maps one two-dimensional harmonic function (1), (2) into another with the same form and different value of λ. Consequently, from the existence of expression (3) for $\lambda \neq n$, $n \in N$, follows the existence of the three-dimensional harmonic function (3) for all values of λ, because (3) has one plane of symmetry with the relation of two-dimensional harmonic functions of the form $r^\lambda U(\Omega)$ in (28) by the conformal mapping in the plane of symmetry.

We can repeat the logic for the expression of three-dimensional harmonic function (28) having axial symmetry and find an analogous result: the existence of three-dimensional harmonic function (4) for all values of λ follows from the existence of expression (4) for $\lambda \neq n$, $n \in N$. [22] $\diamondsuit$

4. Solutions of Boundary Value Problems in the Vicinity of Angular or Conical Point as Mapping of Solutions for Domain with a Smooth Boundary

4.1. **Two-dimensional problems.** We compare below the integrals of (5) and the integrals of normal derivative of (5) with the same integrals on the boundary of a half plane. At the end of the subsection, these comparisons are used to determin the existence conditions of solutions.

Dirichlet problem

Let us consider the limiting expression (5) in a two-dimensional case. This was proved in Section 1 for $S \in C_1$, $u \in C_2(\Theta) \cap C_1(\overline{\Theta})$ under the condition of radiation. Consequently, this is still true for a regular function u. [23] A regular two-dimensional harmonic function u in the vicinity of $p \in S$ in the polar system of coordinates with center in p has to have expression by a sum of functions (1), (2), $\lambda = n$, $n \in N$, and a constant. If we consider the integrals (19), (20), (21), (22) in (5) in the vicinity of $p \in S$, we find that (5) may present regular u in p when the function u is the sum of functions (1), (2) and a constant. The corresponding integrals (19), (20), (21), (22) on the half plane's boundary converge for $\lambda = n$, $n \in N$, $\alpha = \dfrac{\pi}{2}$, $|\theta| = \dfrac{\pi}{2}$ for zero boundary values at the points of the boundary of the half-plane. [24] If we perform conformal mapping by a power function with center in p, the part S_R of the S inside the circle with small radius and center in p transforms in a part of the infinite wedge's boundary, the integrals (19), (20), (21), (22) transform in the expressions for $\lambda \neq n$, $n \in N$, for the infinite wedge.

Because the conformal mapping is not conformal at one point p [9], we have to consider this point separately. The functions (1), (2) have zero values in p which not change after the mapping. Therefore we have to consider the constant function only. At the point p, the mapping of the constant can be calculated by an extended determination of Gauss's theorem and is determined by the value of aperture angle of the wedge. [25] At all other points, the mapping of constant is itself the constant. Consequently, after the mapping we will have expression (5) in which $\delta \neq 1$ at the point $p \in S$.

Neumann problem

The existence of limiting values of the normal derivative of u (5) in $S \in C_1$, $u \in C_2(\Theta) \cap C_1(\overline{\Theta})$ under the condition of radiation has been proved in Section 1. This is still true for a regular function u. If we consider (5) for $p \in \Theta \backslash S$ and limiting expression of normal derivative of u at approach the point p to the point of the normal vector in S, we find that the corresponding integrals of the normal derivative of (19), (20), (21), (22) exist (integrals converge) in the boundary of the half-plane for $\lambda = n$, $n \in N$,

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[22] In this case the third summand in 27 is equal to zero.

[23] In the paper the term "regular function" is equivalent of "infinitely differentiable function".

[24] The corresponding integrals (19), (20), (21), (22) are equal to zero at $\alpha = \dfrac{\pi}{2}$, $|\theta| = \dfrac{\pi}{2}$ despite the denominator $\sin(\lambda\pi) = 0$.

[25] If $p \in S$ is angular or conical point in (8) the coefficient -1 will be replaced by $-\chi$, $\chi \neq 1$, $0 < \chi < 2$. In two-dimensional case χ is equal to the aperture angle of the wedge divided by π, in three-dimensional case χ is equal to a solid angle in the vertex of cone divided by 2π.

$\alpha = \dfrac{\pi}{2}$, $|\theta| = \dfrac{\pi}{2}$ for zero boundary values at the points of the boundary of the half-plane. It becomes obvious if we recall the relations of the derivatives

$$\left.\frac{\partial}{\partial n}\right|_{\theta=\pi/2} = \frac{1}{r}\left.\frac{\partial}{\partial\theta}\right|_{\theta=\pi/2}, \qquad \left.\frac{\partial}{\partial n}\right|_{\theta=-\pi/2} = -\frac{1}{r}\left.\frac{\partial}{\partial\theta}\right|_{\theta=-\pi/2},$$
$$\frac{\partial\sin(\theta)}{\partial\theta} = \cos(\theta), \qquad \frac{\partial\cos(\theta)}{\partial\theta} = -\sin(\theta), \tag{42}$$

owing to which in the expressions of normal derivatives of (19), (19), (20), (21), (22), ($\alpha = \pi/2$). [26] We can obtain conformal mapping by a power function with center at p of the limiting expression of the normal derivative of (5) for $\lambda \neq n$, $n \in N$, when S_R transforms in a part of the infinite wedge's boundary [27] and the expressions (19), (20), (21), (22) correspond to the integrals on infinite boundary of the infinite wedge. [28]

