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SOME COINCIDENCE AND COMMON FIXED POINT RESULTS IN CONE METRIC SPACES OVER BANACH ALGEBRAS VIA WEAK g-φ-CONTRACTIONS

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Abstract. Recently, B. Li and H. Huang [20] introduced the notion of weak φ-contractions on cone metric spaces over Banach algebras. The purpose of this paper is to generalize the main result of Li and Huang [20] by proving some coincidence and common fixed point results in cone metric spaces over Banach algebras via weak g-φ-contractions for weakly compatible mappings. Some examples are presented which verify and illustrate the results proved herein.

1. Introduction

The notion of metric spaces is generalized by several authors in various directions. One such generalization of metric spaces is a cone metric space given by L.G. Huang and X. Zhang [12]. In usual metric spaces, the metric function d is defined from $X \times X$ in the real number system, where X is a nonempty set. When generalizing the metric spaces to a cone metric space, the metric function d is defined from the product $X \times X$ into a Banach space (instead of the real number system). Thus, in cone metric spaces the distance is a vector belongs to the Banach space. Huang and Zhang [12] defined the cone metric spaces and proved some fixed point theorems for various types of contractive mappings and generalized the famous Banach contraction principle [2] in various ways. An example of Huang and Zhang [12] shows that there may be mappings which are contractive in a cone metric space, but fails to be a contraction with usual metric, i.e., the contractive conditions in cone metric spaces are more general, than those in usual metric spaces.

Common fixed point theorems have applications, e.g., in establishing the existence of a common solution for a class of functional equations arising in dynamic programming, in establishing the existence of solution of system of nonlinear integral equations, in establishing the existence of a solution for an implicit integral equation, etc. (see, e.g., [24, 13, 30] and the references therein). G. Jungck [15], introduced a common fixed point theorem for two commutating mappings in such a way that if we take one of them as identity mapping, then we obtain the Banach contraction principle. Although, Jungck's theorem generalizes the Banach contraction principle, but has a drawback that the involved mappings commutate. S. Sessa [28] introduced the notion of weakly commuting mappings and weakened the commutativity of mappings. Further, Jungck [16] introduced the notion of compatible mappings which generalizes the concept of a weak commutativity and showed that weakly commuting maps are compatible but the converse is not true. R.P. Pant [23] introduced R-weakly commuting mappings for proving the common fixed point results. Latter, in [17, 18], Jungck defined weakly compatible mappings which need not to be commuting, but commutes only at their coincidence points. M. Abbas and G. Junck [1] initiated the study of common fixed point theorems in cone metric spaces and proved some common fixed point results for two weakly compatible mappings in cone metric spaces.

Although, some recent papers (see, e.g., [4, 6, 7, 19]) show that the fixed point results in cone metric spaces can be directly derived from their corresponding usual metric versions. To overcome this drawback, Liu and Xu [21] introduced the concept of a cone metric space over the Banach algebra, and proved some fixed point results in such spaces. In the fixed point results of Liu and Xu [21] the scalar control constants of contractive mappings were replaced by a vector control constant. They

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discussed the benefit of taking a vector instead of a scaler as a contractive constant and showed that the fixed point results in this new setting cannot be derived from their usual metric versions. Several authors generalized the results of results obtained by Liu and Xu [21] (see, e.g., [11, 22, 29], etc.). In [3, 26, 27], I.A. Rus and V. Berinde introduced the notion of φ-contraction and also generalized the Banach contraction principle in usual metric spaces. Inspired with these papers, recently, B. Li and H. Huang [20] introduced the notion of weak φ-contractions on cone metric spaces over the Banach algebras and generalized the results of H. Liu and S. Xu [21]. Li and Huang [20] used a vector-valued function as a control function in contractive conditions. They proved some fixed point results for weak φ-contractions and showed that these results on cone metric spaces over the Banach algebra can be applied in finding the solution of elementary system of equations and integral equation.

Inspired by Li and Huang [20], in this paper, we prove some coincidence and common fixed point results in cone metric spaces over the Banach algebras via weak g-φ-contractions for weakly compatible mappings. Our results generalize and unify the results of Huang and Zhang [12], Abbas and Jungck [1] and Li and Huang [20] in cone metric spaces over the Banach algebras. Some example are provided which illustrate the new results.

2. Preliminaries

In this section, we state some known definitions and facts which will be used throughout the paper.

