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SOLUTION FOR SYSTEM OF IMPLICIT ORDERED VARIATIONAL INCLUSIONS

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Abstract. The purpose of this paper is to present an existence theorem for a new class of *system of implicit ordered variational inclusions* in real ordered Banach spaces. Using the concept of resolvent operator, we prove the convergence of sequences generated by an algorithm.

1. Brief Prehistory

Generalized nonlinear ordered variational inclusions have wide applications in many fields including, for example, mathematical physics, optimization and control theory, mathematical programming, economics and engineering sciences. Recently, nonlinear mappings, fixed point theory and their applications have been extensively studied in ordered Banach spaces. In 2008, H.G. Li [6] introduced the generalized nonlinear ordered variational inequalities, studied an approximation algorithm and an approximate solution for a class of generalized nonlinear ordered variational inequalities in ordered Banach spaces. In 2009, by using the B-restricted accretive method of the mapping A with constants α_1, α_2, Li [7] studied a new class of general nonlinear ordered variational equations and established an existence theorem and an approximation algorithm of solutions for this kind of generalized nonlinear ordered variational equations in ordered Banach spaces.

Motivated and inspired by the recent research works [1–5,9,13], in this paper, we consider a *system of implicit ordered variational inclusions* in real ordered Banach spaces. We design an iterative algorithm based on the resolvent operator for solving a *system of implicit ordered variational inclusions*. We prove an existence, as well as a convergence theorem for our problem.

2. Prelude

Definition 2.1. Let $C(\neq \emptyset)$ be a closed, convex subset of X. C is said to be a pointed cone if

(i) for $x \in C$ and $\lambda > 0$, $\lambda x \in C$;
(ii) if x and $-x \in C$, then $x = \theta$,

where θ is a zero vector in X.

Definition 2.2 ([4]). C is called a normal cone if and only if there exists a constant $\lambda_C > 0$ such that $\theta \leq x \leq y$ implies $\|x\| \leq \lambda_C \|y\|$, where λ_C is called the normal constant of C.

Definition 2.3 ([12]). For arbitrary elements $x, y \in X$, $x \leq y$ if and only if $x - y \in C$, then the relation $\leq$ is a partial ordered relation in X. The real Banach space X with the ordered relation $\leq$ defined by C is called a real ordered Banach space.

Throughout this paper, we assume X to be a real ordered Banach space with norm $\|\cdot\|$, an order pair $\langle \cdot, \cdot \rangle$ and partial ordered relation $\leq$ defined by the normal cone C with a normal constant λ_C. Let $CB(X)$ be the family of all nonempty closed and bounded subsets of X, and $\mathfrak{D}$ be the Hausdorff metric defined on $CB(X)$ by

$$\mathfrak{D}(A\,, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(A, y)\},$$

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where $A, B \in CB(X), d(x, B) = \inf_{y \in B} d(x, y)$.

Definition 2.4 ([13]). For arbitrary elements $x, y \in X$, if $x \leq y$ or $y \leq x$, then x and y are called comparable and this is denoted by $x \propto y$.

Lemma 2.5 ([13]). Let X be an ordered Banach space. For arbitrary $x, y \in X$, $lub\{x, y\}$ and $glb\{x, y\}$ express the least upper bound of the set $\{x, y\}$ and the greatest lower bound of the set $\{x, y\}$ on the partial ordered relation $\leq$, *respectively.* Suppose $glb\{x, y\}$ and $lub\{x, y\}$ exist. Some binary operators can be defined as follows:

- $x \vee y = lub\{x, y\}$;
- $x \wedge y = glb\{x, y\}$;
- $x \oplus y = (x - y) \vee (y - x)$.

$\vee$, $\wedge$ and $\oplus$ are called OR, AND and XOR operation, *respectively.* For arbitrary $x, y, w \in X$, the following relations hold:

(i) if $x \leq y$, then $x \vee y = y$, $x \wedge y = x$;
(ii) if x and y are comparable, then $\theta \leq x \oplus y$;
(iii) $(x + w) \vee (y + w)$ exists and $(x + w) \vee (y + w) = (x \vee y) + w$;
(iv) $(x + w) \wedge (y + w)$ exists and $(x + w) \wedge (y + w) = (x \wedge y) + w$;
(v) $(x \wedge y) = (x + y) - (x \vee y)$;
(vi) if $\lambda \geq 0$, then $\lambda(x \vee y) = \lambda x \vee \lambda y$;
(vii) if $\lambda \leq 0$, then $\lambda(x \wedge y) = \lambda x \vee \lambda y$;
(viii) $x \wedge y = -(-x \vee -y)$ and $(-x) \wedge (x) \leq \theta \leq (-x) \vee x$;
(ix) if $x \leq y$ and $s \leq t$ then $x + s \leq y + t$;
(x) if $\theta \leq x$ and $x \neq \theta$, and $\alpha > 0$ then $\theta \leq \alpha x$ and $\alpha x \neq \theta$;
(xi) if X is an ordered Banach space, and if for any $x, y \in X$, either $x \vee y$ and $x \wedge y$ exist, then X is a Banach lattice.

Definition 2.6 ([8]). Let $A : X \longrightarrow X$ be a single-valued mapping.

