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LOCAL VARIATION FORMULAS OF SOLUTIONS FOR THE NONLINEAR CONTROLLED DIFFERENTIAL EQUATION WITH THE DISCONTINUOUS INITIAL CONDITION AND WITH DELAY IN THE PHASE COORDINATES AND CONTROLS

M. IORDANISHVILI

Abstract. In the present work, local variation formulas of solutions are given, in which the effects of the discontinuous initial condition and perturbations of delays containing in the phase coordinates and controls are revealed.

1. Introduction

As is known, the real processes contain information about their behavior in the past and are described by the delay differential equation [3], [6], [4]. Linear representation of the main part of the increment of a solution with respect to perturbations of the initial data of a differential equation is called the variation formula of a solution (variation formula). In this paper, the essential novelty is that here the local variation formula is given when there occur simultaneously perturbations of the initial moment and delays both in the phase coordinates and in controls.

The term "variation formula of solution" has been introduced by R. V. Gamkrelidze and proved in [2] for the ordinary differential equation. The effects of perturbation of the initial moment and the discontinuous initial condition in the variation formulas were for the first time revealed by T. A. Tadumadze in [8] for the delay differential equation.

The variation formula plays a basic role in proving the necessary conditions of optimality [2], [5], [9] and in the sensitivity analysis of mathematical models [6] . Moreover, the variation formula allows one to construct an approximate solution of the perturbed equation.

In the present work, for the controlled delay differential equation

$$\dot{x}(t) = f(t, x(t), x(t-\tau_1), \dots, x(t-\tau_s), u(t), u(t-\theta_1), \dots, u(t-\theta_k))$$

with the discontinuous initial condition the variation formulas are given. The discontinuity of the initial condition means that the values of the initial function and the trajectory, in general, do not coincide at the initial moment.

The variation formulas for various classes of controlled delay differential equations without perturbations of delay occurring in controls are derived in [1], [7], [11], [10], [12].

2. Formulation of the Main Results

Let $I = [a, b]$ be a finite interval and $0 < h_{i1} < h_{i2}, i = \overline{1, s}$; let $0 < q_{i1} < q_{i2}$, $i = \overline{1, k}$ be the given numbers; suppose that $O \subset \mathbb{R}^n$ and $U_0 \subset \mathbb{R}^r$ are open sets. Let the n-dimensional function $f(t, x, x_1, \dots, x_s, u, u_1, \dots, u_k)$, $(t, x, x_1, \dots, x_s, u, u_1, \dots, u_k) \in I \times O^{1+s} \times U_0^{1+k}$ satisfy the following conditions:

a) for almost all fixed $t \in I$, the function $f(t, x, x_1, \dots, x_s, u, u_1, \dots, u_k)$ is continuously differentiable with respect to $(x, x_1, \dots, x_s, u, u_1, \dots, u_k) \in O^{1+s} \times U_0^{1+k}$;

b) for each fixed $(x, x_1, \dots, x_s, u, u_1, \dots, u_k) \in O^{1+s} \times U_0^{1+k}$, the functions

$$f(t, x, x_1, \dots, x_s, u, u_1, \dots, u_s), f_x(t, \cdot), f_{x_i}(t, \cdot), \quad i = \overline{1, s}$$

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and

$$f_u(t,\cdot), f_{u_i}(t,\cdot), \quad i=\overline{1,k}$$

are measurable on I;

c) for any compacts $K \subset O$ and $U \subset U_0$, there exists a function $m_{K,U}(t) \in L_1(I,[0,\infty))$ such that for any $(x,x_1,\ldots,x_s,u,u_1,\ldots,u_k) \in K^{1+s} \times U^{1+k}$ and for almost all $t \in I$, we have

$$|f(t,x,x_1,\ldots,x_s,u,u_1,\ldots,u_k)| + |f_x(t,\cdot)| + \sum_{i=1}^{s} |f_{x_i}(t,\cdot)|$$

$$+|f_u(t,\cdot)| + \sum_{i=1}^{k} |f_{u_i}(t,\cdot)| \le m_{K,U}(t).$$

Let Φ and Ω be sets of continuously differentiable functions $\varphi : I_1 = [\tau,b] \to O$ and $u : [\theta,b] \to U_0$, respectively, where $\tau = a - \max\{h_{12},\ldots,h_{s2}\}$ and $\theta = a - \max\{q_{12},\ldots,q_{k2}\}$.

