Transactions of A. Razmadze
Mathematical Institute
Vol. **173** (2019), 21–29

PROJECTIVITY AND UNIFICATION IN LOCALLY FINITE VARIETIES OF MONADIC MV-ALGEBRAS

A. DI NOLA[1], R. GRIGOLIA[2], AND G. LENZI[1]

Abstract. A duality between the category of finite monadic MV-algebras and a category of labelled finite Boolean spaces is given. A characterization of projectivity in some locally finite varieties of monadic MV-algebras is provided. Finally, we show that the unification type of these varieties is unitary.

1. Introduction

The finitely-valued propositional calculi, which have been described by Lukasiewicz and Tarski in [15], are extended to the corresponding predicate calculi. The predicate Łukasiewicz (infinitely-valued) logic QL is defined in the following standard way. The existential (universal) quantifier is interpreted as a supremum (infimum) in a complete MV-algebra. Then the valid formulas of predicate calculus are defined as all formulas having value 1 for any assignment. The functional description of the predicate calculus is given by Rutledge in [16]. Scarpellini in [17] has proved that the set of valid formulas is not recursively enumerable. We also refer the reader to papers [10, 18, 19] concerning the Łukasiewicz predicate calculus.

Monadic MV-algebras were introduced and studied by Rutledge in [16] as an algebraic model for the predicate calculus QL of Łukasiewicz infinite-valued logic, in which there occurs only a single individual variable. Rutledge followed P.R. Halmos' study of monadic Boolean algebras. In view of the incompleteness of the predicate calculus, the result of Rutledge in [16], showing the completeness of the monadic predicate calculus, has been of great interest.

Let L denote a first-order language based on $\cdot, +, \to, \neg$ (intended as propositional connectives) and let L_m denote a propositional language based on the propositional connectives $\cdot$, $+$, $\to$, $\neg$, $\exists$ (where $\exists$ denotes a unary propositional connective). Let $Form(L)$ and $Form(L_m)$ be the set of all formulas of L and L_m, respectively. We fix a variable x in L, associate with each propositional letter p in L_m a unique monadic predicate $p^*(x)$ in L and define by induction a translation $\Psi : Form(L_m) \to Form(L)$ by putting:

- $\Psi(p) = p^*(x)$ if p is a propositional variable;
- $\Psi(\neg\alpha) = \neg\Psi(\alpha)$;
- $\Psi(\alpha \circ \beta) = \Psi(\alpha) \circ \Psi(\beta)$, where $\circ = \cdot, +, \to$;
- $\Psi(\exists\alpha) = \exists x \Psi(\alpha)$.

Through this translation Ψ, we can identify the formulas of L_m with monadic formulas of L containing the variable x. Moreover, it is routine to check that $\Psi(MLPC) \subseteq QL$, where $MLPC$ is the monadic Lukasiewicz propositional calculus [8].

For a detailed consideration of Łukasiewicz predicate calculus we refer to [1, 2, 14, 15].

2. Preliminaries on Monadic MV-algebras

The characterization of monadic MV-algebras as pairs of MV-algebras, where one of them is a special kind of subalgebra (m is a relatively complete subalgebra), is given in [3, 8]. The MV-algebras were introduced by Chang in [4] as an algebraic model for infinitely-valued Łukasiewicz logic.

2010 *Mathematics Subject Classification.* 03B50, 08B05.
Key words and phrases. MV-algebras; Monadic MV-algebras; Finitely valued Lukasiewicz logic.

An MV-algebra is an algebra $(A, \oplus, \odot, ^*, 0, 1)$, where $(A, \oplus, 0)$ is an abelian monoid, and the following identities hold for all $x, y \in A:$ $x \oplus 1 = 1$, $x^{**} = x$, $0^* = 1$, $x \oplus x^* = 1$, $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$, $x \odot y = (x^* \oplus y^*)^*$.

Every MV-algebra has an underlying ordered structure defined by

$$x \leq y \quad \text{iff} \quad x^* \oplus y = 1.$$

$(A, \leq, 0, 1)$ is a bounded distributive lattice. Moreover, in any MV-algebra, the property

$$x \odot y \leq x \wedge y \leq x \vee y \leq x \oplus y holds.$$

The unit interval of real numbers $[0, 1]$ endowed with the operations $x \oplus y = \min(1, x + y), x \odot y = \max(0, x + y - 1), x^* = 1 - x$, becomes an MV-algebra. It is well known that the MV-algebra $S = ([0, 1], \oplus, \odot, ^*, 0, 1)$ generates the variety **MV** of all MV-algebras, i. e., $\mathcal{V}(S) = \mathbf{MV}$.

Let $\mathbb{Q}$ denote a set of rational numbers; then $[0, 1] \cap \mathbb{Q}$ is an MV-subalgebra of $[0, 1]$.