(i) A is said to be comparison mapping if for each $x, y \in X$, $x \propto y$, then $A(x) \propto A(y), x \propto A(x)$ and $y \propto A(y)$;
(ii) A is said to be strongly comparison mapping if A is a comparison mapping and $A(x) \propto A(y)$, if and only if $x \propto y$;
(iii) A is said to be β-ordered compression mapping if it is a comparison mapping and there exists a constant $0 < \beta < 1$ such that
$$A(x) \oplus A(y) \leq \beta(x \oplus y);$$
(iv) A is said to be γ-order non-extended mapping if there exists a constant $\gamma > 0$ such that
$$\gamma(x \oplus y) \leq A(x) \oplus A(y), \forall x, y \in X.$$

Lemma 2.7 ([4]). If x and y are comparable, then $lub\{x, y\}$ and $glb\{x, y\}$ exist,
$$x - y \propto y - x, \text{ and } \theta \leq (x - y) \vee (y - x).$$

Lemma 2.8 ([4]). If for any natural number $n, x \propto y_n$ and $y_n \longrightarrow y(n \longrightarrow \infty)$, then $x \propto y$.

Lemma 2.9 ([4]). Let C be a normal cone with a normal constant λ_C in X, then for each $x, y \in X$, we have the relations:

(i) $\|\theta \oplus \theta\| = \|\theta\| = \theta$;
(ii) $\|x \wedge y\| \leq \|x\| \wedge \|y\| \leq \|x\| + \|y\|$;
(iii) $\|x \oplus y\| \leq \|x - y\| \leq \lambda_C \|x \oplus y\|$;
(iv) if $x \propto y$, then $\|x \oplus y\| = \|x - y\|$.

Lemma 2.10 ([8,9]). Let $\leq$ be a partial order relation defined by the cone C with a normal constant λ_C in X in Definition 2.3. Then the following relations are satisfied:

(i) $x \oplus y = y \oplus x$, $x \oplus x = \theta$;
(ii) $\theta \leq x \oplus \theta$;

(iii) $(x \oplus \theta) - (y \oplus \theta) \leq (x - y) \oplus \theta$;
(iv) if $x \propto \theta$, then $-x \oplus \theta \leq x \leq x \oplus \theta$;
(v) if $x \propto y$, then $(x \oplus \theta) \oplus (y \oplus \theta) \leq (x \oplus y) \oplus \theta$;
(vi) allow λ to be real, then $(\lambda x) \oplus (\lambda y) =| \lambda | (x \oplus y)$;
(vii) if x, y and w are comparable, then $(x \oplus y) \leq (x \oplus w) + (w \oplus y)$;
(viii) if x, y, r, w are comparable, then

$$(x \wedge y) \oplus (r \wedge w) \leq ((x \oplus r) \vee (y \oplus w)) \wedge ((x \oplus w) \vee (y \oplus r));$$

(ix) let $(x + y) \vee (s + t)$ exist, and if $x \propto s, t$, and $y \propto s, t$, then

$$(x + y) \oplus (s + t) \leq (x \oplus s + y \oplus t) \wedge (x \oplus t + y \oplus s).$$

Definition 2.11 ([11])**.** Allow $A : X \longrightarrow X$ and $M : X \longrightarrow CB(X)$ to be the mappings.

(i) M is called a weak-comparison mapping, if given comparable $x, y \in X$ and $t_x \in M(x)$, there exists $t_y \in M(y)$ such that t_x and t_y are comparable.
(ii) M is called an α-weak-non-ordinary difference mapping associated with A, if it is a weak comparison and there exists $\alpha > 0$, and $t_x \in M(A(x))$ and $t_y \in M(A(y))$ such that

$$(t_x \oplus t_y) \oplus \alpha(A(x) \oplus A(y)) = \theta.$$

(iii) M is called a λ-order different weak-comparison mapping associated with A if for the given comparable $x, y \in X$, there exist $\lambda > 0$, and $t_x \in M(A(x)), t_y \in M(A(y))$ such that

$$\lambda(t_x - t_y) \propto x - y.$$

(iv) M (a weak-comparison map) is called an ordered (α_A, λ)-weak-ANODM mapping, if it is α-weak-non-ordinary difference mapping and λ-order different weak-comparison mapping associated with A, and $(A + \lambda M)(X) = X$, for $\alpha, \lambda > 0$.

Definition 2.12 ([11])**.** Let $M : X \longrightarrow CB(X)$ be γ-order non-extended mapping and α-non-ordinary difference mapping with respect to a mapping $A : X \longrightarrow X$. The resolvent operator $R^M_{A,\lambda} : X \longrightarrow X$ associated with both A and M is defined by

$$R^M_{A,\lambda}(x) = (A + \lambda M)^{-1}(x), \text{ for all } x \in X, \tag{2.1}$$

where $\gamma, \alpha, \lambda > 0$ are the constants.

Definition 2.13 ([6])**.** A bi-mapping $B : X \times X \longrightarrow X$ is called (α_1, α_2)-restricted-accretive if it is comparison and there exist constants $0 \leq \alpha_1, \alpha_2 \leq 1$ such that

$$B(x_1, y_1) \oplus B(x_2, y_2) \leq \alpha_1(B(x_1) \oplus B(x_2)) + \alpha_2(B(y_1) \oplus B(y_2)), \text{ for all } x_1, x_2, y_1, y_2 \in X.$$

Lemma 2.14 ([11])**.** Let $M : X \longrightarrow CB(X)$ be γ-order non-extended and α-weak non-ordinary difference mapping associated with a mapping $A : X \longrightarrow X$, and $\alpha\gamma \neq 1$, then $M_\theta = \{\theta \oplus x \mid x \in M\}$ is α-weak non-ordinary difference mapping associated with A and the resolvent operator $R^{M_\theta}_{A,\lambda} = (A + \lambda M_\theta)^{-1}$ of $(A + \lambda M_\theta)$ is a single valued for $\alpha, \lambda > 0$, *i.e.*, $R^{M_\theta}_{A,\lambda} : X \longrightarrow X$ of M_θ holds.