Dirichlet-Neumann mixed problem

Let the boundary S consist of two parts $S = S_u \cup S_t$, we have the Dirichlet boundary conditions on S_u and Neumann boundary conditions on S_t. Let the point p, $p \in S$, be a common point of S_u and S_t; S_R is part of S inside of the circle of small radius with center at p which we replace by a segment of straight line, since $S \in C_1$. If we suppose that the integrals (19), (20), (21), (22) correspond to the integrals of (5) on S_R we find that the solution of the mixed problem exists when the function $u \in L_2^{(1)}(\Theta)$ consists of the functions (1), (2) and has the expression

$$u(r,\theta) = Ar^{\lambda}\sin(\lambda\theta) - Ar^{\lambda}\cos(\lambda\theta), \tag{43}$$
$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \quad \lambda = \frac{4n+1}{2}, \quad n = 0, 1, 2, 3, dots,$$

where A is a constant; the part S_{Ru} of S_R belongs to S_u ($\theta = \dfrac{\pi}{2}$); the part S_{Rt} of S_R belongs to S_t ($\theta = -\dfrac{\pi}{2}$); $S_R = S_{Ru} \cup S_{Rt}$.

If we consider the integrals of (5) for (43), for points of S_{Rt} and values of the normal derivative of the integrals of (5) for the points of $S_{Ru}$with by using (42), we will find that values of the corresponding integrals of (19), (20), (21), (22) on the both parts of the boundary of the half-plane: on the part with the Dirichlet boundary conditions and on the part with Neumann boundary conditions, have to be equal to zero at the points of the boundary, only under these conditions there is a finite limiting value of normal derivative of u (5) at a point belonging to S_{Ru} for $0 < \lambda < 1$. [29]

Indeed, let the boundary of a half-plane consist of two rays with one vertex at the point p: the ray $\breve{S}_u$ for $\theta = \dfrac{\pi}{2}$ corresponds to the Dirichlet boundary conditions, the ray $\breve{S}_t$ for $\theta = -\dfrac{\pi}{2}$ corresponds to the Neumann boundary conditions. Let us consider the integrals in the right-hand side of (5) on the infinite boundary of the half-plane $W(u), V\left(\dfrac{\partial u}{\partial\Omega}\right)$ (43), at a point of observation belonging to the

---

[26] The corresponding integrals of norm,(21), (22) the $\cos(\theta)$ "swap" $\sin(\theta)$ and the $\sin(\theta)$ "swap" $\cos(\theta)$ in comparison with the initial expressions all derivative of (19), (20), (21), (22) are equal to zero for $\alpha = \dfrac{\pi}{2}$, $|\theta| = \dfrac{\pi}{2}$ despite the denominator $\sin(\lambda\pi) = 0$.

[27] The integrals (21), (22) of the constant density function create the functions of the form: cr, on the straight line $\widehat{S}$, $S_R \in \widehat{S}$, where c is the constant. These functions are equal to zero at p, $r = 0$, therefore this point has not to be considered separately under the conformal mapping.

[28] At the points of the wedge's boundary in two dimensions and at the points of wedge's boundary or cone's boundary in three dimensions there are expressions similar to (42) for $|\theta| \neq \dfrac{\pi}{2}$. Consequently, we do not need to consider the expressions of limiting values of the normal derivative at these points, since the boundary values of the derivative are determined by $\dfrac{\partial}{\partial\theta}$.

[29] The solution for the value $0 < \lambda < 1$ is most important in applications because it has singularity of the derivative $r^{\lambda-1}$ and belongs to $L_2^{(1)}(\Theta)$ (see footnote 8, [26, pp. 305,309]).

boundary of the half-plane:

$$\Big(- W_{\breve{S}_t}\big(- 2Ar^{\lambda}\sin(\lambda\pi/2)\big) + V_{\breve{S}_t}(0) - W_{\breve{S}_u}(0) + V_{\breve{S}_u}\big(2Ar^{\lambda-1}\lambda\sin(\lambda\pi/2)\big)\Big)\Big|_{\theta=-\frac{\pi}{2}} = 0, \tag{44}$$

$$\frac{\partial}{\partial n}\Big(- W_{\breve{S}_t}\big(- 2Ar^{\lambda}\sin(\lambda\pi/2)\big) + V_{\breve{S}_t}(0) - W_{\breve{S}_u}(0) + V_{\breve{S}_u}\big(2Ar^{\lambda-1}\lambda\sin(\lambda\pi/2)\big)\Big)\Big|_{\theta=\frac{\pi}{2}} = 0. \tag{45}$$

Equality (44) was obtained by (43) taking into account (42) and expressions (18) and (15); in expression (15), the sign corresponds to the approach to the boundary in direction of the normal vector (upper sign in (12), $\Omega = 2\pi$ in (15)). Equality (45) can be obtained by (44) taking into account (42).