Moreover, for $(0 \neq) n \in \omega$, we denote by S_n the subalgebra of $[0, 1]$ whose domain is

$$A_n = \left\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\right\}.$$

For any positive integer n, an algebra $(A, \oplus, \odot, ^*, \exists, 0, 1)$ is said to be a monadic MV-algebra (MMV-algebra, for short) if $(A, \oplus, \odot, ^*, 0, 1)$ is an MV-algebra and, in addition, $\exists$ is a unary function and the following identities hold:

E1: $x \leq \exists x$,
E2: $\exists(x \vee y) = \exists x \vee \exists y$,
E3: $\exists(\exists x)^* = (\exists x)^*$,
E4: $\exists(\exists x \oplus \exists y) = \exists x \oplus \exists y$,
E5: $\exists(x \odot x) = \exists x \odot \exists x$,
E6: $\exists(x \oplus x) = \exists x \oplus \exists x$ hold.

Sometimes we shall denote a monadic MV-algebra $(A, \oplus, \odot, ^*, \exists, 0, 1)$ by $(A, \exists)$, for brevity. We can define a unary operation $\forall x = (\exists x^*)^*$, corresponding to the universal quantifier.

Let A_1 and A_2 be any MMV-algebras. A mapping $h : A_1 \to A_2$ is an MMV-homomorphism if h is an MV-homomorphism, and for every $x \in A_1$, $h(\exists x) = \exists h(x)$. Denote by **MMV** the variety and the category of MMV-algebras and MMV-homomorphisms.

From the variety of monadic MV-algebras **MMV** we select the subvariety $\mathbf{K_n}$ for $0 \neq n \in \omega$, which is defined by the following equation [8]:

$$(K_n) \quad x^n = x^{n+1},$$

that is, $\mathbf{K_n} = \mathbf{MMV} + (K_n)$. The main object of our interest are the varieties $\mathbf{K_n}$, which are locally finite, see [8].

A subalgebra A_0 of an MV-algebra A is said to be *relatively complete*, if for every $a \in A$ the set $\{b \in A_0 : a \leq b\}$ has a least element.

A subalgebra A_0 of an MV-algebra A is said to be *m-relatively complete* [8], if A_0 is relatively complete and two additional conditions

(#): $(\forall a \in A)(\forall x \in A_0)(\exists v \in A_0)(x \geq a \odot a \Rightarrow v \geq a \& v \odot v \leq x)$,
(##): $(\forall a \in A)(\forall x \in A_0)(\exists v \in A_0)(x \geq a \oplus a \Rightarrow v \geq a \& v \oplus v \leq x)$ hold.

By [8], there exists a one-to-one correspondence between

(1) the monadic MV-algebras $(A, \exists)$;
(2) the pairs (A, A_0), where A_0 is an m-relatively complete subalgebra of A.

In fact, A_0 and $\exists$ can be uniquely recovered from each other in the following way: A_0 is the range of $\exists$, and $\exists a = inf\{b \in A_0 : a \leq b\}$.

3. Monadic Operators on Finite MV-algebras

In this section, we recall the characterization of all monadic operators over an arbitrary finite MV-algebra given in [3]. In other words, given any finite MV-algebra, we characterize the set of monadic operators which make it an MMV-algebra.

Suppose that A is a finite MV-algebra. Then $A \cong S_{n_1} \times S_{n_2} \times \cdots \times S_{n_k}$, where the $n_i \geq 1$. Let $\Pi = \{K_1, K_2, \ldots, K_m\}$ be a partition of $\{1, 2, \ldots, k\}$. We shall say that Π is homogeneous if $i, j \in K_l$ implies $S_{n_i} = S_{n_j}$. Given such a Π, each K_i has associated a unique S_{n_j}, which we shall denote by A_i. We clearly have

$$A \cong A_1^{K_1} \times \cdots \times A_m^{K_m}. \tag{1}$$

Since each K_i is finite, there is a monadic operator $\exists_i$ defined on $A_i^{K_i}$ such that $(A_i^{K_i}, \exists_i)$ is an MMV-algebra with $\exists_i(A_i^{K_i}) = A_i$. Setting $\exists = \exists_1 \times \cdots \times \exists_m$ and acting pointwise, we obtain a monadic operator $\exists$ on A, that is, $(A, \exists)$ is an MMV-algebra.

If a $K_i \in \Pi$ has at least two members, then the determined monadic operator will not be trivial, that is, it will not be the identity operator. From this we can see that a given homogeneous partition may give up to $2^m - 1$ non-trivial MMV-algebras.

If we say that $n_1 = n_2 = \cdots = n_k = n$, so $A = S_n^k$, then every partition of $\{1, 2, \ldots, k\}$ will be homogeneous. The question arises as to whether or not every monadic operator on $A \cong S_{n_1} \times S_{n_2} \times \cdots \times S_{n_k}$ is obtained from some homogeneous partition of $\{1, 2, \ldots, k\}$.

Let $(A, \exists)$ be a finite MMV-algebra. Then by [16], $(A, \exists)$ is a subdirect of the product of MMV-algebras $(A_i, \exists_i)$, where $\exists_i A_i$ is totally ordered. Moreover, by [8], each $(A_i, \exists_i)$ is a direct power of $\exists_i A_i$, that is, $(\exists_i A_i)^{K_i}$ for some finite set K_i.