Let $\mathcal{A}$ be a Banach algebra with a unit e and a zero element θ. A nonempty closed subset P of $\mathcal{A}$ is called a cone if the following conditions hold:

(1) $\{\theta, e\} \subset P$;
(2) $\forall \alpha, \beta \in [0, \infty) \Rightarrow \alpha P + \beta P \subseteq P$;
(3) $P^2 = PP \subset P$;
(4) $P \cap (-P) = \{\theta\}$.

A cone P is called a solid cone if $P^\circ \neq \emptyset$, where P° stands for the interior of P.

We can always define a partial ordering $\preceq$ with respect to P by $x \preceq y$, if and only if $y - x \in P$. We shall write $x \ll y$ to indicate that $y - x \in P^\circ$. We shall also write $\|\cdot\|$ as the norm on $\mathcal{A}$. A cone P is called normal if there is a number $K > 0$ such that for all $x, y \in \mathcal{A}$, $\theta \preceq x \preceq y$ implies $\|x\| \leq K\|y\|$.

We always suppose that $\mathcal{A}$ is a Banach algebra with a unit e, P is a solid cone in $\mathcal{A}$, and $\preceq, \ll$ are partial orderings with respect to P.

Definition 2.1 ([26])**.** A function $\varphi : [0, \infty) \to [0, \infty)$ is called a comparison if it satisfies the following two conditions:

(1) φ is monotone nondecreasing, i.e., $0 \leq t_1 \leq t_2 \Rightarrow \varphi(t_1) \leq \varphi(t_2)$;
(2) $\{\varphi^n(t)\}$ $(t > 0)$ converges to 0 as $n \to \infty$.

It is obvious that $\varphi(t) < t$ for each $t > 0$, $\varphi(0) = 0$ and $\lim_{t\to 0} \varphi(t) = 0$.

Definition 2.2 ([26])**.** Let (X, d) be a metric space. A mapping $T : X \to X$ is called a φ-contraction if there exists a comparison $\varphi : [0, \infty) \to [0, \infty)$ such that

$$d(Tx, Ty) \leq \varphi(d(x, y)), \quad \text{for all} \quad x, y \in X.$$

Rus [26] proved the following fixed point theorem for φ-contractions and generalized the Banach contraction principle.

Theorem 2.3 ([26])**.** *Let (X, d) be a metric space and $T : X \to X$ be a φ-contraction. Then T has a unique fixed point in X. Moreover, for any $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point.*

Definition 2.4 ([21])**.** Let X be a nonempty set and $\mathcal{A}$ be a Banach algebra. A mapping $d : X \times X \to \mathcal{A}$ is called a cone metric if it satisfies:

(i) $\theta \preceq d(x, y)$ for all $x, y \in X$, $d(x, y) = \theta \Leftrightarrow x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y \in X$.

In this case, the pair (X, d) is called a cone metric space over the Banach algebra $\mathcal{A}$.

Definition 2.5 ([5]). A sequence $\{u_n\}$ in a Banach algebra $\mathcal{A}$ is said to be a c-sequence if for each $c \gg \theta$, there exists $N \in \mathbb{N}$ such that $u_n \ll c$, for all $n > N$.

Definition 2.6 ([10]). Let (X, d) be a cone metric space over a Banach algebra $\mathcal{A}$ and $\{x_n\}$ be a sequence in X. We say that

(i) $\{x_n\}$ converges to $x \in X$ if $\{d(x_n, x)\}$ is a c-sequence and in this case we write $x_n \to x$ as $n \to \infty$;
(ii) $\{x_n\}$ is a Cauchy sequence if $\{d(x_n, x_m)\}$ is a c-sequence for n, m;
(iii) (X, d) is complete if every Cauchy sequence in X is convergent.

It is obvious that the limit of a convergent sequence in a cone metric space (X, d) over a Banach algebra $\mathcal{A}$ is unique.