To each element

$$\mu = (t_0,\tau_1,\ldots,\tau_s,\theta_1,\ldots,\theta_k,x_0,\varphi,u) \in \Lambda = [a,b) \times [h_{11},h_{12}] \times \cdots \times [h_{s1},h_{s2}]$$
$$\times [q_{11},q_{12}] \times \cdots \times [q_{k1},q_{k2}] \times O \times \Phi \times \Omega$$

we assign the controlled delay differential equation

$$\dot{x}(t) = f(t,x(t),x(t-\tau_1),\ldots,x(t-\tau_s),u(t),u(t-\theta_1),\ldots,u(t-\theta_k)) \tag{1}$$

with the discontinuous initial condition

$$x(t) = \varphi(t), \quad t \in [\tau,t_0), \quad x(t_0) = x_0. \tag{2}$$

Condition (2) is called discontinuous because, in general, $x(t_0) \ne \varphi(t_0)$.

Definition 1. Let $\mu = (t_0,\tau_1,\ldots,\tau_s,\theta_1,\ldots,\theta_k,x_0,\varphi,u) \in \Lambda$. A function $x(t) = x(t;\mu) \in O$, $t \in [\tau,t_1]$, $t_1 \in (t_0,b]$ is called a solution of equation (1) with the initial condition (2) or a solution corresponding to the element μ and defined on the interval $[\tau,t_1]$ if it satisfies condition (2) and is absolutely continuous on the interval $[t_0,t_1]$ and satisfies equation (1) almost everywhere on $[t_0,t_1]$.

Let $\mu_0 = (t_{00},\tau_{10},\ldots,\tau_{s0},\theta_{10},\ldots,\theta_{k0},x_{00},\varphi_0,u_0) \in \Lambda$ be a fixed element and let $x_0(t)$ be a solution corresponding to the element μ_0 and defined on the interval $[\tau,t_{10}]$, where

$$t_{00},t_{10} \in (a,b), \quad t_{00} < t_{10}; \quad \tau_{i0} \in (h_{i1},h_{i2}), i=\overline{1,s}; \quad \theta_{i0} \in (q_{i1},q_{i2}), \quad i=\overline{1,k}.$$

Thus, $x_0(t)$ is the solution of the equation

$$\dot{x}(t) = f(t,x(t),x(t-\tau_{10}),\ldots,x(t-\tau_{s0}),u_0(t),u_0(t-\theta_{10}),\ldots,u_0(t-\theta_{k0})), \quad t \in [t_{00},t_{10}]$$

with the initial condition

$$x(t) = \varphi_0(t), \quad t \in [\tau,t_{00}), \quad x(t_{00}) = x_{00}.$$

Let us introduce the set of variations:

$$V = \Big\{\delta\mu = (\delta t_0,\delta\tau_1,\ldots,\delta\tau_s,\delta\theta_1,\ldots,\delta\theta_k,\delta x_0,\delta\varphi,\delta u) : \delta t_0 \in (a,b) - t_{00},$$
$$\delta\tau_i \in (h_{i1},h_{i2}) - \tau_{i0}, \quad i=\overline{1,s}; \quad \delta\theta_i \in (q_{i1},q_{i2}) - \theta_{i0}, \quad i=\overline{1,k};$$
$$\delta x_0 \in O - x_{00}, |\delta\tau_i| \le \alpha, \quad i=\overline{1,s}; \quad |\delta\theta_i| \le \alpha, \quad i=\overline{1,k}, \quad |\delta x_0| \le \alpha,$$
$$\delta\varphi = \sum_{i=1}^{m} \lambda_i\delta\varphi_i, \quad \delta u = \sum_{i=1}^{m} \lambda_i\delta u_i, \quad \delta\varphi_i \in \Phi - \varphi_0, \quad \delta u_i \in \Omega - u_0, \quad |\lambda_i| \le \alpha, \quad i=\overline{1,m}\Big\},$$

where $(a,b) - t_{00} := \{\delta t_0 = t_0 - t_{00} : t_0 \in (a,b)\}$ and $\alpha > 0$ is a fixed number.