From this we obtain that $(A, \exists)$ is a subdirect product of MMV-algebras, $(A, \exists) \hookrightarrow \prod_{i=1}^{m} ((\exists_i A_i)^{K_i}, \exists_i)$ for some integer m.

4. Labelled Boolean Spaces

A *Boolean space (or Stone space)* is a compact, zero-dimensional and Hausdorff topological space. Boolean spaces form a category whose objects are the Boolean spaces and morphisms are the continuous maps. When a Boolean space is finite, then the topology of the Boolean space is discrete. It is well known that there exists a categorical duality between the category of Boolean algebras **Bool** and the category of Boolean spaces **BS**. Then the category of finite Boolean algebras $\mathbf{Bool}_{fin}$ is dually equivalent to the category of finite Boolean spaces $\mathbf{BS}_{fin}$.

The functors establishing the duality between $\mathbf{Bool}_{fin}$ and $\mathbf{BS}_{fin}$ are as follows. The functor $\mathfrak{C} : \mathbf{Bool}_{fin} \to \mathbf{BS}_{fin}$ sends every finite Boolean algebra B to the set of all ultrafilters of B. The functor $\mathfrak{B} : \mathbf{BS}_{fin} \to \mathbf{Bool}_{fin}$ sends every object $T \in \mathbf{BS}_{fin}$ to the powerset of T.

We now define another category, the category of *labelled Boolean spaces* $\mathbf{BS}^L_{fin}$. Let $X \in \mathbf{BS}_{fin}$ and $\lambda : X \to \omega$. The the set $X_\lambda = \{(x, \lambda(x)) : x \in X\}$ is said to be a *labelled Boolean space*. The map $f : X_\lambda \to Y_{\lambda'}$ is said to be a λ-*map* if for every x, we have $f((x, \lambda(x))) = (f(x), \lambda'(f(x)))$, where $\lambda'(f(x))$ divides $\lambda(x)$. Denote this category by $\mathbf{BS}^L_{fin}$.

Let A be any finite MV-algebra. Then A contains a greatest Boolean subalgebra $B(A) \subseteq A$. The set of ultrafilters of $B(A)$ and the set of MV-ultrafilters of A have the same cardinality. In fact, if $F \subseteq A$ is an MV-ultrafilter of A, then $F \cap B(A)$ is an ultrafilter of $B(A)$. Conversely, if $F \subseteq B(A)$ is an ultrafilter of $B(A)$, then the MV-filter $[F)_A$ generated by F in the MV-algebra A is an MV-ultrafilter of A. So, we have one-to-one correspondence between the set of ultrafilters of $B(A)$ and the set of MV-ultrafilters of A. So, we can identify the corresponding elements of the set of ultrafilters of $B(A)$ and the set of MV-ultrafilters of A. We observe that two different finite MV-algebras A_1 and A_2 may have isomorphic Boolean subalgebras $B(A_1)$ and $B(A_2)$. For example, $B(S_1^2) \cong B(S_1 \times S_2)$ and, so, $\mathfrak{C}(B(A_1)) \cong \mathfrak{C}(B(A_2))$.

Let A be a finite MV-algebra. Label the elements of $\mathfrak{C}(B(A))$ as follows: $\lambda(F) = k$ if $A/[F)_A \cong S_k$. Then let

$$\mathfrak{C}_\lambda(B(A)) = \{(F, \lambda(F)) : F \in \mathfrak{C}(B(A))\}$$

be the resulting labelled Boolean space. We observe that if $A_1 \ncong A_2$, then $\mathfrak{E}_\lambda(B(A_1)) \ncong \mathfrak{E}_\lambda(B(A_2))$. From this observation we can define the functor $\mathfrak{E}_\mathfrak{L}$ from the set $\mathbf{MV}_{fin}$ of finite MV-algebras to the labelled Boolean spaces $\mathbf{BS}^L_{fin}$ in the following way:

$$\mathfrak{E}_\mathfrak{L}(A) = \{(F, \lambda(F)) : \lambda(F) = k \in \omega, F \in \mathfrak{E}(B(A)), A/[F)_A \cong S_k\}.$$

Now let X_λ be a labelled Boolean space. We define the functor $\mathfrak{L}$ from $\mathbf{BS}^L_{fin}$ to $\mathbf{MV}_{fin}$ as follows:

$$\mathfrak{L}(X_\lambda) = \prod_{x \in X} S_{\lambda(x)}.$$

It is easy to verify that $\mathfrak{L}(\mathfrak{E}_\mathfrak{L}(A)) \cong A$ and $\mathfrak{E}_\mathfrak{L}(\mathfrak{L}(X_\lambda)) \cong X_\lambda$. So, we arrive to

Theorem 1. *The category of finite MV-algebras $\boldsymbol{MV}_{fin}$ is dually equivalent to the category of labelled Boolean spaces $\boldsymbol{BS}^L_{fin}$.*

Any subalgebra of a finite Boolean algebra is relatively complete. If a Boolean algebra B_1 embeds into a Boolean algebra B_2, then to this embedding we can associate a surjective map $f : \mathfrak{E}(B_2) \to \mathfrak{E}(B_1)$. The surjective map defines a corresponding partition $E(= Kerf)$. Conversely, any partition E on the Boolean space defines a corresponding subalgebra. Namely, if X is a Boolean space, then the Boolean algebra of all subsets of the set X is the Boolean algebra corresponding to the Boolean space X. Then the set of all E-saturated subsets[1] forms a Boolean subalgebra of the given Boolean algebra.