From (45) follows the existence of limiting values of the normal derivative of (5) at the points of S_{Ru} for $0 < \lambda < 1$. Let us show that conditions (43) are exclusive ones for this existence and also existence of solution of the mixed problem. We can rewrite expression of (5) and expression of normal derivative of (5) in the form

$$u + W_{S_t}(u) - V_{S_u}\left(\frac{\partial u}{\partial n}\right) = -W_{S_u}(u) + V_{S_t}\left(\frac{\partial u}{\partial n}\right), \quad p \in S_t, \tag{46}$$

$$\frac{\partial u}{\partial n} + \frac{\partial}{\partial n}\big(W_{S_t}(u)\big) - \frac{\partial}{\partial n}\left(V_{S_u}\left(\frac{\partial u}{\partial n}\right)\right) = -\frac{\partial}{\partial n}\big(W_{S_u}(u)\big) + \frac{\partial}{\partial n}\left(V_{S_t}\left(\frac{\partial u}{\partial n}\right)\right), \quad p \in S_u. \tag{47}$$

We can obtain (47) as a limiting expression of the derivative in direction of normal vector n_{p1} of expression (5) at the point $p \in \Theta\backslash S$ ($\delta = 2$) as $p \to p_1$, $p_1 \in S(C_1)$, taking into account (9), (10). In the mixed problem, we know u on S_u and $\dfrac{\partial u}{\partial n}$ on S_t, thus (46), (47) is a system of resolving equations of the Dirichlet-Neumann mixed problem. In the resolving equation (47), at the points of S_{Ru}, $0 < \lambda < 1$, the divergent integral of u on S_{Rt} has to be "compensated" by the divergent integral of $\dfrac{\partial u}{\partial n_p}$ on S_{Ru}, for the existence of the equation, we can use only "half" of S_R for the integral with unknown density u, because another "half" is "migrated" in the right-hand side of equation (47). The integral of the second "half" of S_R with density $\dfrac{\partial u}{\partial n}$ has to "compensate" the integral of the first "half" with density u, thus the sum of the integrals may be finite in the resolving equation (47), this is possible for (45) only. Consequently, conditions (43) are exclusive for the existence of a solution of the mixed problem having singularity of derivative $r^{\lambda-1}$, $0 < \lambda < 1$. (The transition from the integrals on two rays (44), (45) to the integrals on S_R will be discussed below at the end of the subsection.) Any harmonic function (1), (2) $\lambda \neq n$, $n \in N$, has relation with a regular one through the conformal mapping by a power function, if we mean the harmonic function u has this relation as well we get (43).

Let us denote the boundary line of the half-plane as S_L and mean S_R as a part of the boundary of the half-plane inside of the circle of small radius with center in p, $p \in S_L$, coinciding with the part of S in (5) inside of the circle of small radius, with center at p, $p \in S$. As all three problems: the Dirichlet problem, the Neumann problem and the Dirichlet-Neumann mixed problem have the corresponding integrals (19), (20), (21), (22), having zero values at the points of the half- plane's boundary S_L, the conformal mapping of (5) by a power function with center in p, $p \in S$, corresponds to infinite wedge, having zero values at the points of its boundary.

Integrals (19), (20), (21), (22) on S_L under zero boundary conditions in all three problems have finite values at a point of the half=plane, where $r < \infty$. The parts of integrals (19), (20), (21), (22) on S_R have finite values at all points of the half-plane. [30] Consequently, the parts of the integrals (19), (20), (21), (22) on $S_L\backslash S_R$ have finite values in the vicinity of p, these parts of the integrals are harmonic functions which are infinitely differentiable at a point p_1, $p_1 \notin S_L\backslash S_R$. Consequently, the parts of integrals (19), (20), (21), (22) on S_R in the vicinity of p correspond to a solution of one of three problems for the half-plane under zero boundary conditions with addition of an infinitely differentiable function, after the mapping in the vicinity of p it corresponds to the solution for an

[30] We mean direct calculation of $V_{S_R}(r^{\lambda})$, $W_{S_R}(r^{\lambda})$.

infinite wedge under zero boundary conditions and addition of an infinitely differentiable function. The parts of integrals (5) on $S\backslash S_R$ before and after the mapping create in the vicinity of p an infinitely differentiable function, therefore the integrals of (5) after the mapping in the vicinity of p correspond to the solution for an infinite wedge for zero boundary conditions and addition of an infinitely differentiable function.