There exist the numbers $\delta_1 > 0$ and $\varepsilon_1 > 0$ such that for arbitrary $(\varepsilon,\delta\mu) \in (0,\varepsilon_1) \times V$, we have $\mu_0 + \varepsilon\delta\mu \in \Lambda$, and a solution $x(t;\mu_0+\varepsilon\delta\mu)$ defined on the interval $[\tau,t_{10}+\delta_1] \subset I_1$ corresponds to it (see [9, Theorem 1.4, p. 17]).

We note that $x(t;\mu_0+\varepsilon\delta\mu)$ is the solution of the perturbed equation

$$\dot{x}(t) = f(t,x(t),x(t-\tau_{10}-\varepsilon\delta\tau_1),\ldots,x(t-\tau_{s0}-\varepsilon\delta\tau_s),u_0(t)+\varepsilon\delta u(t),u_0(t-\theta_{10}-\varepsilon\delta\theta_1)$$

$$+\varepsilon\delta u(t-\theta_{10}-\varepsilon\delta\theta_1),\dots,u_0(t-\theta_{k0}-\varepsilon\delta\theta_k)+\varepsilon\delta u(t-\theta_{k0}-\varepsilon\delta\theta_k))$$

with the perturbed initial condition

$$x(t)=\varphi_0(t)+\varepsilon\delta\varphi(t), t\in[\tau,t_{00}+\varepsilon\delta t_0), x(t_{00}+\varepsilon\delta t_0)=x_{00}+\varepsilon\delta x_0.$$

Due to the uniqueness, the solution $x(t;\mu_0)$ is a continuation of the solution $x_0(t)$ on the interval $[\tau,t_{10}+\delta_1]\subset I_1$. Therefore, in the sequel, the solution $x_0(t)$ is assumed to be defined on the interval $[\tau,t_{10}+\delta_1]$.

For arbitrary $(t,\varepsilon,\delta\mu)\in[\tau,t_{10}+\delta_1]\times(0,\varepsilon_1)\times V$ we define the increment of the solution $x_0(t)=x(t;\mu_0)$:

$$\Delta x(t;\varepsilon\delta\mu)=x(t;\mu_0+\varepsilon\delta\mu)-x_0(t).$$

Theorem 1. *Let the following conditions hold:*

1) $\tau_{s0}>\dots>\tau_{10}$ and $t_{00}+\tau_{s0}<t_{10}$;

2) *the function* $f(w,u,u_1,\dots,u_k)$, *where* $w=(t,x,x_1,\dots,x_s)$, *is bounded on* $I\times O^{1+s}\times U_0^{1+k}$;

3) *there exists the finite limit*

$$\lim_{w\to w_0} f_0(w)=f_0^-, w\in(a,t_{00}]\times O^{1+s},$$

where $f_0(w)=f(w,u_0(t),u_0(t-\theta_{10}),\dots,u_0(t-\theta_{k0}))$, $w_0=(t_{00},x_{00},\varphi_0(t_{00}-\tau_{10}),\dots,\varphi_0(t_{00}-\tau_{s0}))$;

4) *there exist the finite limits*

$$\lim_{(w_{1i},w_{2i})\to(w_{1i}^0,w_{2i}^0)}[f_0(w_{1i})-f_0(w_{2i})]=f_i,\quad i=\overline{1,s},$$

where $w_{1i},w_{2i}\in(a,b)\times O^{1+s}, i=\overline{1,s}$,

$$w_{1i}^0=\Big(t_{00}+\tau_{i0},x_0(t_{00}+\tau_{i0}),x_0(t_{00}+\tau_{i0}-\tau_{10}),\dots,x_0(t_{00}+\tau_{i0}-\tau_{i-10}),$$
$$x_{00},x_0(t_{00}+\tau_{i0}-\tau_{i+10}),\dots,x_0(t_{00}+\tau_{i0}-\tau_{s0})\Big),$$
$$w_{2i}^0=\Big(t_{00}+\tau_{i0},x_0(t_{00}+\tau_{i0}),x_0(t_{00}+\tau_{i0}-\tau_{10}),\dots,x_0(t_{00}+\tau_{i0}-\tau_{i-10}),$$
$$\varphi_0(t_{00}),x_0(t_{00}+\tau_{i0}-\tau_{i+10}),\dots,x_0(t_{00}+\tau_{i0}-\tau_{s0})\Big);$$