We are interested in m-relatively complete subalgebras of a finite MV-algebra A. Note that not every subalgebra of a finite MV-algebra A is m-relatively complete.

A partition E of a labelled Boolean space is said to be *correct*, if for any set $U \in E$ and any two elements $x, y \in U$ we have $\lambda(x) = \lambda(y)$. Note that every correct partition is a homogeneous partition in the sense defined above. So, we have

Theorem 2. *Let A be a finite MV-algebra and X_λ be the labelled Boolean space corresponding to it. Then every correct partition of X_λ defines a subalgebra of A which is m-relatively complete, or equivalently, a monadic operator on A.*

Proof. Any correct partition of X_λ defines a decomposition $A = A_1^{K_1} \times \cdots \times A_m^{K_m}$, where $A_1, \ldots, A_m$ are finite MV-chains. From this decomposition, a monadic operator on A can be obtained as that described after equation (1). □

Now we define a category $\mathbf{BS}^{LM}_{fin}$ of monadic labelled Boolean spaces, the objects of which are the pairs (X_λ, E), where X_λ is a labelled Boolean space and E is an equivalence relation which is a correct partition of X_λ.

Let $(A, \exists)$ be a finite monadic MV-algebra. Then $X_\lambda = \mathfrak{E}_\mathfrak{L}(A)$ is a labelled Boolean space. On X_λ there is a homogeneous (correct) partition E corresponding to the monadic operator $\exists$ (see [3]).

Conversely, suppose we have a labelled Boolean space X_λ and a homogeneous (correct) partition E. Let $E(x) = \{y \in X : \textit{there is } U \in E \textit{ such that } x \in U \wedge y \in U\}$.

Then this partition E defines a monadic operator $\exists$ on $A = \mathfrak{L}(X_\lambda)$.

Now define a morphism $f : (X_\lambda, E) \to (X_{\lambda'}, E')$ (similarly to the monadic Boolean algebras) to be a λ-map $f : X_\lambda \to X_{\lambda'}$ which satisfies the following condition: $f(E(x)) = E'(f(x))$, being the condition of strong isotonicity. So, we arrived at

Theorem 3. *The category of monadic labelled Boolean spaces $\boldsymbol{BS}^{LM}_{fin}$ with strongly isotone λ-maps is dually equivalent to the category of finite monadic MV-algebras.*

Remark 4. Let us note that a duality between the category of multisets and the category of finite MV-algebras is established in [5]. The duality established in this section is a particular case of the one given in [5], but represented in another way. We also mention the related paper [7] (especially, Theorem 1.5).

[1]A subset of X is E-saturated if it coincides with the union of E-equivalence classes.

5. Projective Monadic MV-algebras

Now we come back to the subvariety $\mathbf{K_n}$ $(\mathbf{MMV} + (K_n))$ for $1 \leq n \in \omega$.

There is a unique monadic operator $\exists$ on S_n^k , which corresponds to an m-relatively complete totally ordered MV-subalgebra, converting the algebra S_n^k into a simple monadic MV-algebra [8]. This subalgebra coincides with the greatest diagonal subalgebra, i.e., $d(S_n^k) = \{(x, \ldots, x) \in S_n^k : x \in S_n\}$. Denote this monadic MV-algebra by $(S_n^k, \exists_d)$. In this case, the "diagonal" monadic operator $\exists_d$ is defined as follows:

$$\exists_d(x_1, \ldots, x_k) = (x_j, \ldots, x_j),$$

where $x_j = \max(x_1, \ldots, x_k)$. The operator $\forall_d$ is defined dually:

$$\forall_d(x_1, \ldots, x_k) = (x_i, \ldots, x_i),$$

where $x_i = \min(x_1, \ldots, x_k)$.

Notice that $\mathbf{K_n}$ is generated by $(S_p^k, \exists_d)$, $p = 1, \ldots, n$ and $k \in \omega$. Moreover, $\mathbf{K_n}$ is locally finite and there exists a maximal $k \in \omega$ depending on p and m such that $(S_p^k, \exists_d)$ is m-generated. There exists also a maximal positive number $r(k, p, m)$ depending on k p and m such that $(S_p^k, \exists_d)^{r(k,p,m)}$ is m-generated.

We emphasize that for every m there is a finite number of simple m-generated monadic MV-algebras from $\mathbf{K_n}$.