Lemma 2.15 ([11])**.** Let $A : X \longrightarrow X$ be a mapping and $M : X \longrightarrow CB(X)$ be (α_A, λ)-weak-ANODD set-valued and strongly comparison mapping associated with $R^M_{A,\lambda}$. Then the resolvent operator $R^M_{A,\lambda} : X \longrightarrow X$ is a comparison mapping.

Lemma 2.16 ([11])**.** Let $A : X \longrightarrow X$ be a mapping and $M : X \longrightarrow CB(X)$ be ordered (α_A, λ)-weak-ANODD and γ-ordered non-extended mapping associated with $R^M_{A,\lambda}$, for $\alpha_A > \frac{1}{\lambda}$. Then the following relation

$$R^M_{A,\lambda}(x) \oplus R^M_{A,\lambda}(y) \leq \frac{1}{\gamma(\alpha_A\lambda - 1)}(x \oplus y), \text{ for all } x, y \in X. \tag{2.2}$$

holds.

3. Formulation of the Problem

Let X be a real Banach space and C be a normal cone having the normal constant λ_C. Suppose $f_i, g_i : X \longrightarrow X$ $(i = 1, 2)$ and $Q_i : X \times X \longrightarrow X$ $(i = 1, 2)$ are single-valued mappings. Assume that $T_1, T_2, F_1, F_2 : X \longrightarrow CB(X)$ and $M, N : X \times X \longrightarrow CB(X)$ are set-valued mappings. Now we look at the problem: for some $(w_1, w_2) \in X \times X$ and $\rho_1, \rho_2 > 0$, find $x, y \in X$, $u \in T_1(x)$, $v \in T_2(y)$, $p \in F_1(x)$, $q \in F_2(y)$ such that

$$\begin{aligned} w_1 &\in Q_1(f_1(x), v) + \rho_1 M(g_1(x), q), \\ w_2 &\in Q_2(u, f_2(y)) + \rho_2 N(p, g_2(y)). \end{aligned} \tag{3.1}$$

Problem (3.1) is called *a system of implicit ordered variational inclusions.*

Special Cases:

(1) If T_1, T_2, F_1, F_2 are single-valued mappings, then (3.1) reduces to the problem of finding some $(w_1, w_2) \in X \times X$, $\rho_1, \rho_2 > 0$, and $x, y \in X$ such that

$$\begin{aligned} w_1 &\in Q_1(f_1(x), T_2(y)) + \rho_1 M(g_1(x), F_2(y)), \\ w_2 &\in Q_2(T_1(x), f_2(y)) + \rho_2 N(F_1(x), g_2(y)), \end{aligned} \tag{3.2}$$

called *a system of generalized ordered variational inclusions.*

(2) If T_1, T_2, F_1, F_2 are identity mappings, then (3.2) reduces to the problem of finding some $(w_1, w_2) \in X \times X$, $\rho_1, \rho_2 > 0$, and $x, y \in X$ such that

$$\begin{aligned} w_1 &\in Q_1(f_1(x), y) + \rho_1 M(g_1(x), y), \\ w_2 &\in Q_2(x, f_2(y)) + \rho_2 N(x, g_2(y)), \end{aligned} \tag{3.3}$$

called *a system of general ordered variational inclusions.*

(3) If $g_1 = f_2 = I$ (the identity mapping on X), M and N are single-valued mappings and $M(g_1(x), y) = M(x, y)$, then (3.3) reduces to the problem of finding some $w_1, w_2 \in X$, and $x, y \in X$ such that

$$\begin{aligned} w_1 &\in Q_1(f_1(x), y) + \rho_1 M(y, x), \\ w_2 &\in Q_2(x, y) + \rho_2 N(x, g_2(y)), \end{aligned} \tag{3.4}$$

a variant form studied in [9].

(4) If $w_2 = 0$, $Q_2 = f_2 = N = g_2 = 0$, then problem (3.4) is to find $x, y \in X$ such that

$$w_1 \in Q_1(f_1(x), y) + \rho_1 M(y, x), \tag{3.5}$$

a variant form of generalized variational inclusions.

(5) If $\rho_1 = 1$, $w_1 = 0$, then problem (3.5) reduces to finding $x, y \in X$ such that

$$0 \in Q_1(f_1(x), y) + M(y, x), \tag{3.6}$$

considered and studied in [14].

(6) If $\rho_1 = \rho$, $w_1 = w$, $Q_1(f_1(x), y) = f(x)$ and $M(y, x) = M(x)$, then problem (3.5) becomes that of finding $x \in X$ such that

$$w \in f(x) + \rho M(x). \tag{3.7}$$

Problem (3.7) was studied in [11].

(7) If $f = 0$ is a zero mapping, then problem (3.7) reduces to finding $x \in X$ such that

$$w \in \rho M(x). \tag{3.8}$$

Problem (3.9) was initiated and studied in [10].