As a consequence, the described conditions are necessary for representation (5), $S \in C_1$, of a regular harmonic function near the considered point in the Dirichlet and Neumann problems. Since (5) is the representation of any harmonic function, satisfying the radiation condition, the described conditions are necessary for the existence of solutions of these two boundary problems. [31] In the Dirichlet-Neumann mixed problem the described conditions are necessary for the existence of the representation of limiting values on a part of the boundary, adjacent to a point of change of type of the boundary conditions for $u \in L_2^{(1)}(\Theta)$.

Since a domain with an angular point is connected by a conformal mapping by a power function with a domain with a smooth boundary, the existence conditions for the solutions of boundary problems for domain with an angular point are obtained through this mapping.

If the functions of solutions (5) of the Dirichlet and Neumann problems for the boundary S, $S \in C_1$, are regular or belong to $C_2(\Theta) \cap C_1(\overline{\Theta})$, the results of the mapping belong to $L_2^{(1)}(\Theta)$. The initial function of solution (5) of the Dirichlet-Neumann mixed problem for S, $S \in C_1$, and the result of the mapping together belong to $L_2^{(1)}(\Theta)$.

4.2. **Three-Dimensional Problems.** Since two-dimensional harmonic functions are special case of three-dimensional harmonic functions, all functions (1), (2), (3), (4) for $\lambda = n$, $n \in N$, and a constant are the forms of any regular three-dimensional harmonic function which has expression (5). [32] Because of relations (37), (38), (39), (40) with (19), (20), (21), (22), through the forms of two-dimensional harmonic function (1), (2), we can consider three-dimensional boundary value problems with using (5) and β-mappings as generalization of two-dimensional problems.

Let S_R be a part of S, $S \in C_1$, inside of the sphere of small radius and center at p, $p \in S$. As $S \in C_1$, we can replace S_R by the circle with center at p. If we repeat the reasoning of the previous subsection, we get: the potentials $\overline{W}_{S_R}(u)$, $\overline{V}_{S_R}\left(\dfrac{\partial u}{\partial n_p}\right)$ of (5) in the Dirichlet and Neumann problems correspond to solutions for the half-space under zero boundary conditions and an infinite differentiable function. The β_0-mapping of (5) for solutions of the Dirichlet and Neumann problems in the vicinity of p corresponds to solutions for an infinite wedge under zero boundary conditions and an infinitely differentiable function. The β_1-mapping of (5) for solutions of the Dirichlet and Neumann problems in the vicinity of p corresponds to solutions for an infinite cone under zero boundary conditions and

[31] Integrals (19), (20), (21), (22) of the solutions for a half-plane have density functions corresponding to the density functions of integrals (5) in p, $p \in S(C_1)$: in the Dirichlet and Neumann problems, the density functions correspond to the terms of Taylor's decomposition in p of regular u and regular $\dfrac{\partial u}{\partial n_p}$.

[32] If the density function in three dimensions $\breve{\varphi}$ has constant values in the direction of Ox_3-axis, there is the relation between potentials W and $\overline{W}$ at the point p located at the origin of the local system of coordinates $p(0,0,0)$:

$$\overline{W}(\breve{\varphi}) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\int\limits_{L}\frac{n_{1q}x_{1q}+n_{2q}x_{2q}}{(x_{1q}^2+x_{2q}^2+x_{3q}^2)^{\frac{3}{2}}}\,\breve{\varphi}(q)dLdx_{3q} = \frac{1}{2\pi}\int\limits_{L}\frac{(n_{1q}x_{1q}+n_{2q}x_{2q})x_{3q}}{(x_{1q}^2+x_{2q}^2+x_{3q}^2)^{\frac{1}{2}}(x_{1q}^2+x_{2q}^2)}\,\breve{\varphi}(q)dL_q\Bigg|_{x_{3q}=-\infty}^{x_{3q}=\infty}$$
$$= \frac{1}{\pi}\int\limits_{L}\frac{n_{1q}x_{1q}+n_{2q}x_{2q}}{x_{1q}^2+x_{2q}^2}\,\breve{\varphi}(q)dL_q = W(\breve{\varphi}),$$

where L is a two-dimensional closed line in the plane Ox_1x_2, n is the normal vector of L. There is the same relation between $\dfrac{\partial\overline{V}(\breve{\varphi})}{\partial n_p}$ and $\dfrac{\partial V(\breve{\varphi})}{\partial n_p}$.