Observe that, since the variety $\mathbf{K_n}$ is locally finite, the free object in m generators, denoted by $F_{\mathbf{K_n}}(m)$, is finite, and the labelled Boolean space $X_\lambda(m)$ of $F_{\mathbf{K_n}}(m)$ is a finite cardinal sum of one-element labelled points. So, $F_{\mathbf{K_n}}(m)$ is a finite product of simple monadic MV-algebras, where one of the factors coincides with $(S_1^1, \exists_d)$. Therefore we can represent $F_{\mathbf{K_n}}(m)$ as $(S_1^1, \exists_d) \times \prod_{i \in I} A_i$ for some finite set I, where A_i is a simple m-generated monadic MV-algebra from $\mathbf{K_n}$.

Recall now that a *projective* object of a variety is an object which is a retract of a free object. We will give a characterisation of projective finitely generated MMV-algebras and give two proofs of the assertion - algebraic and in dual category.

Theorem 5. *An m-generated MMV-algebra A from $\mathbf{K_n}$ is projective, iff A is isomorphic to $(S_1^1, \exists_d) \times A'$ for some finite MMV-algebra A'.*

Proof. Firstly, we give an algebraic proof. Let A have the form $A' \times (S_1^1, \exists_d)$. Since the m-generated free MMV-algebra in $\mathbf{K_n}$ is a finite product of subdirectly irreducible simple MMV-algebras, we find that any homomorphism of $F_{\mathbf{K_n}}(m)$ is a projection on the factors. Let us suppose that A (in its representation as product) has k factors. Let us permute the factors of $F_{\mathbf{K_n}}(m)$ in such a way that the first k factors are isomorphic to the first k factors of A. So, A is a homomorphic image of $F_{\mathbf{K_n}}(m)$, which is an isomorphic copy of A. Let this homomorphism be a projection $\pi : F_{\mathbf{K_n}}(m) \to A$. So, $\pi(x_1, \ldots, x_k, \ldots, x_q) = (x_1, \ldots, x_k)$ and let us suppose that $x_1 \in S_1^1$.

Let $\overline{\pi}$ be the projection whose image gives the rest part of the product $(S_1^1, \exists_d) \times \prod_{i \in I} A_i$. Then $(S_1^1, \exists_d)$ is a subalgebra of every non-trivial MMV-algebra. So, $\overline{\pi}(F_{\mathbf{K_n}})$ contains a subalgebra which is isomorphic to $(S_1^1, \exists_d)$. In other words, we have an embedding $\varepsilon : A \to F_{\mathbf{K_n}}(m)$ such that $\varepsilon(x_1, \ldots, x_k) = (x_1, \ldots, x_k, x_1, \ldots, x_1)$. Therefore A is a subalgebra of $F_{\mathbf{K_n}}(m)$ such that $\pi\varepsilon = Id_A$. It means that A is a retract of $F_{\mathbf{K_n}}(m)$.

Conversely, if A does not have the form $A' \times (S_1^1, \exists_d)$, then A cannot be embedded into $F_{\mathbf{K_n}}(m)$. From here we conclude the proof of the theorem.

Now we give another proof of this theorem using duality. Let X_λ be the labelled Boolean space of the MMV-algebra A and $Y_{\lambda'}$ the labelled Boolean space of $F_{\mathbf{K_n}}(m)$. We have to show that X_λ is a retract of $Y_{\lambda'}$. Since A has the form $(S_1^1, \exists_d) \times A'$, we find that X_λ has the form of cardinal sum $(x, 1) \sqcup \coprod_{j=1}^{k-1} (x, i_j)$, i. e., X_λ contains the labelled point $(x, 1)$. Since A is a homomorphic image of $F_{\mathbf{K_n}}$, we find that there exists an injective λ-map $f : X_\lambda \to Y_{\lambda'}$. Notice that for every $(S_i, \exists)$ there exists an embedding of $(S_1^1, \exists_d)$ into $(S_i, \exists)$. In the dual picture we have a λ-map from U_λ into $V_{\lambda'}$, where $U_\lambda = \mathfrak{E}_{\mathfrak{L}}((S_i, \exists))$ and $V_{\lambda'} = \mathfrak{E}_{\mathfrak{L}}((S_1^1, \exists_d))$, since $\lambda(x)$ divides $\lambda'(y)$.

Now we construct a λ-map $h : Y_{\lambda'} \to X_\lambda$ in the following way: let $hf((x,i)) = (x,i)$ and for every $(y,j) \in Y_{\lambda'} - f(X_\lambda)$ $h((y,j)) = (x,1) \in X_\lambda$. It is clear that $hf = Id_{X_\lambda}$. Therefore, X_λ is a retract of $Y_{\lambda'}$. It means that A is a retract of $F_{\mathbf{K_n}}(m)$.

