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ON NEW VERSIONS OF THE LINDEBERG–FELLER'S LIMIT THEOREM

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Abstract. It is well known that the classical Lindeberg condition is sufficient for the validity of the central limit theorem. It will be also a necessary if the summands satisfy the condition of infinite smallness (Feller's theorem). The limit theorems for the distributions of sums of independent random variables which do not use the condition of infinite smallness were called non-classical.

The exact bounds for the Lindeberg, Rotar characteristics using the difference of distribution of sum of independent random variables and a standard normal distribution are established. These results improve Feller's theorem.

Introduction

Let $X_{n1}, X_{n2}, \dots, X_{nn}$, $n = 1, 2, \dots$ be an array of independent random variables (r.v.'s). Assume that

$$EX_{nj} = 0, \quad EX_{nj}^2 = \sigma_{nj}^2, \quad j = 1, 2, \dots,$$
$$S_n = X_{n1} + \dots + X_{nn}, \quad \sum_{j=1}^{n} \sigma_{nj}^2 = 1.$$

Set

$$F_n(x) = P\left(S_n < x\right), \quad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du,$$
$$\Delta_n = \sup_x \left|F_n(x) - \Phi(x)\right|.$$

It is well-known that the following condition (Feller's characteristic)

$$\max_{1 \le j \le n} \sigma_{nj} \to 0, \quad n \to \infty \tag{F}$$

is called uniform of infinite smallness condition of a sequence of independent r.v.'s. $\{X_{nj},\, j \ge 1\}$. We say that this sequence satisfies Lindeberg condition if for any $\varepsilon > 0$

$$L_n(\varepsilon) = \sum_{j=1}^{n} E(X_{nj}^2 I(|X_{nj}| > \varepsilon)) \to 0, \quad n \to \infty. \tag{L}$$

Here, $I(A)$ denotes an indicator of the event A.

It is well-known that under the condition L,

$$\Delta_n \to 0, \quad n \to \infty,$$

which means a central limit theorem (CLT). The Lindeberg–Feller's theorem improves the above theorem and can be represented in the form of the following implication:

$$(F)\&(CLT) \Leftrightarrow (L),$$

i.e., under the condition (F), Lindeberg's condition is necessary one for CLT.

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1. Estimation of Numerical Characteristics Used in CLT

Following V. M. Zolotarev [8, ch. 5, §5.2], we call the limit theorems non-classical in which the condition (F) is not used. The first non-classical variants of CLT were proved by Zolotarev in 1967 and Rotar in 1975 [6].

In [3], [5], the following estimates of $L_n(\varepsilon)$ $(\varepsilon > 0)$ were obtained.

Theorem A. *There exists an absolute constant $C > 0$ such that for any $\varepsilon > 0$,*

$$\left(1 - e^{-\varepsilon^2/4}\right)\sum_{j=1}^{n} E(X_{nj}^2 I(|X_{nj}| > \varepsilon)) \leq C\left(\Delta_n + \sum_{j=1}^{n} \sigma_{nj}^4\right). \tag{1}$$

Note. It is obvious that under the condition (F) and $\sum\limits_{j=1}^{n} \sigma_{nj}^2 = 1$,

$$\sum_{j=1}^{n} \sigma_{nj}^4 \leq \max_j \sigma_{nj}^2 \to 0, \quad n \to \infty.$$

Thus (1) implies that if the sequence of independent r.v.'s $\{X_{nj},\, j \geq 1\}$ satisfies CLT (i.e. $\Delta_n \to 0$, $n \to \infty$), then the Lindeberg condition

$$\sum_{j=1}^{n} E(X_{nj}^2 I(|X_{nj}| > \varepsilon)) \to 0$$

holds for any $\varepsilon > 0$ by $n \to \infty$.