Lemma 2.7 ([14]). *Let $\mathcal{A}$ be a Banach algebra and $u, v, w \in \mathcal{A}$. Then*
(1) *$u \ll w$ if $u \preceq v \ll w$ or $u \ll v \preceq w$;*
(2) *$u = \theta$ if $\theta \preceq u \ll c$ for each $c \gg \theta$.*

Lemma 2.8 ([25]). *Let $\mathcal{A}$ be a Banach algebra with its unit e. Then the spectral radius of $u \in \mathcal{A}$ equals to $\rho(u) = \lim\limits_{n\to\infty} \|u^n\|^{\frac{1}{n}}$.*

Lemma 2.9 ([9]). *Let P be a cone in a Banach algebra $\mathcal{A}$, $\{u_n\}$ and $\{v_n\}$ be two c-sequences in $\mathcal{A}$, and $\alpha, \beta \in P$ be vectors, then $\{\alpha u_n + \beta v_n\}$ is a c-sequence in $\mathcal{A}$.*

Lemma 2.10 ([8]). *Let P be a cone and $k \in P$ with $\rho(k) < 1$. Then $\{k^n\}$ is a c-sequence.*

Definition 2.11 ([20]). Let $\mathcal{A}$ be a Banach algebra and P be a cone in $\mathcal{A}$. A mapping $\varphi : P \to P$ is called a weak comparison if the following conditions hold:
(WC1) φ is nondecreasing with respect to $\preceq$, namely, $t_1, t_2 \in P$, $t_1 \preceq t_2 \Rightarrow \varphi(t_1) \preceq \varphi(t_2)$;
(WC2) $\{\varphi^n(t)\}$ $(t \in P)$ is a c-sequence in P;
(WC3) if $\{u_n\}$ is a c-sequence in P, then $\{\varphi(u_n)\}$ is also a c-sequence in P.

Remark 2.12 ([20]). By Definition 2.11, we have $\varphi(\theta) = \theta$. Indeed, by (i) of Definition 2.11, we have $\theta \preceq \varphi(\theta) \preceq \varphi^n(\theta)$. Since $\{\varphi^n(\theta)\}$ is a c-sequence, by Lemma 2.7, it may be verified that $\varphi(\theta) = \theta$.

If $\mathcal{A} = \mathbb{R}$ and $P = [0, \infty)$, then the above definition is reduced to the Definition 2.1.

Example 2.13 ([20]). Let $\mathcal{A}$ be a Banach algebra, P be a cone in $\mathcal{A}$, and $k \in P$. Take $\varphi(t) = kt$ $(t \in P)$, where $\rho(k) < 1$. Then by Lemma 2.9 and Lemma 2.10, φ is a weak comparison.

Example 2.14 ([20]). Let M be a compact set of $\mathbb{R}^n$ and $\mathcal{A} = C(M)$, where $C(M)$ denotes the set of all continuous functions on M. Let $P = \{u \in \mathcal{A} : u(t) \geq 0, t \in M\}$ and define a mapping $\varphi : P \to P$ by $\varphi(u) = \frac{u}{u+1}$. Then φ is a weak comparison.

Definition 2.15 ([20]). Let (X, d) be a cone metric space over a Banach algebra $\mathcal{A}$. Let P be a cone and $\varphi : P \to P$ be a weak comparison. Then a mapping $T : X \to X$ is called a weak φ-contraction if

$$d(Tx, Ty) \preceq \varphi(d(x, y)), \quad \text{for all} \quad x, y \in X. \tag{1}$$

Clearly, the above definition generalizes Definition 2.2. The following theorem is the main result of [20].

Theorem 2.16 ([20]). *Let (X, d) be a complete cone metric space over a Banach algebra, and $T : X \to X$ be a weak φ-contraction. Then T has a unique fixed point in X. Moreover, for any $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point.*

Definition 2.17 ([1]). Let X be a nonempty set and $f, g \colon X \to X$ be two mappings. A point $z \in X$ is called the coincidence point of the pair (f, g) if $fz = gz$. A point $w \in X$ is called the point of coincidence of the pair (f, g) if $fz = gz = w$ for some $z \in X$. The pair (f, g) is called weakly compatible if f and g commute at each coincidence point.

3. Main Results

In this section, we introduce weak g-φ-contractions in the framework of a cone metric space over Banach algebra and obtain some coincidence and common fixed point theorems.

Definition 3.1. Let (X, d) be a cone metric space over a Banach algebra $\mathcal{A}$, P a cone, $\varphi : P \to P$ be a weak comparison and $g\colon X \to X$ be a mapping. Then a mapping $f : X \to X$ is called a weak g-φ-contraction if the following condition

$$d(fx, fy) \preceq \varphi\big(d(gx, gy)\big), \quad \text{for all} \ \ x, y \in X. \tag{2}$$

is satisfied.