Now, we mention the fixed point formulation of (3.1).

Lemma 3.1. Let $x, y \in X$, $u \in T(x) \in CB(X)$, $v \in T(y) \in CB(X)$, $p \in F_1(x) \in CB(X)$, $q \in F_2(y) \in CB(X)$ be a solution of (3.1) if and only if $x, y \in X$, $u \in T_1(x) \in CB(X)$, $v \in T_2(y) \in CB(X)$, $p \in F_1(x) \in CB(X)$, $q \in F_2(y) \in CB(X)$ fulfill the following relations:

$$x = R_{A,\lambda}^{M(g_1(\cdot),q)}\Big[A(x) + \frac{\lambda}{\rho_1}(w_1 - Q_1(f_1(x), v))\Big],$$
$$y = R_{A,\lambda}^{N(p,g_2(\cdot))}\Big[A(y) + \frac{\lambda}{\rho_2}(w_2 - Q_2(u, f_2(y)))\Big]. \tag{3.9}$$

Proof. The proof follows from the definition of the resolvent operator (2.1). □

4. Main Results

In this section, we present existence results for the *system of implicit ordered variational inclusions* under some suitable conditions. Let us also discuss the convergence of sequences suggested by an iterative algorithm.

Theorem 4.1. Let C be a normal cone having a normal constant λ_C in a real ordered Banach space X. Let $A, f_i, g_i : X \longrightarrow X$ be single-valued mappings such that A is a λ_A-compression, f_i is λ_{f_i}-compression and g_i is comparison mappings for $i = 1, 2$. Let $Q_i : X \times X \longrightarrow X$ $(i = 1, 2)$ be single-valued mappings such that Q_1 is an (α_1, α_2)-restricted-accretive mapping with respect to f_1, and Q_2 is (β_1, β_2)-restricted accretive mapping with respect to f_2. Suppose that $T_i, F_i : X \longrightarrow CB(X)$ $(i = 1, 2)$ be the $\mathfrak{D}$-Lipschitz continuous mappings with respect to the constants $\varrho_i, \sigma_i > 0$. Suppose $M, N : X \times X \longrightarrow CB(X)$ are the mappings such that M is (α_A, λ)-weak-ANODD and N is $(\alpha_{A'}, \lambda)$-weak-ANODD set-valued mappings.
In addition, if $x_i \propto y_i$, $u_i \propto v_i$, $p_i \propto q_i$, $R^M_{A,\lambda}(x_i) \propto R^M_{A,\lambda}(y_i)$, $R^N_{A,\lambda_2}(x_i) \propto R^N_{A,\lambda}(y_i)$ $(i = 1, 2)$ and for all $\lambda_i, \delta_i > 0$ $(i = 1, 2)$, the following condition

$$R_{A,\lambda}^{M(g_1(\cdot),q_1)}(x_1) \oplus R_{A,\lambda}^{M(g_1(\cdot),q_2)}(x_1) \le \delta_2(q_1 \oplus q_2),$$
$$R_{A,\lambda}^{N(p_1,g_2(\cdot))}(y_1) \oplus R_{A,\lambda}^{N(p_2,g_2(\cdot))}(y_1) \le \delta_1(p_1 \oplus p_2), \tag{4.1}$$

and

$$\frac{\lambda_C}{\rho_1\rho_2}[\rho_2\mu_1\lambda\alpha_1\lambda_{f_1} + \rho_1\mu_2\lambda\beta_1\varrho_1] < 1 - \lambda_C(\mu_1\lambda_A + \delta_1\sigma_1),$$
$$\frac{\lambda_C}{\rho_1\rho_2}[\rho_1\mu_2\lambda\beta_2\lambda_{f_2} + \rho_2\mu_1\lambda\alpha_2\varrho_2] < 1 - \lambda_C(\mu_2\lambda_A + \delta_2\sigma_2) \tag{4.2}$$

are satisfied. Then (3.1) grants a solution (x, y, u, v, p, q).

Proof. From Lemma 2.16, we know that the resolvent operators $R^M_{A,\lambda}(\cdot)$ and $R^N_{A,\lambda}(\cdot)$ are Lipschitz continuous with the constants $\mu_1 = \frac{1}{\gamma_1(\alpha_A\lambda - 1)}$ and $\mu_2 = \frac{1}{\gamma_2(\alpha_{A'}\lambda - 1)}$, *respectively.*
Now, define a mapping $P : X \times X \longrightarrow X \times X$ by

$$P(x, y) = (G(x_i, y_i), S(x_i, y_i)), \ \forall (x, y) \in X \times X, \ (i = 1, 2) \tag{4.3}$$

where $G, S : X \times X \longrightarrow X$ are the mappings defined as

$$G(x_i, y_i) = R_{A,\lambda}^{M(g_1(\cdot),q_i)}\Big[A(x_i) + \frac{\lambda}{\rho_1}(w_1 - Q_1(f_1(x_i), v_i))\Big], \tag{4.4}$$

and

$$S(x_i, y_i) = R_{A,\lambda}^{N(p_i,g_2(\cdot))}\Big[A(y_i) + \frac{\lambda}{\rho_2}(w_2 - Q_2(u_i, f_2(y_i)))\Big]. \tag{4.5}$$