The integral of the potential $\overline{V}$ of an infinite straight line does not converges, therefore the potential V is defined through the relation of $\dfrac{\partial\overline{V}(\breve{\varphi})}{\partial x_i}$ and $\dfrac{\partial V(\breve{\varphi})}{\partial x_i}$, $i = 1, 2$, [28, pp. 351-353]. Because these derivatives are equal, the potentials of infinite straight line may have difference in a constant only despite the integral $\overline{V}$ cannot perform the expression of the potential of the line since it is not converges.

an infinitely differentiable function. For the β_0-mapping, the initial regular harmonic function has to be of the form (1), (2), for β_1-mapping it has to be of the form (3), (4).

The conditions of the existence of a solution of the three-dimensional Dirichlet-Neumann mixed problem for the wedge coincide with those of the two-dimensional problem, and the β_0-mapping maps a wedge into the other one with a different aperture angle.

A solution of the three-dimensional mixed Dirichlet-Neumann problem for a cone does not exist, since the sum of nonzero functions (3), (4) having zero values of derivative by θ of the sum in a part of the cone's boundary with nonzero square does not exist.

Indeed, if we consider the equation analogous to the two-dimensional one (47) in three-dimensional case for $0 < \lambda < 1$, we find that the integrals analogous to (47) in the left-hand side must "compensate" each other on S_R, since the integrals diverge separately. Consequently, in the right-hand side the integrals must have zero densities on S_R, in other case, the "compensation" for two three-dimensional sectors (35), (36) at the boundary of a half-space in the left-hand side does not occur analogously to (44), (45) for two rays (15 (18). The sum of nonzero functions (3), (4), equal to zero on S_{R_u}, having zero values of the normal derivative on S_{R_t}, does not exist (42), $S_R = S_{R_u} \cup S_{R_t}$, hence there is no resolving equation in the three-dimensional case for a cone for $0 < \lambda < 1$ analogous to (47). Since (3), (4) for $0 < \lambda < 1$ have relation through the β_1-mapping with (3), (4) for $\lambda \neq n$, $n \in N$, the resolving equation for a cone does not exist for these values of λ. The values $\lambda = n$, $n \in N$, correspond to solutions of the Dirichlet or Neumann problem in the boundary of the half-space, the "compensation" of sectors in the Dirichlet-Neumann mixed problem is impossible. Finally, we get: the resolving equation in the three-dimensional case for a cone, analogous to (47), does not exist.

The rest reasoning of representation of solutions of the Dirichlet, Neumann problems and the Dirichlet-Neumann mixed problem by (5) in three dimensions is analogous to the reasoning for two dimensions.

If three-dimensional functions of solutions (5) of the Dirichlet and Neumann problems for the boundary S, $S \in C_1$, are regular or belong to $C_2(\Theta) \cap C_1(\overline{\Theta})$, the results of the mapping belong to $L_2^{(1)}(\Theta)$. The initial function of solution (5) of the Dirichlet-Neumann mixed problem for S, $S \in C_1$, and the result of the β_0-mapping together belong to $L_2^{(1)}(\Theta)$.

5. Some Effects of the Model of an Ideal Incompressible Fluid

Expression (5) is true for any regular three-dimensional harmonic function under the condition of radiation. As is stated above, a regular three-dimensional harmonic function is a sum of functions (1), (2), (3), (4) and a constant, all these forms of three-dimensional harmonic functions include the two-dimensional harmonic functions (1), (2) as multiplier. The two-dimensional harmonic functions (1), (2) for $\lambda = n$, $n \in N$, have relations with the functions of the same forms for $\lambda \neq n$, $n \in N$, through the conformal mapping, consequently, three-dimensional functions (1), (2), (3), (4) for $\lambda = n$, $n \in N$, have relations with the functions of the same forms for $\lambda \neq n$, $n \in N$, through the β_0-mapping or β_1-mapping. Therefore expression (5) of a regular three-dimensional harmonic function has to have relation with nonregular three-dimensional harmonic function through the β_0-mapping or β_1-mapping. [33] If the local geometry of S in (5) in the vicinity of $p \in S$ is not a wedge or a cone, expression (5) cannot be the expression of nonregular in p three-dimensional harmonic function, thus the geometry of the S admits a regular at p solution of a boundary value problem only. [34]

Let us consider the surface (4). Since the local geometry of the surface (4) at a point belonging to $]DE[$ is not a wedge or a cone, a solution of the boundary value problem in the vicinity of the point is a regular function when the surface (Figure 4) is part of S in (5). The situation is the same in the vicinity of P when surface (5) is a part of S in (5). Consequently, in the framework of the model of an ideal incompressible fluid the sharp edges of surfaces (4), (5) are the alternative of rounded edges. Moreover, if the regular three-dimensional harmolic function u is the function of velocity in some direction which is a derivative in this direction of another regular harmonic function of the potential

[33] The conformal mapping by a power function at the point $r = 0$, is not conformal. See footnote (25).