Then there exist the numbers $\varepsilon_2\in(0,\varepsilon_1)$ *and* $\delta_2\in(0,\delta_1)$, *with* $t_{10}-\delta_2>t_{00}+\tau_{s0}$ *such that for arbitrary*

$$(t,\varepsilon,\delta\mu)\in[t_{10}-\delta_2,t_{10}+\delta_2]\times(0,\varepsilon_2)\times V^-,$$

where $V^-=\{\delta\mu\in V:\delta t_0\le 0\}$, *we have*

$$\Delta x(t;\varepsilon\delta\mu)=\varepsilon\delta x_-(t;\delta\mu)+o(t;\varepsilon\delta\mu), \tag{3}$$

where $\delta x_-(t;\delta\mu)$ *has the form*

$$\delta x_-(t;\delta\mu)=-Y(t_{00};t)f_0^-\delta t_0+\beta(t;\delta\mu), \tag{4}$$

$$\beta(t;\delta\mu)=Y(t_{00};t)\delta x_0-\Big[\sum_{i=1}^{s}Y(t_{00}+\tau_{i0};t)f_i\Big]\delta t_0-\sum_{i=1}^{s}\Big[Y(t_{00}+\tau_{i0};t)f_i$$
$$+\int_{t_{00}}^{t}Y(\xi;t)f_{0x_i}[\xi]\Big(\chi_i(\xi)\dot\varphi_0(\xi-\tau_{i0})+(1-\chi_i(\xi))\dot x_0(\xi-\tau_{i0})\Big)d\xi\Big]\delta\tau_i$$
$$+\sum_{i=1}^{s}\int_{t_{00}-\tau_{i0}}^{t_{00}}Y(\xi+\tau_{i0};t)f_{0x_i}[\xi]\delta\varphi(\xi)d\xi-\sum_{i=1}^{k}\Big[\int_{t_{00}}^{t}Y(\xi;t)f_{0u_i}[\xi]\dot u_0(\xi-\theta_{i0})d\xi\Big]\delta\theta_i$$
$$+\int_{t_{00}}^{t}Y(\xi;t)\Big[f_{0u}[\xi]\delta u(\xi)+\sum_{i=1}^{k}f_{0u_i}[\xi]\delta u(\xi-\theta_{i0})\Big]d\xi, \tag{5}$$

where $\chi_i(\xi)$ is the characteristic function of the interval $[t_{00}, t_{00}+\tau_{i0}]$; furthermore, $Y(s;t)$ is the $n\times n$-matrix function satisfying the equation

$$Y_\xi(\xi;t) = -Y(\xi;t)f_{0x}[\xi] - \sum_{i=1}^{s} Y(\xi+\tau_{i0};t)f_{0x_i}[\xi+\tau_{i0}], \xi\in[t_{00},t]$$

and the condition

$$Y(\xi;t) = \begin{cases} H, & \text{for} \quad \xi = t, \\ \Theta, & \text{for} \quad \xi > t. \end{cases} \tag{6}$$

Here, $f_{0x}[t] = f_{0x}(t, x_0(t), x_0(t-\tau_{10}), \dots, x_0(t-\tau_{s0}))$, H is the identity matrix and Θ is the zero matrix;

$$\lim_{\varepsilon\to 0}\frac{o(t;\varepsilon\delta\mu)}{\varepsilon} = 0 \quad \text{uniformly for} \quad (t,\delta\mu)\in[t_{10}-\delta_2, t_{10}+\delta_2]\times V^-.$$

Some comments. Theorem 1 corresponds to the case where variation at the point t_{00} is performed on the left. The function $\delta x(t;\delta\mu)$ is called the first variation of the solution $x_0(t)$, $t\in[t_{10}-\delta_2, t_{10}+\delta_2]$ and expression (4) is called the local variation formula.