For any $x_i, y_i \in X$ and $x_i \propto y_j$, $u_i \propto v_j$, $p_i \propto q_j$ $(i, j = 1, 2)$. By using (4.4), Definition 2.6, Definition 2.13 and Lemmas 2.16 and 2.10, we have

$$0 \le G(x_1, y_1) \oplus G(x_2, y_2)$$
$$= R_{A,\lambda}^{M(g_1(\cdot),q_1)}\Big[A(x_1) + \frac{\lambda}{\rho_1}(w_1 - Q_1(f_1(x_1), v_1))\Big] \oplus R_{A,\lambda}^{M(g_1(\cdot),q_2)}\Big[A(x_2) + \frac{\lambda}{\rho_1}(w_1 - Q_1(f_1(x_2), v_2))\Big]$$

$$\le R_{A,\lambda}^{M(g_1(\cdot),q_1)}\Big[A(x_1)+\frac{\lambda}{\rho_1}(w_1-Q_1(f_1(x_1),v_1))\Big]\oplus R_{A,\lambda}^{M(g_1(\cdot),q_1)}\Big[A(x_2)+\frac{\lambda}{\rho_1}(w_1-Q_1(f_1(x_2),v_2))\Big]$$
$$\oplus R_{A,\lambda}^{M(g_1(\cdot),q_1)}\Big[A(x_2)+\frac{\lambda}{\rho_1}(w_1-Q_1(f_1(x_2),v_2))\Big]\oplus R_{A,\lambda}^{M(g_1(\cdot),q_2)}\Big[A(x_2)+\frac{\lambda}{\rho_1}(w_1-Q_1(f_1(x_2),v_2))\Big]$$
$$\le \mu_1\Big[A(x_1)\oplus A(x_2)+\frac{\lambda}{\rho_1}(Q_1(f_1(x_1),v_1)\oplus Q_1(f_1(x_2),v_2))\Big]\oplus\delta_2(q_1\oplus q_2)$$
$$\le \mu_1\Big[A(x_1)\oplus A(x_2)+\frac{\lambda}{\rho_1}(\alpha_1(f_1(x_1)\oplus f_1(x_2))+\alpha_2(v_1\oplus v_2))\Big]\oplus\delta_2(q_1\oplus q_2)$$
$$\le \mu_1\Big[\lambda_A(x_1\oplus x_2)+\frac{\lambda}{\rho_1}(\alpha_1\lambda_{f_1}(x_1\oplus x_2)+\alpha_2(v_1\oplus v_2))\Big]\oplus\delta_2(q_1\oplus q_2)$$
$$\le \mu_1\Big[\Big(\lambda_A+\frac{\lambda\alpha_1\lambda_{f_1}}{\rho_1}\Big)(x_1\oplus x_2)+\frac{\lambda\alpha_2}{\rho_1}(v_1\oplus v_2)\Big]\oplus\delta_2(q_1\oplus q_2). \tag{4.6}$$

From Definition 2.2 and Lemma 2.9, we have

$$\|G(x_1,y_1)\oplus G(x_2,y_2)\| = \|G(x_1,y_1)-G(x_2,y_2)\|$$
$$\le \lambda_C\Big\|\mu_1\Big[\Big(\lambda_A+\frac{\lambda\alpha_1\lambda_{f_1}}{\rho_1}\Big)(x_1\oplus x_2)+\frac{\lambda\alpha_2}{\rho_1}(v_1\oplus v_2)\Big]\oplus\delta_2(q_1\oplus q_2)\Big\|$$
$$\le \lambda_C\Big\{\mu_1\Big\|\Big(\lambda_A+\frac{\lambda\alpha_1\lambda_{f_1}}{\rho_1}\Big)(x_1\oplus x_2)\Big\|+\mu_1\Big\|\frac{\lambda\alpha_2}{\rho_1}(v_1\oplus v_2)\|+\delta_2\|q_1\oplus q_2\Big\|\Big\}$$
$$\le \lambda_C\Big\{\mu_1\Big(\lambda_A+\frac{\lambda\alpha_1\lambda_{f_1}}{\rho_1}\Big)\|x_1-x_2\|+\mu_1\frac{\lambda\alpha_2}{\rho_1}\|v_1-v_2\|+\delta_2\|q_1-q_2\|\Big\}$$
$$\le \lambda_C\Big\{\mu_1\Big(\lambda_A+\frac{\lambda\alpha_1\lambda_{f_1}}{\rho_1}\Big)\|x_1-x_2\|+\frac{\mu_1\lambda\alpha_2}{\rho_1}\mathfrak{D}(T_2(y_1),T_2(y_2))+\delta_2\mathfrak{D}(F_2(y_1),F_2(y_2))\Big\}$$
$$\le \lambda_C\Big\{\frac{\mu_1(\lambda_A\rho_1+\lambda\alpha_1\lambda_{f_1})}{\rho_1}\|x_1-x_2\|+\frac{\mu_1\lambda\alpha_2\varrho_2}{\rho_1}\|y_1-y_2\|+\delta_2\sigma_2\|y_1-y_2\|\Big\}$$
$$\le \lambda_C\Big\{\frac{\mu_1(\lambda_A\rho_1+\lambda\alpha_1\lambda_{f_1})}{\rho_1}\|x_1-x_2\|+\frac{\mu_1\lambda\alpha_2\varrho_2+\delta_2\sigma_2\rho_1}{\rho_1}\|y_1-y_2\|\Big\}.$$