[34] Despite the fact that the solution of the Dirichlet-Neumann mixed problem (5), $S \in C_1$, is a nonregular function, the β_0-mapping can transform it into a regular function which has no physical sense.

of the potential field, the function u in the vicinity of the points of $]DE[$ (4) and in the vicinity of the point P (5) satisfies the conditions: $u = 0$, $\dfrac{\partial u}{\partial n} = 0$. Indeed, a regular three-dimensional harmonic function u is a sum of functions (1), (2), (3), (4) for $\lambda = n$, $n \in N$, and constant, equal to zero, [35] has expression (5) in which the potentials $\overline{V}_{S_R}\left(\dfrac{\partial u}{\partial n}\right)$, $\overline{W}_{S_R}(u)$ may create a regular harmonic function at these points, if only $S_R \in C_1$, where S_R is a part of S inside of the sphere with of small radius, with center in the considered point. Because at any of the considered points $S_R \notin C_1$ (Figure 4),(Figure 5) and the potentials $\overline{V}_{S\setminus S_R}\left(\dfrac{\partial u}{\partial n}\right)$, $\overline{W}_{S\setminus S_R}(u)$ of (5) create a regular harmonic function at the point, the regularity of u (5) at the point can exist if $u(p) = 0\Big|_{p\in S_R}$, $\dfrac{\partial u(p)}{\partial n} = 0\Big|_{p\in S_R}$ only. [36]

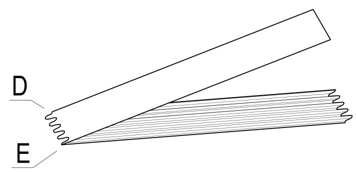


FIGURE 4. Surface with a sharp edge $]DE[$.

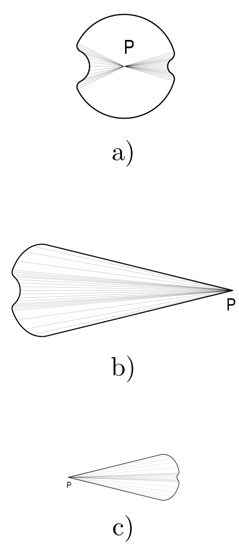


FIGURE 5. Three viewings of surface (a, b, c) with a sharp edge at the point P.

In the ideal fluid model, velocity of the potential flow is a gradient of harmonic function [13]. Despite the fact that this model is the simplest one, it has applications in numerical calculations of

[35]The constant is equal to zero because the function u is a derivative of another regular harmonic function of potential of the potential field which has the same form. The constant differentiation disappears.

[36]We provide regularity of u at the considered point only. The function u, $u \neq 0$, cannot be regular at all poins of $\Theta \cup S$ (5), because in this case the logic can be repeated for all next derivatives of potential of the potential field, since the expression (5) exists for each of them. This is possible if the potential of the potential field is a constant in the local domain of the Θ near the sharp edge, consequently in this case the potential of the potential field is the constant in the whole domain Θ. The boundary conditions for the existence of solutions of the boundary value problems in $L_2^{(1)}(\Theta)$ are discussed in the next section.

the airplane wing [1]. The expressions of harmonic functions by potentials of simple and double layers are used there [28, pp. 146,175]. The problem of impermeability of the wing by the potential flow is the external Neumann problem.

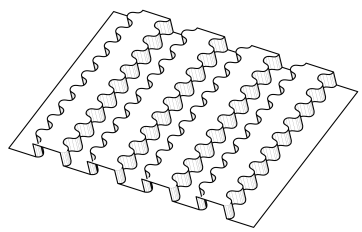


FIGURE 6. Example of a surface, in the vicinity of which the initially appearing vortex in a potential flow is delayed or prevented in the framework of an ideal fluid model.

Let us consider the surface (Figure 6). If it is a part of the boundary of a body in a gas flow or liquid, the surface is in some sense better, than a smooth surface. The velocity in the vicinity of sharp edges is a regular function in the framework of the model, the potentials of expression (5) of the velocity have a zero density function there. Therefore this surface behaves as a body with voids, as a lattice, the breaks of which are on the sharp edges, because in sense of the model the potentials with a zero density function are equivalent to the "absence of boundary" there.

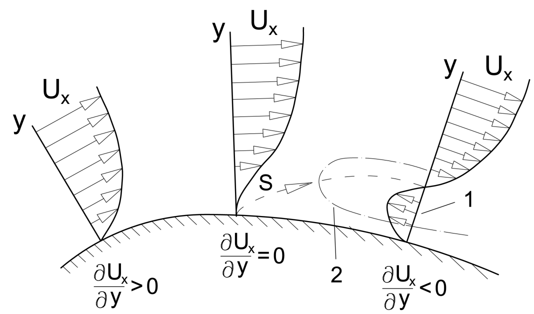


FIGURE 7. Schema of vortex emerging. 1 - zone of backward flow, 2 = vortex, U - velocity, S is a point of vortex formation.