The expression

$$-\Big[Y(t_{00};t)f_0^- + \sum_{i=1}^{s} Y(t_{00}+\tau_{i0};t)f_i\Big]\delta t_0$$

in formula (4) is the effect of the discontinuous initial condition (2) and perturbation of the initial moment t_{00}.

The addend

$$-\sum_{i=1}^{s}\Big[Y(t_{00}+\tau_{i0};t)f_i + \int_{t_{00}}^{t} Y(\xi;t)f_{0x_i}[\xi][\chi_i(\xi)\dot\varphi_0(\xi-\tau_{i0}) + (1-\chi_i(\xi))\dot x_0(\xi-\tau_{i0})d\xi\Big]\delta\tau_i$$

in formula (4) is the effect of perturbations of the delays τ_{i0}, $i=\overline{1,s}$.

The expression

$$-\sum_{i=1}^{k}\Big[\int_{t_{00}}^{t} Y(\xi;t)f_{0u_i}[\xi]\dot u_0(\xi-\theta_{i0})d\xi\Big]\delta\theta_i$$

in formula (4) is the effect of perturbations of delays θ_{i0}, $i=\overline{1,k}$.

The expression

$$\sum_{i=1}^{s}\int_{t_{00}-\tau_{i0}}^{t_{00}} Y(\xi+\tau_{i0};t)f_{0x_i}[\xi]\delta\varphi(\xi)d\xi$$

in formula (4) is the effect of perturbation of the initial function φ_0.

The expression

$$\int_{t_{00}}^{t} Y(\xi;t)\Big[f_{0u}[\xi]\delta u(\xi) + \sum_{i=1}^{k} f_{0u_i}[\xi]\delta u(\xi-\theta_{i0})\Big]d\xi$$

in formula (4) is the effect of perturbation of the control function u_0.

It is clear that if $\varphi_0(t_{00}) = x_{00}$, then $f_i = 0$, $i=\overline{1,s}$.

It is easy to see that (see (4),(5))

$$\delta x_-(t;\delta\mu) = \delta x_-^{(0)}(t;\delta\mu) - \sum_{i=1}^{s}\delta x^{(i)}(t;\delta\mu), t\in[t_{10}-\delta_2, t_{10}+\delta_2],$$

where

$$\delta x_-^{(0)}(t;\delta\mu) = Y(t_{00};t)\Big[\delta x_0 - f_0^-\delta t_0\Big] + \sum_{i=1}^{s}\int_{t_{00}-\tau_{i0}}^{t_{00}} Y(\xi+\tau_{i0};t)f_{0x_i}[\xi]\delta\varphi(\xi)d\xi$$

$$+\int\limits_{t_{00}}^{t} Y(\xi;t)\Big\{-\sum_{i=1}^{s}\Big[f_{0x_i}[\xi]\Big(\chi_i(\xi)\dot{\varphi}_0(\xi-\tau_{i0})+(1-\chi_i(\xi))\dot{x}_0(\xi-\tau_{i0})\Big)\Big]\delta\tau_i$$
$$-\sum_{i=1}^{k} f_{0u_i}[\xi]\dot{u}_0(\xi-\theta_{i0})\delta\theta_i+f_{0u}[\xi]\delta u(\xi)+\sum_{i=1}^{k} f_{0u_i}[\xi]\delta u(\xi-\theta_{i0})\Big\}d\xi$$

and

$$\delta x^{(i)}(t;\delta\mu)=-Y(t_{00}+\tau_{i0};t)f_i\Big(\delta t_0+\delta\tau_i\Big),\quad i=\overline{1,s}.$$