That is,

$$\|G(x_1,y_1)-G(x_2,y_2)\|\le\lambda_C\frac{\mu_1(\lambda_A\rho_1+\lambda\alpha_1\lambda_{f_1})}{\rho_1}\|x_1-x_2\| + \lambda_C\frac{(\mu_1\lambda\alpha_2\varrho_2+\rho_1\delta_2\sigma_2)}{\rho_1}\|y_1-y_2\|. \tag{4.7}$$

Again,

$$0\le S(x_1,y_1)\oplus S(x_2,y_2)$$
$$= R_{A,\lambda}^{N(p_1,g_2(\cdot))}\Big[A(y_1)+\frac{\lambda}{\rho_2}(w_2-Q_2(u_1,f_2(y_1)))\Big]\oplus R_{A,\lambda}^{N(p_2,g_2(\cdot))}\Big[A(y_2)+\frac{\lambda}{\rho_2}(w_2-Q_2(u_2,f_2(y_2)))\Big]$$
$$\le R_{A,\lambda}^{N(p_1,g_2(\cdot))}\Big[A(y_1)+\frac{\lambda}{\rho_2}(w_2-Q_2(u_1,f_2(y_1)))\Big]\oplus R_{A,\lambda}^{N(p_1,g_2(\cdot))}\Big[A(y_2)+\frac{\lambda}{\rho_2}(w_2-Q_2(u_2,f_2(y_2)))\Big]$$
$$\oplus R_{A,\lambda}^{N(p_1,g_2(\cdot))}\Big[A(y_2)+\frac{\lambda}{\rho_2}(w_2-Q_2(u_2,f_2(y_2)))\Big]\oplus R_{A,\lambda}^{N(p_2,g_2(\cdot))}\Big[A(y_2)+\frac{\lambda}{\rho_2}(w_2-Q_2(u_2,f_2(y_2)))\Big]$$
$$\le \mu_2\Big[A(y_1)\oplus A(y_2)+\frac{\lambda}{\rho_2}(Q_2(u_1,f_2(y_1))\oplus Q_2(u_2,f_2(y_2)))\Big]\oplus\delta_1(p_1\oplus p_2)$$
$$\le \mu_2\Big[A(y_1)\oplus A(y_2)+\frac{\lambda}{\rho_2}(\beta_1(u_1\oplus u_2)+\beta_2\lambda_{f_2}(y_1\oplus y_2))\Big]\oplus\delta_1(p_1\oplus p_2)$$
$$\le \mu_2\Big[\lambda_A(y_1\oplus y_2)+\frac{\lambda}{\rho_2}(\beta_1(u_1\oplus u_2)+\beta_2\lambda_{f_2}(y_1\oplus y_2))\Big]\oplus\delta_1(p_1\oplus p_2). \tag{4.8}$$

From Definition 2.2 and Lemma 2.9, we have

$$\begin{aligned}
&\|S(x_1,y_1)\oplus S(x_2,y_2)\| = \|S(x_1,y_1)-S(x_2,y_2)\|\\
&\le \lambda_C\Big\|\mu_2\lambda_A(y_1\oplus y_2)+\frac{\mu_2\lambda\beta_1}{\rho_2}(u_1\oplus u_2)+\frac{\mu_2\lambda\beta_2\lambda_{f_2}}{\rho_2}(y_1\oplus y_2)+\delta_1(p_1\oplus p_2)\Big\|\\
&\le \lambda_C\Big[\mu_2\lambda_A\|y_1-y_2\|+\frac{\mu_2\lambda\beta_2\lambda_{f_2}}{\rho_2}\|y_1-y_2\|+\frac{\mu_2\lambda\beta_1}{\rho_2}|u_1-u_2\|+\delta_1\|p_1-p_2\|\Big]\\
&\le \lambda_C\Big[\Big(\mu_2\lambda_A+\frac{\mu_2\lambda\beta_2\lambda_{f_2}}{\rho_2}\Big)\|y_1-y_2\|+\frac{\mu_2\lambda\beta_1}{\rho_2}\mathfrak{D}(T_1(x_1),T_1(x_2))+\delta_1\mathfrak{D}(F_1(x_1),F_1(x_2))\Big]\\
&\le \lambda_C\Big[\Big(\mu_2\lambda_A+\frac{\mu_2\lambda\beta_2\lambda_{f_2}}{\rho_2}\Big)\|y_1-y_2\|+\frac{\mu_2\lambda\beta_1\varrho_1}{\rho_2}\|x_1-x_2\|+\delta_1\sigma_1\|x_1-x_2\|\Big]\\
&\le \lambda_C\Big(\mu_2\lambda_A+\frac{\mu_2\lambda\beta_2\lambda_{f_2}}{\rho_2}\Big)\|y_1-y_2\|+\lambda_C\Big(\frac{\mu_2\lambda\beta_1\varrho_1}{\rho_2}+\delta_1\sigma_1\Big)\|x_1-x_2\|.
\end{aligned}$$

That is,

$$\|S(x_1,y_1)-S(x_2,y_2)\|\le\frac{\lambda_C\mu_2(\lambda_A\rho_2+\lambda\beta_2\lambda_{f_2})}{\rho_2}\|y_1-y_2\|+\frac{\lambda_C(\mu_2\lambda\beta_1\varrho_1+\rho_2\delta_1\sigma_1)}{\rho_2}\|x_1-x_2\|. \quad (4.9)$$