These voids act against the powers, which are created by the initial vortex (Figure 7) [11]. The point S of maximum of the harmonic function of potential of the potential field, in which velocity U_x changes the sign (Figure 7), may be in the boundary of the domain only because of the "principle of maximum", but it cannot be near the voids (Figures 6,8). [37] [38]

Evidently, because of the regularity of solutions in the vicinity of the sharp edges of this type (Figures 4,5), they can be the alternative for rounded edges. It should be pointed out that the region

[37]The flow ceases to be a potential one from the moment when the point S "moves away" from the boundary (Figure 7). In theory, a surface with spaced arrays of sharp edges of this type serves in a potential flow for delaying or preventing the formation of initial vortex.

[38]Likely, special applications described in the patents: US5540406, US20090304511, US8256846, US5171623, US2261558, US5378524, US5289997, US8141936, US4776535 are based on this effect.

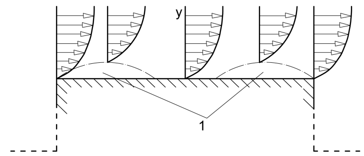
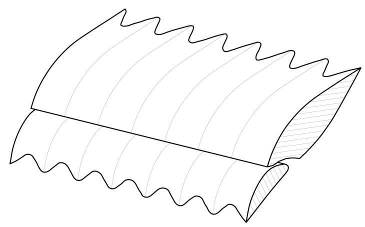
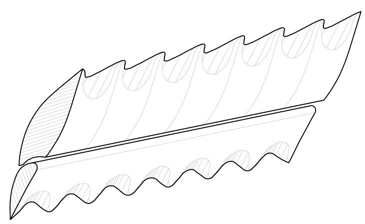


FIGURE 8. Schema of supposed distribution of velocity U_x in a cross-section of a part of the surface (Figure 6) 1 - zone of the zero velocities.

of resistance is smaller, than that of a rounded edge, therefore theoretically the resistance of flow can be decreased in the framework of the model. Let us consider few examples.



a)



b)

FIGURE 9. Prototype an airplane wing (a, b).

The prototype of an airplane wing with sharp edges of type (Figure 4) is shown in (Figure 9). [39] As it has been shown above, in the vicinity of sharp edges of this wing the formation of initial vortex in potential flow is impossible in the framework of the ideal fluid model. Theoretically, the resistance of the flow can be decreased in comparison with rounded edges.

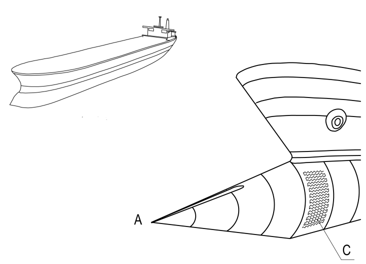


FIGURE 10. Prototype of bulbous bow for a bulk carrier.

[39] In trailing edge of the wing of this form, the Chaplygin-Zhukovskii hypothesis (Kutta condition) is true because of the described effect.

Possible application of the surface (Figure 5) at point A for a leading sharp edge of bulbous bow for a bulk carrier is shown in (Figure 10). Theoretically, the resistance of flow can be decreased in comparison with a rounded edge without appearance of vortexes. In the region marked by letter C, where formation of initial vortex is possible because of the pressure drop, the surface (Figure 6) can be used to prevent or delay the occurrence of this vortex.

6. Boundary Conditions on Piecewise Smooth Boundary for Solutions of Boundary Value Problems in $L_2^{(1)}$

Let $\widetilde{S}$ be a closed piecewise smooth boundary of a simply-connected domain $\widetilde{\Theta}$ in the form of a union of surfaces S_i, $i = 1, \ldots, N$, $S_i \in C_1$. It follows from the above discussion that the form of harmonic function $L_2^{(1)}(\widetilde{\Theta})$ for a solution of the Dirichlet and Neumann problems in polar coordinates in a two-dimensional case and in spherical coordinates in a three-dimensional case at any point of the piecewise smooth boundary $\widetilde{S}$ is determined by representations (1), (2) and (3), (4), respectively, and the existence of conformal mapping by a power function or β-mapping to regular function for which representation (5) exists. For the mixed problem, the form of this function is determined by the existence of the mapping to the solution function of the problem with a smooth boundary at the considered point and the corresponding representation (5).

In the two-dimensional case boundary values of the solution in the vicinity of any point of $\widetilde{S}$ has to be the values of the sum of the functions (1), (2), $\lambda > 0$, and a constant. These functions (1), (2) are the solution of the boundary value problem for an infinite wedge with aperture angle τ under zero boundary conditions, where the angle τ is the internal angle under which the tangent lines to $\widetilde{S}$ intersect at the considered point of $\widetilde{S}$. On a smooth part of the $\widetilde{S}$ $\tau = \pi$. (See 4.1.)