Conversely, if A does not have the form $A' \times (S_1^1, \exists_d)$, then X_λ does not contain a point with label 1, i. e., a point $(x,1)$. But $Y_{\lambda'}$ contains points of such kind. In this case, there is no any λ-map from $Y_{\lambda'}$ to X_λ sending this point, because this point must be sent to the point labelled by 1. So, X_λ will not be a retract of $Y_{\lambda'}$. □

Corollary 6. *Any subalgebra of the m-generated free algebra $F_{\mathbf{K_n}}(m)$ is projective.*

Proof. The proof immediately follows from the fact that any subalgebra of the free m-generated algebra $F_{\mathbf{K_n}}(m)$ contains as a factor the algebra which is isomorphic to $(S_1^1, \exists_d)$. □

Consider the variety of MV-algebras $\mathbf{V_n}$, which is generated by $\{S_1, \ldots, S_n\}$. Let us observe that

$$A = \prod_{p=1}^{n} (S_p^1, \exists)^{r(1,p,m)}$$

is an algebra with a trivial monadic operator $\exists$ (i. e. $\exists x = x$) which is isomorphic as an MV-algebra to the m-generated free MV-algebra $F_{\mathbf{V_n}}(m)$, by Lemma 2.2 in [6], and Theorem 1 in [9]. Hence we write $A = (F_{\mathbf{V_n}}(m), \exists)$.

Since $\prod_{p=1}^{n} (S_p^1, \exists)^{r(1,p)}$ contains as a factor an algebra isomorphic to $(S_1^1, \exists_d)$, by Theorem 5 it holds.

Theorem 7. *The MMV-algebra $A = (F_{\mathbf{V_n}}(m), \exists)$ is projective.*

6. Unification Problem

Let E be an equational theory. The E-unification problem is formulated as follows: given two terms s, t, to find a unifier for them, that is, a uniform replacement of the variables occurring in s and t by other terms that makes s and t equal modulo E. For detailed information on unification problems we refer to [11, 12].

Let us be more precise. Let Φ be a set of functional symbols and let V be a set of variables. Let $T_V(\Phi)$ be the term algebra built from Φ and V, and $T_V(\Phi_m)$ be the term algebra of m-variable terms. Let E be a set of equations $p(x) = q(x)$, where $p(x), q(x) \in T_V(\Phi_m)$.

Let $\mathbf{V}$ be the variety of algebras over Φ, axiomatized by the equations in E.

A *unification problem modulo E* is a finite set of pairs

$$\mathcal{E} = \{(s_j, t_j) : s_j, t_j \in T_V(\Phi_m), j \in J\}$$

for some finite set J. A *solution to* (or *unifier for*) $\mathcal{E}$ is a substitution σ (i.e., an endomorphism of the term algebra $T_V(\Phi_m)$) such that the equality $\sigma(s_j) = \sigma(t_j)$ holds in every algebra of the variety $\mathbf{V}$. The problem $\mathcal{E}$ is *solvable* (or *unifiable*) if it admits at least one unifier.

Let $(X, \preceq)$ be a quasi-ordered set (i. e., a reflexive and transitive relation). A *μ-set* for $(X, \preceq)$ (see [12]) is a subset $M \subseteq X$ such that: (1) every $x \in X$ is less than, or equal to some $m \in M$; (2) all elements of M are mutually $\preceq$-incomparable.

There might be no μ-set for $(X, \preceq)$ (in this case we say that $(X, \preceq)$ has *type* 0), or there might be many of them, due to the lack of antisymmetry. However, all μ-sets for $(X, \preceq)$, if any, must have the same cardinality. We say that $(X, \preceq)$ has *type* $1, \omega, \infty$, iff it has a μ-set of cardinality 1, of finite (greater than 1) cardinality or of infinite cardinality, respectively.

Substitutions are compared by instantiation in the following way: we say that $\sigma : T_V(\Phi_m) \to T_V(\Phi_m)$ is *more general, than* $\tau : T_V(\Phi_m) \to T_V(\Phi_m)$ (written as $\tau \preceq \sigma$), iff there is a substitution $\eta : T_V(\Phi_m) \to T_V(\Phi_m)$ such that for all $x \in V_m$, we have $E \vdash \eta(\sigma(x)) = \tau(x)$. The relation $\preceq$ is a quasi-order.

Let $U_E(\mathcal{E})$ be the set of unifiers for the unification problem $\mathcal{E}$; then $(U_E(\mathcal{E}), \preceq)$ is a quasi-ordered set.

We say that an equational theory E has:

1. Unification type 1, iff for every solvable unification problem $\mathcal{E}$, $U_E(\mathcal{E})$ has type 1;
2. Unification type ω, iff for every solvable unification problem $\mathcal{E}$, $U_E(\mathcal{E})$ has type ω;
3. Unification type ∞, iff for every solvable unification problem $\mathcal{E}$, $U_E(\mathcal{E})$ has type 1, or ω, or ∞, and there is a solvable unification problem $\mathcal{E}$ such that $U_E(\mathcal{E})$ has type ∞;
4. Unification type nullary, if none of the preceding cases applies.

An algebra A is called *finitely presented* if A is finitely generated, with the generators $a_1, \ldots, a_m \in A$, and there exist a finite number of equations $P_1(x_1, \ldots, x_m) = Q_1(x_1, \ldots, x_m), \ldots, P_n(x_1, \ldots, x_m) = Q_n(x_1, \ldots, x_m)$ holding in A on the generators $a_1, \ldots, a_m \in A$ such that if there exists an m-generated algebra B, with generators $b_1, \ldots, b_m \in B$, such that the equations $P_1(x_1, \ldots, x_m) = Q_1(x_1, \ldots, x_m)$, $\ldots, P_n(x_1, \ldots, x_m) = Q_n(x_1, \ldots, x_m)$ hold in B on the generators $b_1, \ldots, b_m \in B$, then there exists a homomorphism $h : A \to B$ sending a_i to b_i.