On the basis of the Cauchy formula (see [9, Lemma 2.3, p. 31], the function

$$\delta x_0(t)=\begin{cases}\delta\varphi(t), & t\in[\tau,t_{00}),\\ \delta x_-^{(0)}(t;\delta\mu), & t\in[t_{00},t_{10}]\end{cases}$$

is the solution of the equation in “variations”

$$\dot{\delta}x(t)=f_{0x}[t]\delta x(t)+\sum_{i=1}^{s} f_{0x_i}[t]\delta x(t-\tau_{i0})-\sum_{i=1}^{s}\Big[f_{0x_i}[t]\Big(\chi_i(t)\dot{\varphi}_0(t-\tau_{i0})$$
$$+(1-\chi_i(t))\dot{x}_0(t-\tau_{0i})\Big)\Big]\delta\tau_i-\sum_{i=1}^{k} f_{0u_i}[t]\dot{u}_0(t-\theta_{i0})\delta\theta_i+f_{0u}[t]\delta u(t)+\sum_{i=1}^{k} f_{0u_i}[s]\delta u(t-\theta_{i0})$$

with the discontinuous initial condition

$$\delta x(t)=\delta\varphi(t), t\in[\tau,t_{00}), \delta x(t_{00})=\delta x_0-f_0^-\delta t_0$$

and the function

$$\delta x_i(t)=\begin{cases}0, & t\in[\tau,t_{00}+\tau_{i0}),\\ \delta x^{(i)}(t;\delta\mu), & t\in[t_{00}+\tau_{i0},t_{10}]\end{cases}$$

is the solution of the equation in “variations”

$$\dot{\delta}x(t)=f_{0x}[t]\delta x(t)+\sum_{i=1}^{s} f_{0x_i}[t]\delta x(t-\tau_{i0})$$

with the discontinuous initial condition

$$\delta x(t)=0, t\in[\tau,t_{00}+\tau_{i0}), \delta x(t_{00}+\tau_{i0})=-f_i\Big(\delta t_0+\delta\tau_i\Big).$$

The variation formula allows us to obtain an approximate solution of the perturbed equation in the analytical form. In fact, for a small $\varepsilon>0$, from

$$x(t;\mu_0+\varepsilon\delta\mu)-x_0(t)=\Delta x(t;\varepsilon\delta\mu)=\varepsilon\delta x_-(t;\delta\mu)+o(t;\varepsilon\delta\mu)$$

(see (3)), it follows that

$$x(t;\mu_0+\varepsilon\delta\mu)\approx x_0(t)+\varepsilon\delta x_-(t;\delta\mu).$$

Theorem 2. *Let conditions* 1), 2) *and* 4) *of Theorem* 1 *hold. Moreover, there exists the finite limit*

$$\lim_{w\to w_0} f_0(w)=f_0^+, w\in[t_{00},b)\times O^{1+s}. \tag{7}$$

Then there exist the numbers $\varepsilon_2\in(0,\varepsilon_1)$ *and* $\delta_2\in(0,\delta_1)$, *with* $t_{10}-\delta_2>t_{00}+\tau_{s0}$ *such that for arbitrary*

$$(t,\varepsilon,\delta\mu)\in[t_{10}-\delta_2,t_{10}+\delta_2]\times(0,\varepsilon_2)\times V^+,$$

where $V^+=\{\delta\mu\in V:\delta t_0\geq 0\}$, *we have*

$$\Delta x(t;\varepsilon\delta\mu)=\varepsilon\delta x_+(t;\delta\mu)+o(t;\varepsilon\delta\mu),$$

where $\delta x_+(t;\delta\mu)$ *has the form*

$$\delta x_+(t;\delta\mu)=-Y(t_{00};t)f_0^+\delta t_0+\beta(t;\delta\mu).$$

Theorem 2 corresponds to the case, where variation at the point t_{00} is performed on the right. Theorems 1 and 2 are proved by the scheme given in [10].

Theorem 3. *Let conditions* 1)–4) *and condition* (7) *hold. Moreover,*

$$f_0^- = f_0^+ := f_0.$$

Then there exist the numbers $\varepsilon_2 \in (0, \varepsilon_1)$ *and* $\delta_2 \in (0, \delta_1)$, *with* $t_{10} - \delta_2 > t_{00} + \tau_{s0}$ *such that for arbitrary*

$$(t, \varepsilon, \delta\mu) \in [t_{10} - \delta_2, t_{10} + \delta_2] \times (0, \varepsilon_2) \times V,$$

we have

$$\Delta x(t; \varepsilon\delta\mu) = \varepsilon\delta x(t; \delta\mu) + o(t; \varepsilon\delta\mu),$$

where $\delta x(t; \delta\mu)$ *has the form*

$$\delta x(t; \delta\mu) = -Y(t_{00}; t) f_0 \delta t_0 + \beta(t; \delta\mu).$$

Theorem 3 corresponds to the case, where variation at the point t_{00} is carried out from double sided and is a corollary to Theorems 1 and 2.