Remark 3.2. Take $g = I_X$, the identity mapping of X, a weak g-φ-contraction reduces into weak φ-contraction. Therefore, a weak g-φ-contraction is a generalization of a weak φ-contraction. On the other hand, if we take $\varphi(t) = k$, where $k \in P$ and $\rho(k) < 1$, then we get an improved version of contraction mappings of Abbas and Jungck [1].

In the following theorem, we obtain a coincidence point result for two mappings satisfying the weak g-φ-contractivity condition.

Theorem 3.3. *Let (X, d) be a cone metric space over a Banach algebra $\mathcal{A}$, and $g : X \to X$ be a mapping. Suppose $f\colon X \to X$ is a weak g-φ-contraction and $f(X) \subseteq g(X)$. If $g(X)$ or $f(X)$ is a complete subspace of X, then $f$and g have a point of coincidence.*

Proof. Let $x_0 \in X$ be arbitrary, then $fx_0 \in f(X)$ and since $f(X) \subseteq g(X)$, there exists $x_1 \in X$ such that $gx_1 = fx_0 = y_1$ (say). Again, since $fx_1 \in f(X)$ and $f(X) \subseteq g(X)$, there exists $x_2 \in X$ such that $gx_2 = fx_1 = y_2$ (say). epeating this process, we obtain the so-called Jungck sequence defined by $\{y_n\} = \{gx_n\} = \{fx_{n-1}\}$.

If $c \in P^\circ$, then by (WC2), the sequence $\{\varphi^n(c)\}$ is a c-sequence, and so, there exists $n_0 \in \mathbb{N}$ such that

$$\varphi^{n_0}(c) \ll c. \tag{3}$$

Now, since f is a weak g-φ-contraction, we obtain

$$d(y_n, y_{n+n_0}) = d\big(fx_{n-1}, fx_{n+n_0-1}\big) \preceq \varphi\big(d(gx_{n-1}, gx_{n+n_0-1})\big) = \varphi\big(d(y_{n-1}, y_{n+n_0-1})\big). \tag{4}$$

Using (WC1), we obtain from the above inequality that

$$\varphi\left(d(y_n, y_{n+n_0})\right) \preceq \varphi\left(\varphi\left(d(y_{n-1}, y_{n+n_0-1})\right)\right) = \varphi^2\big(d(y_{n-1}, y_{n+n_0-1})\big).$$

Replacing n by $n-1$, the above inequality gives $\varphi\big(d(y_{n-1}, y_{n-1+n_0})\big) \preceq \varphi^2\big(d(y_{n-2}, y_{n+n_0-2})\big)$. On using this inequality in (4), we obtain $d(y_n, y_{n+n_0}) \preceq \varphi^2\left(d(y_{n-2}, y_{n+n_0-2})\right)$. Repetition of these arguments yields the following inequality:

$$d(y_n, y_{n+n_0}) \preceq \varphi^n\left(d(y_0, y_{n_0})\right). \tag{5}$$

By (WC2), we have $\{\varphi^n\left(d(y_0, y_{n_0})\right)\}$ is a c-sequence, and so, by Lemma 2.7 and (5) the sequence $\{d(y_n, y_{n+n_0})\}$ is also a c-sequence. Therefore, there exists $n_1 \in \mathbb{N}$ such that

$$d(y_n, y_{n+n_0}) \ll c - \varphi^{n_0}(c), \quad \text{for all} \quad n \geq n_1. \tag{6}$$

We shall show that

$$d(y_n, y_{n+kn_0}) \ll c \ \ \text{for all} \ \ k \in \mathbb{N}, n \geq n_1. \tag{7}$$

To this end, we use mathematical induction.

By (6), the result is true for $k = 1$. Suppose $d(y_n, y_{n+jn_0}) \ll c$, for all $n \geq n_1$ is the induction hypothesis. Then we have

$$\begin{aligned} d\left(y_n, y_{n+(j+1)n_0}\right) &\preceq d\left(y_n, y_{n+n_0}\right) + d\left(y_{n+n_0}, y_{n+(j+1)n_0}\right) \\ &\ll c - \varphi^{n_0}(c) + d\left(fx_{n+n_0-1}, fx_{n+(j+1)n_0-1}\right) \\ &\preceq c - \varphi^{n_0}(c) + \varphi\left(d(gx_{n+n_0-1}, gx_{n+(j+1)n_0-1})\right) \\ &= c - \varphi^{n_0}(c) + \varphi\left(d(y_{n+n_0-1}, y_{n+(j+1)n_0-1})\right). \end{aligned}$$

Repeating the process, we obtain

$$d\left(y_n, y_{n+(j+1)n_0}\right) \ll c - \varphi^{n_0}(c) + \varphi^{n_0}\left(d(y_n, y_{n+jn_0})\right) \preceq c - \varphi^{n_0}(c) + \varphi^{n_0}(c) = c.$$

This completes the inductive proof of (7).