From (4.7) and (4.9), we have

$$\begin{aligned}
&\|G(x_1,y_1)-G(x_2,y_2)\|+\|S(x_1,y_1)-S(x_2,y_2)\|\le\lambda_C\frac{\mu_1(\lambda_A\rho_1+\lambda\alpha_1\lambda_{f_1})}{\rho_1}\|x_1-x_2\|\\
&+\lambda_C\frac{(\mu_1\lambda\alpha_2\varrho_2+\rho_1\delta_2\sigma_2)}{\rho_1}\|y_1-y_2\|+\lambda_C\frac{\mu_2(\lambda_A\rho_2+\lambda\beta_2\lambda_{f_2})}{\rho_2}\|y_1-y_2\|\\
&+\lambda_C\frac{(\mu_2\lambda\beta_1\varrho_1+\delta_1\sigma_1\rho_2)}{\rho_2}\|x_1-x_2\|\\
&\le\lambda_C\Big[\frac{\mu_1(\lambda_A\rho_1+\lambda\alpha_1\lambda_{f_1})}{\rho_1}+\frac{(\mu_2\lambda\beta_1\varrho_1+\delta_1\sigma_1\rho_2)}{\rho_2}\Big]\|x_1-x_2\|\\
&+\lambda_C\Big[\frac{\mu_2(\lambda_A\rho_2+\lambda\beta_2\lambda_{f_2})}{\rho_2}+\frac{(\mu_1\lambda\alpha_2\varrho_2+\rho_1\delta_2\sigma_2)}{\rho_1}\Big]\|y_1-y_2\|\\
&\le\frac{\lambda_C}{\rho_1\rho_2}[\mu_1\rho_2(\lambda_A\rho_1+\lambda\alpha_1\lambda_{f_1})+\rho_1(\mu_2\lambda\beta_1\varrho_1+\delta_1\sigma_1\rho_2)]\|x_1-x_2\|\\
&+\frac{\lambda_C}{\rho_1\rho_2}[\mu_2\rho_1(\lambda_A\rho_2+\lambda\beta_2\lambda_{f_2})+\rho_2(\mu_1\lambda\alpha_2\varrho_2+\rho_1\delta_2\sigma_2)]\|y_1-y_2\|\\
&\le\Omega_1\|x_1-x_2\|+\Omega_2\|y_1-y_2\|\\
&\le\max\{\Omega_1,\Omega_2\}(\|x_1-x_2\|+\|y_1-y_2\|), \quad (4.10)
\end{aligned}$$

where

$$\Omega_1=\frac{\lambda_C}{\rho_1\rho_2}\Big[\mu_1\rho_2(\lambda_A\rho_1+\lambda\alpha_1\lambda_{f_1})+\rho_1(\mu_2\lambda\beta_1\varrho_1+\delta_1\sigma_1\rho_2)\Big]$$

and

$$\Omega_2=\frac{\lambda_C}{\rho_1\rho_2}\Big[\mu_2\rho_1(\lambda_A\rho_2+\lambda\beta_2\lambda_{f_2})+\rho_2(\mu_1\lambda\alpha_2\varrho_2+\rho_1\delta_2\sigma_2)\Big].$$

Now, we define $\|(x,y)\|_*$ on $X\times X$ by

$$\|(x,y)\|_*=\|x\|+\|y\|,\ \forall(x,y)\in X\times X. \quad (4.11)$$

One can easily show that $(X\times X,\|\cdot\|)$ is a Banach space. Hence from (4.3), (4.10) and (4.11), we have

$$\|P(x_1,y_1)-P(x_2,y_2)\|_*\le\max\{\Omega_1,\Omega_2\}(\|x_1-x_2\|+\|y_1-y_2\|). \quad (4.12)$$

By (4.2), we know that $\max\{\Omega_1,\Omega_2\}<1$. It follows from (4.12) that P is a contraction mapping. Hence there exists unique $(x,y)\in X\times X$ such that

$$P(x,y)=(x,y).$$

This leads to

$$x = R_{A,\lambda}^{M(g_1(\cdot),q)}\Big[A(x) + \frac{\lambda}{\rho_1}(w_1 - Q_1(f_1(x), v))\Big],$$

and

$$y = R_{A,\lambda}^{N(p,g_2(\cdot))}\Big[A(y) + \frac{\lambda}{\rho_2}(w_2 - Q_2(u, f_2(y)))\Big].$$

It is determined by Lemma 3.1 that (x, y, u, v, p, q) is a solution of (3.1). □

Now, we suggest an iterative scheme for problem (3.1).