It is clear that if the function $f(t, x, x_1, \dots, x_s, u, u_1, \dots, u_k)$ is continuous then

$$f_0 = f(t_{00}, x_{00}, \varphi_0(t_{00} - \tau_{10}), \dots, \varphi_0(t_{00} - \tau_{s0}), u_0(t_{00}), u_0(t_{00} - \theta_{10}), \dots, u_0(t_{00} - \theta_{k0})).$$

Theorem 4. *Let*

$$f(t, x, x_1, \dots, x_s, u, u_1, \dots, u_k) = A(t)x + \sum_{i=1}^{s} B_i(t) x_i + C(t)u + \sum_{i=1}^{k} D_i(t) u_i,$$

where $A(t)$, $B_i(t), i = \overline{1, s}$, $C(t)$ *and* $D_i(t), i = \overline{1, k}$ *are continuous matrix functions. Then there exist the numbers* $\varepsilon_2 \in (0, \varepsilon_1)$ *and* $\delta_2 \in (0, \delta_1)$, *with* $t_{10} - \delta_2 > t_{00} + \tau_{s0}$ *such that for arbitrary*

$$(t, \varepsilon, \delta\mu) \in [t_{10} - \delta_2, t_{10} + \delta_2] \times (0, \varepsilon_2) \times V,$$

we have

$$\begin{aligned}
\delta x(t; \delta\mu) = {} & Y(t_{00}; t)\Big[\delta x_0 - \Big(A(t_{00}) x_{00} + \sum_{i=1}^{s} B_i(t_{00}) \varphi_0(t_{00} - \tau_{i0}) + C(t_{00}) u_0(t_{00}) \\
& + \sum_{i=1}^{k} D_i(t_{00}) u_0(t_{00} - \theta_{i0})\Big)\delta t_0\Big] - \sum_{i=1}^{s} Y(t_{00} + \tau_{i0}; t) B_i(t_{00} + \tau_{i0})\Big(x_{00} - \varphi_0(t_{00})\Big)\delta t_0 \\
& - \sum_{i=1}^{s} \Big[Y(t_{00} + \tau_{i0}; t) B_i(t_{00} + \tau_{i0})\Big(x_{00} - \varphi_0(t_{00})\Big) + \int_{t_{00}}^{t} Y(\xi; t) B_i(\xi)\Big(\chi_i(\xi)\dot{\varphi}_0(\xi - \tau_{i0}) \\
& + (1 - \chi_i(\xi))\dot{x}_0(\xi - \tau_{i0})\Big) d\xi\Big]\delta\tau_i + \sum_{i=1}^{s} \int_{t_{00} - \tau_{i0}}^{t_{00}} Y(\xi + \tau_{i0}; t) B_i(\xi + \tau_{i0}) \delta\varphi(\xi) d\xi \\
& - \sum_{i=1}^{k} \Big[\int_{t_{00}}^{t} Y(\xi; t) D(\xi) \dot{u}_0(\xi - \theta_0) d\xi\Big]\delta\theta_i + \int_{t_{00}}^{t} Y(\xi; t)\Big[C(\xi)\delta u(\xi) \\
& + \sum_{i=1}^{k} D_i(\xi) \delta u(\xi - \theta_{i0})\Big] d\xi.
\end{aligned}$$

Here, $Y(\xi; t)$ *is the* $n \times n$*-matrix function satisfying the equation*

$$Y_\xi(\xi; t) = -Y(\xi; t) A(\xi) - \sum_{i=1}^{s} Y(\xi + \tau_{i0}; t) B_i(\xi + \tau_{i0}), \xi \in [t_{00}, t]$$

and condition (6).

Theorem 4 is a simple corollary to Theorem 3.

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I. Javakhishvili Tbilisi State University, Department of Computer Scinces, Tbilisi, Georgia
E-mail address: imedea@yahoo.com