Set

$$F_{nj}(x) = P(X_{nj} < x),$$

$\Phi_{nj}(x)$ is a distribution function of normal r.v. with parameters $(0, \sigma_{nj}^2)$ $(j = 1, 2, \dots)$ and for any $\varepsilon > 0$,

$$R_n(\varepsilon) = \sum_{j=1}^{\infty} \int\limits_{|x|>\varepsilon} |x||F_{nj}(x) - \Phi_{nj}(x)|dx.$$

Theorem B (V. I. Rotar [6])**.** *The condition*

$$R_n(\varepsilon) \to 0, \quad n \to \infty, \tag{2}$$

for any $\varepsilon > 0$ is sufficient and necessary for CLT.

The above Theorem B is a nonclassical version of CLT and it generalizes Lindeberg–Feller's theorem. Indeed, in Theorem B we do not use the condition (F). The proof of the necessity of condition (2) is based on the following statement (note that a proof of the necessity of condition (2) given in [6] is rather complicated and it uses the properties of probabilistic metrics).

The following theorem holds.

Theorem 1. *For some $C = C(\varepsilon)$, the following estimation*

$$R_n(\varepsilon) \leq C\left(L_n(\varepsilon) + \sum_{j=1}^{n} \sigma_{nj}^{2s}\right) \tag{3}$$

for any $\varepsilon > 0$ and $s \geq 2$, is true.

Proof. In [3], the inequality

$$R_n(\varepsilon) \leq \sum_{j=1}^{n} \int\limits_{|x|>\varepsilon} x^2 dF_{nj}(x) + \sum_{j=1}^{n} \int\limits_{|x|>\varepsilon} x^2 d\Phi_{nj}(x) = L_n(\varepsilon) + \Phi_n(\varepsilon) \tag{4}$$

is proved. Further, it is not hard to prove that

$$\Phi_n(\varepsilon) = \sum_{j=1}^{n} \int\limits_{|x|>\varepsilon/\sigma_{nj}} \sigma_{nj}^2 x^2 d\Phi(x) \le \varepsilon^2 \sum_{j=1}^{n} \int\limits_{|x|>\varepsilon/\sigma_{nj}} \left(\frac{\sigma_{nj}^2 x^2}{\varepsilon^2}\right)^s d\Phi(x)$$

$$\le \varepsilon^{-2(s-1)} \left(\sum_{j=1}^{n} \sigma_{nj}^{2s}\right) \int\limits_{|x|>\varepsilon/\max\limits_{1\le j\le n} \sigma_{nj}} x^{2s} d\Phi(x).$$

Hence, for any $s \ge 2$,

$$\Phi_n(\varepsilon) \le C(\varepsilon)\left(\sum_{j=1}^{n} \sigma_{nj}^{2s}\right). \tag{5}$$

Now, the fairness of estimation (3) and Theorem 1 follows from relations (4) and (5).

It can easily be checked that the above-proven Theorem 1 has the following corollaries.

1) We say that Rotar's condition holds if

$$R_n(\varepsilon) \to 0, \quad \forall\varepsilon > 0, \quad n \to \infty. \tag{R}$$

It is easy to prove that the following estimation (6)

$$\max_{1\le j\le n} \sigma_{nj}^2 \le \varepsilon^2 + \max_{1\le j\le n} \int\limits_{|x|>\varepsilon} x^2 dF_{nj}(x) \le \varepsilon^2 + \sum_{j=1}^{n} \int\limits_{|x|>\varepsilon} x^2 dF_{nj}(x) = \varepsilon^2 + L_n(\varepsilon) \tag{6}$$

is true. Directly from the estimation (3), for $s = 2$, we have

$$R_n(\varepsilon) \le C(\varepsilon)\Big(L_n(\varepsilon) + \max_{1\le j\le n} \sigma_{nj}^2\Big). \tag{7}$$

Thus from relations (6) and (7) it follows that if $L_n(\varepsilon) \to 0$, $\forall\varepsilon > 0$, $n \to \infty$, then $R_n(\varepsilon) \to 0$, $\forall\varepsilon > 0$, $n \to \infty$. Therefore the implication $(L) \Rightarrow (R)$ is true. In turn, from the last it follows that Theorem 1 generalizes classic version of the Lindeberg-Feller limit theorem.