If $\mathbf{V}$ is a variety of algebras and Ω is a finite set of m-ary $\mathbf{V}$-equations, then we denote by $F_{\mathbf{V}}(m, \Omega)$ the object, free over $\mathbf{V}$ with respect to the conditions Ω on the generators (see [13]). If $\Omega = \emptyset$, then $F_{\mathbf{V}}(m, \Omega) = F_{\mathbf{V}}(m)$. $F_{\mathbf{V}}(m, \Omega)$ is a finitely presented algebra.

Now we will give a characterization of finitely presented MMV-algebras.

A filter F of an algebra $(A, \exists) \in \mathbf{MMV}$ is called a *monadic filter* (which is dual to an ideal, see [16]) if for every $a \in A$ we have $a \in F \Rightarrow \forall a \in F$.

For any set $X \subseteq A$, let $[X)$ denote the monadic filter generated by X. It is easy to check that $[X) = \{a \in A : a \geq \forall x_1 \odot \ldots \odot \forall x_n : x_1, \ldots, x_n \in X\}$.

Theorem 8. *Let p be an m-ary term. Then there is a principal monadic filter F such that $F_{\mathbf{MMV}}(m, p = 1) \cong F_{\mathbf{MMV}}(m)/F$.*

Proof. Let
$$F = \{x : x \in F_{\mathbf{MMV}}(m) \text{ and } x \geq \forall p^n(g_1, \ldots, g_m) \text{ for some } n \in \omega\},$$
where $g_1, \ldots, g_m$ are free generators of $F_{\mathbf{MMV}}(m)$. Then $g_1/F, \ldots, g_m/F$ are generators of $F_{\mathbf{MMV}}(m)/F$.

Let A be an MMV-algebra generated by $\{a_1, \ldots, a_m\}$ such that $p(a_1, \ldots, a_m) = 1$, and let $f : F_{\mathbf{MMV}}(m) \to A$ be a homomorphism such that $f(g_i) = a_i$, $i = 1, \ldots, m$. Then $\forall p^n(g_1, \ldots, g_m) \in f^{-1}(1)$ for every $n \in \omega$ and therefore $F \subseteq f^{-1}(1)$. By the homomorphism theorem, there is a homomorphism $f' : F_{\mathbf{MMV}}(m)/F \to A$ such $\pi_F f' = f$. It should be clear that f' is the needed homomorphism extending the map $g_i/F \mapsto a_i$. □

From this theorem it follows that if an algebra A is finitely presented, then there exists a principal monadic filter F of the free algebra $F_{\mathbf{MMV}}(m)$ such that $A \cong F_{\mathbf{MMV}}(m)/F$.

Theorem 9. *Let $u \in F_{\mathbf{MMV}}(m)$ such that $\forall u^n \neq 0$ for any $n \in \omega$. Then $F = \{x : x \geq \forall u^n,\ n \in \omega\}$ is a proper principal monadic filter in $F_{\mathbf{MMV}}(m)$ such that $F_{\mathbf{MMV}}(m)/F \cong F_{\mathbf{MMV}}(m, p = 1)$ for some m-ary term p.*

Proof. Let F be a monadic filter satisfying the condition of the theorem. Then $u = p(g_1, \ldots, g_m)$ for some term p, where $g_1, \ldots, g_m$ are free generators of $F_{\mathbf{MMV}}(m)$. We find that $F_{\mathbf{MMV}}(m)/F$ is generated by $g_1/F, \ldots, g_m/F$, and that $p(g_1/F, \ldots, g_m/F) = p(g_1, \ldots, g_m)/F = 1_{F(m)/F}$. The rest can be verified as in the proof of Theorem 8. □

Combining the two theorems, we arrive at

Theorem 10. *An m-generated MMV-algebra A is finitely presented, iff there exists a principal monadic filter F of $F_{\mathbf{MMV}}(m)$ such that $F_{\mathbf{MMV}}(m)/F \cong A$.*

Now we follow Ghilardi [11], who has introduced the relevant definitions for E-unification from an algebraic point of view. Let E be an equational theory. By an *algebraic unification problem* we mean a finitely presented algebra A of the variety associated to E. A solution for it (also called a unifier for A) is a pair given by a projective algebra P and a homomorphism $u : A \to P$. The set of unifiers for A is denoted by $U_E(A)$. A is said to be *unifiable* or solvable, iff $U_E(A)$ is not empty. Given another algebraic unifier $w : A \to Q$, we say that u is more general, than w, written $w \preceq u$, if there is a homomorphism $g : P \to Q$ such that $w = gu$.

The set of all algebraic unifiers $U_E(A)$ of a finitely presented algebra A forms a quasi-ordered set with the quasi-ordering $\preceq$.