Again, since f is a weak g-φ contraction, one can obtain easily that

$$d(y_n, y_{n+1}) \preceq \varphi^n(d(y_0, y_1)), \text{ for all } n \in \mathbb{N}.$$

Therefore

$$\begin{aligned} &d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{n+n_0-1}, y_{n+n_0}) \\ \preceq &\varphi^n(d(y_0, y_1)) + \varphi^n(d(y_1, y_2)) + \cdots + \varphi^n(d(y_{n_0-1}, y_{n_0})). \end{aligned}$$

By (WC2), each of the sequences $\{\varphi^n(d(y_i, y_{i+1}))\}$, where $i = 0, 1, \ldots, n_0 - 1$ is a c-sequence, therefore $\{\varphi^n(d(y_0, y_1)) + \varphi^n(d(y_1, y_2)) + \cdots + \varphi^n(d(y_{n_0-1}, y_{n_0}))\}$ is also a c-sequence. Hence, by Lemma 2.7, the sequence $\{d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{n+n_0-1}, y_{n+n_0})\}$ is also a c-sequence. Thus, for the above $c \gg \theta$, there exists $n_2 \in \mathbb{N}$ such that

$$d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{n+n_0-1}, y_{n+n_0}) \ll c, \text{ for all } n > n_2.$$

Choose $n_3 = \max\{n_1, n_2\}$ and for $m, n > n_3$, let

$$k_m = \left[\frac{m - n_3}{n_0}\right], \ k_n = \left[\frac{n - n_3}{n_0}\right],$$

where $[\cdot]$ stands for the integer part. Then we have

$$n_3 \le m - k_m n_0 < n_3 + n_0, \ n_3 \le n - k_n n_0 < n_3 + n_0.$$

Therefore, for all $n, m \ge n_3$, we have

$$d(y_m, y_n) \preceq d(y_m, y_{m-k_m n_0}) + d(y_{m-k_m n_0}, y_{n-k_n n_0}) + d(y_{n-k_n n_0}, y_n) \ll 3c.$$

Thus, the sequence $\{y_n\} = \{gx_n\} = \{fx_{n-1}\}$ is a Cauchy sequence.

Suppose that $g(X)$ is complete (the proof for the case where $f(X)$ is complete, is same). Then by the completeness of $g(X)$, there exists $x^* \in X$ such that $y_n \to gx^* = y^*$ (say) as $n \to \infty$.

We shall show that y^* is a point of coincidence of f and g. Then since f is a weak g-φ contraction, we obtain

$$\begin{aligned} d(y^*, fx^*) &\preceq d(y^*, y_{n+1}) + d(y_{n+1}, fx^*) = d(y^*, y_{n+1}) + d(fx_n, fx^*) \\ &\preceq d(y^*, y_{n+1}) + \varphi(d(gx_n, gx^*)) = d(y^*, y_{n+1}) + \varphi(d(y_n, y^*)). \end{aligned}$$

Since $y_n \to gx^* = y^*$ as $n \to \infty$, the sequences $\{d(y^*, y_{n+1})\}$ and $\{d(y_n, y^*)\}$ are c-sequences, and so, by (WC3) and Lemma 2.7, there exists $n_4 \in \mathbb{N}$ such that

$$d(y^*, fx^*) \ll c, \quad \text{for all } \ c \in P^\circ, n \ge n_4.$$

Therefore, by Lemma 2.7, $d(y^*, fx^*) = \theta$, i.e., $fx^* = gx^* = y^*$. It shows that x^* is a coincidence point of f and g, and y^* is the corresponding point of coincidence of f and g. □

In the next theorem, we provide a sufficient condition for the existence of a common fixed point of f and g.