2) Since the integration domain in the expression $\Phi_n(\varepsilon)$ contains in the domain $\left\{x : \frac{\left(\max\limits_{1\le j\le n} \sigma_{nj}\right)^2 x^2}{\varepsilon^2} > 1\right\}$, inequality (5) can be written as

$$\Phi_n(\varepsilon) \le C(\varepsilon)\left(\max_{1\le j\le n} \sigma_{nj}\right)^{2s}.$$

Hence, estimation (3) can be rewritten as

$$R_n(\varepsilon) \le C(\varepsilon)\left[L_n(\varepsilon) + \Big(\max_{1\le j\le n} \sigma_{nj}\Big)^{2s}\right], \quad s \ge 2.$$

3) In view of Theorem A, estimation (3) can be rewritten as

$$R_n(\varepsilon) \le C(\varepsilon)\left(\Delta_n + \sum_{j=1}^{n} \sigma_{nj}^4\right).$$

Thus the following implication

$$(F)\&(CLT) \Rightarrow (R)$$

holds. From the above corollaries 1)–3) we have that the characteristic $R_n(\varepsilon)$ is thinner then the Lindeberg characteristic. For example, in the case of the equality of distributions $F_{nj} \equiv \Phi_{nj}$, $j = 1, 2, \dots, n$, it is obvious that the value $R_n(\varepsilon)$ vanishes trivially, but at the same time, $L_n(\varepsilon) > 0$, $\forall\varepsilon > 0$. It should be noted that if the condition F holds, then these conditions are equivalent, i.e.,

$$(F)\&(R) \Leftrightarrow (L).$$

The last limit relation is proved in the book of A. N. Shiryaev [7]. □

2. Ibragimov-Osipov-Essen Characteristic and Some Relations of Equivalence

We put

$$M_n(\alpha)=\sum_{j=1}^{n}\int\limits_{|x|\le 1}|x|^{2+\alpha}\,dF_{nj}(x)+\sum_{j=1}^{n}\int\limits_{|x|>1}x^2dF_{nj}(x)=m_n(\alpha)+L_n,\quad \alpha>0.$$

This numerical characteristic is the first encountered in [1] and [2] for $\alpha=1$. The appearance of this characteristic is due to the impossibility of estimating the remainder term in the CLT by the single Lindeberg characteristic $L_n(\cdot)$ (see [1]). In [2], it is shown that the value $M_n(1)$ can be used in estimating the rate of convergence in CLT. It should be noted that there are the cases where $m_n(1)=o\left(L_n(\cdot)\right)$ or $L_n(\cdot)=o\left(m_n(1)\right)$ as $n\to\infty$. Therefore in the expression $M_n(\cdot)$ it is impossible to confine ourselves to one of the two terms.

We present some asymptotic properties of $M_n(\alpha)$, as $n\to\infty$.

Lemma 1. *If for some $\alpha=\alpha_0>0$,*

$$m_n(\alpha_0)=\sum_{j=1}^{n}\int\limits_{|x|\le 1}|x|^{2+\alpha_0}\,dF_{nj}(x)\to 0,\quad n\to\infty,$$

then $m_n(\alpha)\to 0$, $n\to\infty$ for any $\alpha>0$.

Proof. Let $\alpha<\alpha_0$. Then for any $0<\varepsilon\le 1$,

$$\begin{aligned}m_n(\alpha)&=\sum_{j=1}^{n}\int\limits_{|x|\le\varepsilon}|x|^{2+\alpha}\,dF_{nj}(x)+\sum_{j=1}^{n}\int\limits_{\varepsilon<|x|\le 1}|x|^{2+\alpha}\,dF_{nj}(x)\\&\le\varepsilon^{\alpha}+\sum_{j=1}^{n}\int\limits_{\varepsilon<|x|\le 1}|x|^{2+\alpha_0}\,|x|^{\alpha-\alpha_0}\,dF_{nj}(x)\\&\le\varepsilon^{\alpha}+\left(\sum_{j=1}^{n}\int\limits_{|x|\le 1}|x|^{2+\alpha_0}\,dF_{nj}(x)\right)\cdot\varepsilon^{-(\alpha_0-\alpha)}=\varepsilon^{\alpha}+\frac{m_n(\alpha_0)}{\varepsilon^{\alpha_0-\alpha}}.\end{aligned}$$