The algebraic unification type of an algebraically unifiable finitely presented algebra A in the variety $\mathbf{V}$ is now defined exactly as in the symbolic case, using the quasi-ordering set $(U_E(A), \preceq)$.

Theorem 11. *The unification type of the equational class* $\mathbf{K_n}$ *is* 1*, i. e., unitary.*

Proof. The proof immediately follows from Theorem 5. Indeed, any finitely presented MMV-algebra in the variety $\mathbf{K_n}$ is finite. The finitely presented projective algebras are those of the kind $(S_1^1, \exists_d) \times A'$ (Theorem 5). Let the MMV-algebra A be unifiable. It means that there is a homomorphism from A into a projective algebra, say $B \times (S_1^1, \exists_d)$, hence also a homomorphism $h : A \to (S_1^1, \exists_d)$. Then A is a retract of $A \times (S_1^1, \exists_d)$ which is projective. Indeed, we can take the homomorphism $\varepsilon = (Id_A, h) : A \to (A \times (S_1^1, \exists_d))$ (i. e., $\varepsilon(a) = (a, h(a))$) and the projection $\pi : A \times (S_1^1, \exists_d) \to A$ so that $h\varepsilon = Id_A$ (i. e., identity homomorphisms are most general unifications). It is obvious that any projective algebra is unifiable. Thereby we have shown that an MMV-algebra A is unifiable, iff it is projective. □

Acknowledgements

The second author is supported by the grant (French-Georgian) CNRS-SNRSF # 09/09 and the grant SNRSF # 31/08. We also express our gratitude to the referee for her/his suggestions to improve the readability of this paper.

References

1. L. P. Belluce, Further results on infinite valued predicate logic. *J. Symbolic Logic* **29** (1964), 69–78.
2. L. P. Belluce, C. C. Chang, A weak completeness theorem for infinite valued first-order logic. *J. Symbolic Logic* **28** (1963), 43–50.
3. L. P. Belluce, R. Grigolia and A. Lettieri, Representations of monadic MV-algebras. *Studia Logica* **81** (2005), no. 1, 125–144.
4. C. C. Chang, Algebraic analysis of many valued logics. *Trans. Amer. Math. Soc.* **88** (1958), 467–490.
5. R. Cignoli, E. J. Dubuc, D. Mundici, Extending Stone duality to multisets and locally finite MV-algebras. *J. Pure Appl. Algebra* **189** (2004), 37–59.
6. A. Di Nola, R. Grigolia, G. Panti, Finitely generated free MV-algebras and their automorphism groups. Many-valued logics. *Studia Logica* **61** (1998), no. 1, 65–78.
7. R. Cignoli, L. Monteiro, Maximal subalgebras of MV$_n$-algebras. A proof of a conjecture of A. Monteiro. *Studia Logica* **84** (2006), 393–405.
8. A. Di Nola, R. Grigolia, On Monadic MV-algebras. *Ann. Pure and Appl. Logic* **128** (2004), no. 1–3, 125–139.
9. A. Di Nola, R. Grigolia, Projective MV-Algebras and their automorphism groups. *J. Mult.-Valued Logic Soft Comput.* **9** (2003), 291–317.
10. G. Georgescu, A. Iurgulescu, I. Leustean, Monadic and closure MV-Algebras. *Multi. Val. Logic* **3** (1998), 235–257.
11. S. Ghilardi, Unification through projectivity. *J. Logic Comput.* **7** (1997), 733–752.
12. S. Ghilardi, Unification, finite duality and projectivity in varieties of Heyting algebras. Provinces of logic determined. *Ann. Pure Appl. Logic* **127** (2004), no. 1–3, 99–115.
13. G. Grätzer, *Universal Algebra.* Second edition. Springer-Verlag, New York-Heidelberg, 1979.
14. L. S. Hay, *An axiomatization of the infinitely many-valued calculus.* M.S. Thesis, Cornell University, 1958.
15. J. Łukasiewicz, A. Tarski, Untersuchungen über den aussagenkalkül. *Comptes Rendus des de la Societe des Sciences et des Lettres de Varsovie*, **23** (1930), 51–77.
16. J. D. Rutledge, *A preliminary investigation of the infinitely many-valued predicate calculus.* Thesis (Ph. D.), Cornell University. ProQuest LLC, Ann Arbor, MI, 1959.
17. B. Scarpellini, Die Nichtaxiomatisierbarkeit des unendlichwertigen Predikaten-kalkulus von Łukasiewicz. *J. Symbolic Logic* **27** (1962), 159–170.
18. D. Schwartz, Theorie der polyadischen MR-algebren endlicher Ordnung. *Math. Nachr.* **78** (1977), 131–138.
19. D. Schwartz, Polyadic MV-algebras. *Z. Math. Logik Grundlag. Math.* **26** (1980), no. 6, 561–564.

(Received 26.03.2019)

[1]Department of Mathematics, University of Salerno
E-mail address: adinola@unisa.it
E-mail address: gilenzi@unisa.it

[2]Tbilisi State University
E-mail address: revaz.grigolia@tsu.ge, revaz.grigolia359@gmail.com