Theorem 3.4. *Suppose that all the conditions of Theorem* 3.3 *are satisfied. In addition, suppose that the mappings f and g are weakly compatible, then f and g have a unique common fixed point.*

Proof. The existence of a coincidence point x^* and the corresponding point of coincident y^* follows from Theorem 3.3. Thus, $fx^* = gx^* = y^*$. By weak compatibility of f and g, we have

$$fy^* = fgx^* = gfx^* = gy^*. \tag{8}$$

Since f is a weak g-φ-contraction, we have

$$d(fy^*, y^*) = d(ffx^*, fx^*) \preceq \varphi(d(gfx^*, gx^*)) = \varphi(d(fy^*, y^*))$$

Repetition of this process yields

$$d\left(fy^*, y^*\right) \preceq \varphi^n\left(d\left(fy^*, y^*\right)\right), \text{ for all } n \in \mathbb{N}.$$

By (WC2), the sequence $\{\varphi^n\left(d\left(fy^*, y^*\right)\right)\}$ is a c-sequence, therefore, it follows from Lemma 2.7 and the above inequality that there exists $n_0 \in \mathbb{N}$ such that

$$d\left(fy^*, y^*\right) \ll c, \text{ for all } c \in P^\circ, n \geq n_0.$$

The above inequality with Lemma 2.7 yields that $d\left(fy^*, y^*\right) = \theta$, i.e., $fy^* = y^*$. Now, by (8), we obtain $fy^* = gy^* = y^*$. Thus, y^* is a common fixed point of f and g.

For the uniqueness of y^*, suppose that z^* is another common fixed point of f and g and $y^* \neq z^*$. Then, since f is a weak g-φ-contraction, we have

$$d\left(y^*, z^*\right) = d\left(fy^*, fz^*\right) \preceq \varphi\left(d\left(gy^*, gz^*\right)\right) = \varphi\left(d\left(y^*, z^*\right)\right).$$

Repetition of this process yields

$$d\left(y^*, z^*\right) \preceq \varphi^n\left(d\left(y^*, z^*\right)\right), \text{ for all } n \in \mathbb{N}.$$

Again, repeating the same arguments as above, we get $d\left(y^*, z^*\right) = \theta$, i.e., $y^* = z^*$. This contradiction shows that the common fixed point of f and g is unique. □

In the above theorem, for the existence of a common fixed point of f and g we apply an additional condition on f and g, namely, the condition of a weak compatibility. The following is an interesting example which shows that Theorem 3.3 ensures the existence of a point of coincidence of f and g, but not the existence of common fixed point of f and g, and so, the condition of a weak compatibility in Theorem 3.4 is not superfluous.

Example 3.5. Let $X = \mathbb{R}$, $A = \mathbb{R}^2$ with the norm $\|(x_1, x_2)\| = |x_1| + |x_2|$ and product defined by $(x_1, x_2)(y_1, y_2) = (x_1y_1, x_1y_2 + x_2y_1)$ for all $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$ and the unit $e = (1, 0)$. Define $d\colon X \times X \to \mathcal{A}$ by

$$d(x, y) = \left(|x - y|, \alpha|x - y|\right), \text{ for all } x, y \in X$$

where $\alpha > 0$ is a fixed real number. Then (X, d) is a cone metric space over a Banach algebra $\mathcal{A}$.

Define two mappings $f, g\colon X \to X$ by $fx = 1$ for all $x \in X$ and $gx = 1$ if $x \neq 1$ and $g1 = 2$. Then, note that f is a constant mapping, therefore, $d\left(fx, fy\right) = \theta$ for all $x, y \in X$. Thus, f is a weak g-φ-contraction. All the conditions of Theorem 3.3 are satisfied, and so, we can conclude the existence of a point of coincidence of f and g. Indeed, all the points of X, except 1, are the coincidence points of f and g and 1 is the corresponding point of coincidence of f and g. On the other hand, one can see that f and g have no common fixed point. Note that the mappings f and g are not weakly compatible. For instance, every $x \neq 1$ is a coincidence point of f and g, but

$$fgx = f1 = 1 \neq gfx = g1 = 2.$$

Therefore, f and g do not commute at their coincidence point, and so, are not weakly compatible.

The following corollary is an improvement of one of the main results of Abbas and Jungck [1] (Theorem 2.1 of Abbas and Jungck [1]) and is a special case of Corollary 2.10 of [8].

Corollary 3.6. *Let (X, d) be a cone metric space over a Banach algebra $\mathcal{A}$. Suppose that the mappings $f, g\colon X \to X$ satisfy*

$$d(fx, fy) \preceq kd(gx, gy), \quad \text{for all } \text{x}, \text{y} \in \text{X}$$

where $k \in P$ is such that $\rho(k) < 1$. If the range of g contains the range of f and $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, f and g have a unique common fixed point.