Hence for any $0<\varepsilon\le 1$,

$$\lim_{n\to\infty}\sup m_n(\alpha)\le\varepsilon^{\alpha},\quad \alpha<\alpha_0.$$

Since $0<\varepsilon\le 1$ is arbitrary, from the last relation we get the proof of the relation

$$\{m_n(\alpha_0)\to 0\}\Rightarrow\{m_n(\alpha)\to 0,\ \alpha<\alpha_0\}.$$

Now let $\alpha_0<\alpha$. Then swapping α and α_0 in the previous reasoning, we obtain the proof of the following relation:

$$\{m_n,\alpha_0\to 0\}\Rightarrow\{m_n(\alpha)\to 0,\alpha>\alpha_0\}.\qquad\square$$

Lemma 2. *If for some $\alpha=\alpha_0$, the relation $M_n(\alpha_0)\to 0$, $n\to\infty$ is true, then the Feller conditions (F) hold.*

Proof. Indeed, for $0<\varepsilon\le 1$, we have

$$\begin{aligned}\max_{1\le j\le n}\sigma_{nj}^2&\le\varepsilon^2+\max_{1\le j\le n}\left[\int\limits_{\varepsilon<|x|\le 1}x^2dF_{nj}(x)+\int\limits_{|x|>1}x^2dF_{nj}(x)\right]\\&\le\varepsilon^2+\sum_{j=1}^{n}\int\limits_{\varepsilon<|x|\le 1}x^2dF_{nj}(x)+\sum_{j=1}^{n}\int\limits_{|x|>1}x^2dF_{nj}(x)\\&\le\varepsilon^2+\varepsilon^{-\alpha}\cdot\sum_{j=1}^{n}\int\limits_{|x|\le 1}|x|^{2+\alpha}\,dF_{nj}(x)+L_n\le\varepsilon^2+\varepsilon^{-\alpha}m_n(\alpha)+o(1),\ n\to\infty.\end{aligned}$$

Now, by applying Lemma 1, we obtain

$$\lim_{n\to\infty} \sup \max_{1\le j\le n} \sigma_{nj}^2 \le \varepsilon^2, \quad 0<\varepsilon\le 1. \qquad \square$$

Further, we say that the condition (M_α) holds, if for some $\alpha>0$ the value $M_n(\alpha)\to 0,\ n\to\infty$. Then assertion of Lemma 2 can be written as the implication $(M_\alpha)\Rightarrow(F)$.

Theorem 2. *Let for some* $\alpha=\alpha_0>0$,

$$M_n(\alpha_0)=m_n(\alpha_0)+L_n\to 0,\ \ n\to\infty. \tag{M}$$

Then for any α*, the following implication*

$$\{M_n(\alpha_0)\to 0,\ n\to\infty\}\Leftrightarrow\{L_n(\varepsilon)\to 0,\ \forall\varepsilon>0,\ n\to\infty\}$$

is true.

Remark. In view of Lemma 1, Theorem 2 can be written as an implication of the equivalence

$$(M)\Leftrightarrow(L) \tag{8}$$

Proof of Theorem 2. For simplicity, in the condition M we put $\alpha=1$, i.e.,

$$M_n=M_n(1)=m_n(1)+L_n=\sum_{j=1}^n\int\limits_{|x|\le 1}|x|^3\,dF_{nj}(x)+\sum_{j=1}^n\int\limits_{|x|>1}x^2dF_{nj}(x)\to 0,\ \ n\to\infty.$$

Note that by Lemma 1 this does not limit the generality in the following reasoning.