In the three-dimensional case, boundary values of the solution in the vicinity of any point of $\widetilde{S}$ has to be the values of representation by one of the four following variants.

1) The sum of functions of the form (1), (2), $\lambda > 0$, and a constant if a part of the surface $\widetilde{S}$ inside of the sphere of small radius and center at the considered point tends to the wedge surface whose wedge includes the considered point when the radius tends to zero. The functions (1), (2) are the solution of the boundary-value problem for an infinite wedge for zero boundary conditions. (This corresponds to the two-dimensional case.)

2) The sum of functions of the form (3), (4), $\lambda > 0$, and a constant if a part of the surface $\widetilde{S}$ inside of the sphere of small radius and with center at the considered point tends to the surface of the cone whose vertex is at the considered point; any cone generatrix is tangent to $\widetilde{S}$ at the considered point. The functions (3), (4) correspond to the solution for infinite cone at zero boundary conditions. (See 4.2.)

3) The sum of the functions (1), (2), (3), (4), $\lambda > 0$, and a constant if the considered point is located on the smooth part of $\widetilde{S}$. The functions (1), (2) correspond to the solution for infinite wedge with aperture angle π at zero boundary conditions. The functions (3), (4) correspond to the solution for an infinite cone with aperture angle π at zero boundary conditions.

4) A constant if the conditions of each of the variants 1-3 are not fulfilled at the considered point of $\widetilde{S}$. [40] (Examples of the points of variant 4) are the points of the line DE (4) and the point P (5). Another example is the point of vertex of a pyramid.) [41]

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[40] If the boundary $\widetilde{S}$ includes the points of variant 4), footnote 36 has to be taken into account. When the solution function is infinitely differentiable at all points of $\widetilde{\Theta} \cup \widetilde{S}$, each of its derivatives is a harmonic function having expression (5). Consequently, all derivatives of the solution in direction of the normal vector to $\widetilde{S}$ in vicinity of a point of the variant 4) are equal to zero, all tangents to $\widetilde{S}$ derivatives are equal to zero there, as well. Therefore the solution function is a constant in the vicinity of the considered point, because the first term of an infinite Taylor's decomposition of the function is not equal to zero only. Thus this solution function is a constant in the whole $\widetilde{\Theta}$.

[41] In all four variants the factor δ in expression (5) is equal to the value of the solid angle at the considered point of $\widetilde{S}$ divided by 2π.

For convergence of results of the algorithm of Boundary element method, when the total number of the boundary elements increases at the decreasing of characteristic size of elements, it suffices to make approximation of functions (1), (2), (3), (4), $0 < \lambda < 2$, and a constant at all points of $\widetilde{S}$. Functions (1), (2), (3), (4) for $0 < \lambda < 1$ may have singularity of derivative only, if this feature is approximated, the numerical results are refined, when the characteristic size of the elements decreases.

Statements of this section can be repeated if domain $\widetilde{\Theta}$ is a complement of some simply connected domain with respect to the plane in a two-dimensional case or to the space in a three-dimensional case and the radiation condition is satisfied.

7. Conclusion

We have obtained expressions (3), (4) of harmonic functions in the three-dimensional case. These representations of harmonic functions in three dimensions could be used as an alternative to the well-known Legendre functions and can be applied to various fields of mathematics and technology with the benefit of a more simple form.

The expressions of the summands with possibly infinite derivative in the solutions of the Dirichlet, Neumann and Dirichlet-Neumann mixed problems in $L_2^{(1)}$ in the vicinity of sharp edges of piecewise smooth boundary by potentials of simple or double layer are proposed. These allow us to exclude the key shortcomings of traditional formulation of the Method of potential (traditional form of the Boundary element method). These expressions will allow us to formulate the method for piecewise smooth boundary and mixed boundary conditions. It opens up opportunities for simulation applications of the Laplace equation and can be generalized to solve other equations, solutions of which have representation in the form of a combination of harmonic functions, for example, for the equations of the theory of elasticity.

The suggested technique will bring about simplicity of numerical algorithms and proximity to the analytical methods which would increase the calculation accuracy. This factor presents a definite benefit over other methods and makes it highly competitive.

The found parallel between solutions for smooth and piecewise smooth boundaries in the vicinity of angular or conical points eliminates the key shortcomings of traditional BEM. This relationship is maintained through the conformal mapping in the two-dimensional case or β-mapping in the three-dimension case. The Taylor series of presentations of density functions of potentials in the expression of solution of the Dirichlet or Neumann problem by a sum of potentials of simple and double layers on a smooth boundary are transformed after the mapping into the series of density functions, the potentials of which present the series of Kondrat'ev solutions of these problems in the vicinity of angular or conical point.

Some types of sharp edges allow only regular solutions, and the suggested technique shows a possible alternative to rounding. This alternative may be used to improve the efficiency of technical embodiment in many areas.

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Further reading

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(Received 19.07.2018)

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