Proof. Taking $\varphi(t) = kt$ in Theorem 3.4, we obtain the required result. □

The following corollary is the main result of Li and Huang [20].

Corollary 3.7. *Let (X, d) be a complete cone metric space over a Banach algebra, and $f : X \to X$ be a weak φ-contraction. Then f has a unique fixed point in X. Moreover, for any $x \in X$, the iterative sequence $\{f^n x\}$ converges to the fixed point.*

Proof. Taking $g = I_X$ in Theorem 3.4, we obtain the existence and uniqueness of a fixed point of f. Further, for $g = I_X$, the Jungck sequence $\{y_n\}$ reduces into the iterative sequence $\{f^n x\}$ which converges to the fixed point of f. □

Corollary 3.7 uses the completeness of space X. In the next theorem, we omit the completeness of X by applying a different condition associated with f and the metric d.

Theorem 3.8. *Let (X, d) be a cone metric space over a Banach algebra, and $f : X \to X$ be a weak φ-contraction. Suppose, there exists $x^* \in X$ such that $d(x^*, fx^*) \preceq d(x, fx)$, for all $x \in X$. Then f has a unique fixed point in X and x^* is the fixed point of f.*

Proof. Denote $D(x) = d(x, fx)$ for all $x \in X$. Then by the assumption, we have

$$D(x^*) \preceq D(x), \quad \text{for all} \ \ x \in X. \tag{9}$$

Since f is a weak φ-contraction, we have

$$D(fx^*) = d(fx^*, ffx^*) \preceq \varphi(d(x^*, fx^*)) = \varphi(D(x^*)).$$

Using (WC1) and (9) and the fact that $f\colon X \to X$ we obtain $\varphi(D(x^*)) \preceq \varphi(D(fx^*))$. Therefore, it follows from the above inequality and Lemma 2.7 that $D(fx^*) \preceq \varphi(D(fx^*))$. Repetition of this process yields

$$D(fx^*) \preceq \varphi^n(D(fx^*)), \text{ for all } n \in \mathbb{N}.$$

Again, using (WC2) and Lemma 2.7, we obtain $D(fx^*) = \theta$, which together with (9) yields $D(x^*) = d(x^*, fx^*) = \theta$ i.e., $fx^* = x^*$. Therefore, x^* is the fixed point of f. The uniqueness of the fixed point follows from the weak φ-contractivity of f and (WC2). □

Example 3.9. Let M be a compact set of $\mathbb{R}^n$ and $\mathcal{A} = C(M)$, where $C(M)$ denotes the set of all continuous functions on M. Let $P = \{u(t) \in \mathcal{A} : u(t) \geq 0, t \in M\}$ and define a mapping $\varphi : P \to P$ by $\varphi(u) = \frac{u}{u+1}$. Then φ is a weak comparison. Let $X = \{u(t) \in \mathcal{A}\colon 0 \leq u(t) \in \mathbb{Q} \text{ for all } t \in M\}$ and define a mapping $d : X \times X \to \mathcal{A}$ by

$$d(u(t), v(t)) = |u(t) - v(t)|, \ t \in M.$$

Then (X, d) is a cone metric space over a Banach algebra. Define a mapping $T : X \to X$ by

$$Tu = \frac{u}{u+1} \text{ for all } u \in X.$$

Clearly, T is a weak φ-contraction. Indeed, if $u, v \in X$, we have

$$d(Tu, Tv) = \left|\frac{u}{u+1} - \frac{v}{v+1}\right| \preceq \frac{|v-u|}{|v-u|+1} = \frac{d(u,v)}{d(u,v)+1} = \varphi(d(x,y)).$$

It is easy to see that (X, d) is not a complete cone metric space, therefore, we cannot apply the results of Li and Huang [20]. On the other hand, there exists a point $x^*(t) = 0(t) = 0 \in X$ such that $d(x^*, Tx^*) \preceq d(u, Tu)$, for all $u \in X$. Therefore, all the conditions of Theorem 3.8 are satisfied and we can conclude the existence and uniqueness of fixed point of T by Theorem 3.8. Indeed, $x^*(t) = 0(t) = 0 \in X$ is the unique fixed point of T.

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