Let the condition M holds, i.e., for $n\to\infty$, $M_n\to 0$. Then the following relations are clear: for $0<\varepsilon\le 1$,

$$\begin{aligned}L_n(\varepsilon)&=\sum_{j=1}^n\int\limits_{\varepsilon<|x|\le 1}x^2dF_{nj}(x)+L_n(1)\le\frac{1}{\varepsilon}\sum_{j=1}^n\int\limits_{\varepsilon<|x|\le 1}|x|^3\,dF_{nj}(x)+L_n\\&\le\frac{m_n(1)}{\varepsilon}+o(1)\to 0,\quad n\to\infty.\end{aligned} \tag{9}$$

For $\varepsilon\ge 1$, by monotonicity of $L_n(\varepsilon)$, we obtain

$$L_n(\varepsilon)\le L_n(1)=L_n\to 0,\quad n\to\infty. \tag{10}$$

Relations (9) and (10) prove the justice of the implication $(M)\Rightarrow(L)$.

Now, let the Lindeberg condition L hold. Then for $0<\varepsilon\le 1$,

$$L_n=L_n(1)\le L_n(\varepsilon)\to 0,\quad n\to\infty. \tag{11}$$

In addition, for any $0<\varepsilon\le 1$,

$$m_n(1)=\sum_{j=1}^n\int\limits_{|x|\le 1}|x|^3\,dF_{nj}(x)\le\varepsilon\sum_{j=1}^n\int\limits_{|x|\le\varepsilon}x^2dF_{nj}(x)+\sum_{j=1}^n\int\limits_{\varepsilon<|x|\le 1}x^2dF_{nj}(x)\le\varepsilon+L_n(\varepsilon).$$

Thus, from the last estimation it follows that

$$\lim_{n\to\infty}\sup m_n(1)\le\varepsilon,\quad m_n(1)\to 0,\quad n\to\infty. \tag{12}$$

From relations (11), (12), we conclude that the implication $(L)\Rightarrow(M)$ holds. Hence, the equivalence relation (8) is proved, and thus we obtain the proof of Theorem 2. $\square$

3. The Classical Version of Analogue of the Lindeberg-Feller Theorem

The classical version of the Lindeberg-Feller limit theorem can be represented as equivalence of the implications

$$(F)\&(CLT) \Leftrightarrow (L).$$

We give a theorem in which conditions M are used instead of L.

Theorem 3. *In order for the sequence of series of independent random variables $\{X_{nj}, \quad 1 \le j \le n\}$ to satisfy the Feller condition F and obey (CLT), it is necessary and sufficient that condition M be fulfilled.*

Below, we give the proof of Theorem 3 with the direct use of the condition M.

Proof of Theorem 3. We introduce the notions

$$f_{nj}(t) = \int_{-\infty}^{\infty} e^{itx} dF_{nj}(x), \quad g_{nj}(t) = \int_{-\infty}^{\infty} e^{itx} d\Phi_{nj}(x) = e^{-\sigma_{nj}^2 t^2/2},$$

$$f_n(t) = \prod_{j=1}^{n} f_{nj}(t), \quad \prod_{j=1}^{n} g_{nj}(t) = g_n(t) = e^{-t^2/2}.$$

To prove the validity of the CLT, it suffices to make sure that for any $T > 0$,

$$\sup_{|t|\le T} \left| f_n(t) - e^{-t^2/2} \right| \to 0, \quad n \to \infty. \tag{13}$$

Note first that for all complex numbers satisfying the inequalities $|a_k| \le 1$, $|b_k| \le 1$, $k = 1, 2, \ldots$, the inequality

$$\left| \prod_{k=1}^{n} a_k - \prod_{k=1}^{n} b_k \right| \le \sum_{k=1}^{n} |a_k - b_k| \tag{14}$$

holds. By the last inequality (14), we obtain

$$\left| \prod_{j=1}^{n} f_{nj}(t) - \prod_{k=1}^{n} g_{nj}(t) \right| \le \sum_{j=1}^{n} |f_{nj}(t) - g_{nj}(t)|. \tag{15}$$

Therefore, by (15), we can conclude that relation (13) will be proved if it states that

$$d_n(t) = \sum_{j=1}^{n} |f_{nj}(t) - g_{nj}(t)| \to 0, \quad n \to \infty, \tag{16}$$

for any $t \in \mathrm{R}$.

Further, we use the following equalities:

$$\int_{-\infty}^{\infty} x dF_{nj}(x) = \int_{-\infty}^{\infty} x d\Phi_{nj}(x) = 0,$$

$$\int_{-\infty}^{\infty} x^2 dF_{nj}(x) = \int_{-\infty}^{\infty} x^2 d\Phi_{nj}(x) = \sigma_{nj}^2, \quad j = 1, 2, \ldots.$$

By these equalities, for $j = 1, 2, \ldots$, we can write

$$f_{nj}(t) - g_{nj}(t) = \int_{-\infty}^{\infty} \left[e^{itx} - 1 - itx - \frac{(itx)^2}{2} \right] d\left(F_{nj}(x) - \Phi_{nj}(x)\right). \tag{17}$$

After integrating by parts in the last integral and by virtue of

$$x^2 \left[1 - F_{nj}(x) + F_{nj}(-x)\right] \to 0, \quad x^2 \left[1 - \Phi_{nj}(x) + \Phi_{nj}(-x)\right] \to 0,$$

for $x \to \infty$, we obtain

$$\int_{-\infty}^{\infty} \left[e^{itx} - 1 - itx - \frac{(itx)^2}{2}\right] d\left(F_{nj}(x) - \Phi_{nj}(x)\right)$$
$$= -it \int_{-\infty}^{\infty} \left(e^{itx} - 1 - itx\right)\left(F_{nj}(x) - \Phi_{nj}(x)\right) dx. \quad (18)$$

Now, by (15), (17) and (18), for any $\varepsilon > 0$, we have

$$d_n(t) \le \sum_{j=1}^{n} \left| t \int_{-\infty}^{\infty} \left(e^{itx} - 1 - itx\right)\left(F_{nj}(x) - \Phi_{nj}(x)\right) dx \right|$$
$$\le \frac{|t|^3}{2} \varepsilon \sum_{j=1}^{n} \int_{|x| \le \varepsilon} |x| \left|F_{nj}(x) - \Phi_{nj}(x)\right| dx + 2t^2 \sum_{j=1}^{n} \int_{|x| > \varepsilon} |x| \left|F_{nj}(x) - \Phi_{nj}(x)\right|. \quad (19)$$

By direct integration by parts, it is easy to verify that for any random variable X with the distribution function $F(x)$, the following equality

$$\mathrm{E}\,|X|^n = \int_{-\infty}^{\infty} |x|^n \, dF(x) = n \int_{0}^{\infty} x^{n-1} \left(1 - F(x) + F(-x)\right) dx. \quad (20)$$

holds. Based on formula (20) with $n = 2$, we can verify the following estimate:

$$\sum_{j=1}^{n} \int_{|x| \le \varepsilon} |x| \left|F_{nj}(x) - \Phi_{nj}(x)\right| dx \le 2 \sum_{j=1}^{n} \sigma_{nj}^2 = 2.$$

Hence, inequality (19) can be written as

$$d_n(t) \le |t|^3 \cdot \varepsilon + 2t^2 \cdot R_n(\varepsilon).$$

Now, by Theorem 2, it follows that

$$R_n(\varepsilon) = O\left(L_n(\varepsilon) + \sum_{j=1}^{n} \sigma_{nj}^{2s}\right), \quad s \ge 2.$$

By the last expression, we obtain

$$\overline{\lim_{n \to \infty}} \sup_{|t| \le T} d_n(t) \le T^3 \cdot \varepsilon.$$

Therefore, finally, we have the relation

$$\overline{\lim_{n \to \infty}} \sup_{|t| \le T} \left| f_n(t) - e^{-t^2/2} \right| \le T^3 \cdot \varepsilon. \quad (21)$$

Since $\varepsilon > 0$ is arbitrary, by relation (21) we obtain that for any $T > 0$, relation (13) holds, which proves the sufficiency of the condition M for fulfilling CLT. □

Necessity. By Theorem 2, conditions M and L are equivalent (i.e., $(M) \Leftrightarrow (L)$). Therefore, the necessity of condition M for the validity of CLT follows from Theorem A above.

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