Algorithm 4.2. Let C be a normal cone with a normal constant λ_C in a real ordered Banach space X. Assume that $f_i, g_i : X \longrightarrow X$ and $Q_i : X \times X \longrightarrow X$ are single-valued mappings for $i = 1, 2$. Let $M, N : X \times X \longrightarrow CB(X)$ and $T_i, F_i : X \longrightarrow CB(X)(i = 1, 2)$ be the set-valued mappings.
For any given $x_0, y_0 \in X$, $u_0 \in T_1(x_0)$, $v_0 \in T_2(y_0)$, $p_0 \in F_1(x_0)$, $q_0 \in F_2(y_0)$, let

$$x_1 = (1-\pi)x_0 + \pi R_{A,\lambda}^{M(g_1(\cdot),q_0)}\Big[A(x_0) + \frac{\lambda}{\rho_1}(w_1 - Q_1(f_1(x_0), v_0))\Big],$$
$$y_1 = (1-\pi)y_0 + \pi R_{A,\lambda}^{N(p_0,g_2(\cdot))}\Big[A(y_0) + \frac{\lambda}{\rho_2}(w_2 - Q_2(u_0, f_2(y_0)))\Big],$$

there exist $u_1 \in T_1(x_1) \in CB(X)$, $v_1 \in T_2(y_1) \in CB(X)$, $p_1 \in F_1(x_1) \in CB(X)$, $q_1 \in F_2(y_1) \in CB(X)$, and assume that $x_0 \propto x_1$, $y_0 \propto y_1$, $u_0 \propto u_1$, $v_0 \propto v_1$, $p_0 \propto p_1$, $q_0 \propto q_1$ such that

$$\|u_1 \oplus u_0\| = \|u_1 - u_0\| \le (1+1)\mathfrak{D}(T_1(x_1), T_1(x_0));$$
$$\|v_1 \oplus v_0\| = \|v_1 - v_0\| \le (1+1)\mathfrak{D}(T_2(y_1), T_2(y_0));$$
$$\|p_1 \oplus p_0\| = \|p_1 - p_0\| \le (1+1)\mathfrak{D}(F_1(x_1), F_1(x_0));$$
$$\|q_1 \oplus q_0\| = \|q_1 - q_0\| \le (1+1)\mathfrak{D}(F_2(y_1), F_2(y_0)).$$

Continuing in this way, we can define iterative sequences $\{x_n\}, \{y_n\}, \{u_n\}, \{v_n\}, \{p_n\}, \{q_n\}$ with the supposition that $x_{n+1} \propto x_n, y_{n+1} \propto y_n, u_{n+1} \propto u_n, v_{n+1} \propto v_n, p_{n+1} \propto p_n, q_{n+1} \propto q_n$, for all $n \in \mathbb{R}$. We have the following iterative schemes:

$$x_{n+1} = (1-\pi)x_n + \pi R_{A,\lambda}^{M(g_1(\cdot),q_n)}\Big[A(x_n) + \frac{\lambda}{\rho_1}(w_1 - Q_1(f_1(x_n), v_n))\Big], \tag{4.13}$$
$$y_{n+1} = (1-\pi)y_n + \pi R_{A,\lambda}^{N(p_n,g_2(\cdot))}\Big[A(y_n) + \frac{\lambda}{\rho_2}(w_2 - Q_2(u_n, f_2(y_n)))\Big], \tag{4.14}$$

with

$$u_{n+1} \in T_1(x_{n+1}), \|u_{n+1} \oplus u_n\| = \|u_{n+1} - u_n\| \le \Big(1 + \frac{1}{n+1}\Big)\mathfrak{D}(T_1(x_{n+1}), T_1(x_n));$$
$$v_{n+1} \in T_2(y_{n+1}), \|v_{n+1} \oplus v_n\| = \|v_{n+1} - v_n\| \le \Big(1 + \frac{1}{n+1}\Big)\mathfrak{D}(T_2(y_{n+1}), T_2(y_n));$$
$$p_{n+1} \in F_1(x_{n+1}), \|p_{n+1} \oplus p_n\| = \|p_{n+1} - p_n\| \le \Big(1 + \frac{1}{n+1}\Big)\mathfrak{D}(F_1(x_{n+1}), F_1(x_n));$$
$$q_{n+1} \in F_2(x_{n+1}), \|q_{n+1} \oplus q_n\| = \|q_{n+1} - q_n\| \le \Big(1 + \frac{1}{n+1}\Big)\mathfrak{D}(F_2(y_{n+1}), F_2(y_n)); \tag{4.15}$$

for $n = 0, 1, 2, 3, 4, \ldots$, where $0 \le \pi < 1$ and $\lambda, \rho > 0$ are the constants.

Theorem 4.3. Allow $X, C, M, N, f_i, g_i, Q_i, T_i, F_i$ $(i = 1, 2)$ to be as in Theorem 4.1. Then the sequences $\{(x_n, y_n, u_n, v_n, p_n, q_n)\}$ formulated by Algorithm 4.2, converge strongly to $\{(x, y, u, v, p, q)\}$ of (3.1).

Proof. From Algorithm 4.2, (4.1) and Lemma 2.10, we get

$$\begin{aligned} 0 &\le x_{n+1} \oplus x_n \\ &= (1-\pi)x_n + \pi R_{A,\lambda}^{M(g_1(\cdot),q_n)}\Big[A(x_n) + \frac{\lambda}{\rho_1}(w_1 - Q_1(f_1(x_n), v_n))\Big] \end{aligned}$$

$$\oplus (1-\pi)x_{n-1} + \pi R_{A,\lambda}^{M(g_1(\cdot),q_{n-1})}\Big[A(x_{n-1}) + \frac{\lambda}{\rho_1}(w_1 - Q_1(f_1(x_{n-1}), v_{n-1}))\Big]$$
$$= (1-\pi)(x_n \oplus x_{n-1}) + \pi\Big[R_{A,\lambda}^{M(g_1(\cdot),q_n)}\Big[A(x_n) + \frac{\lambda}{\rho_1}(w_1 - Q_1(f_1(x_n), v_n))\Big]$$
$$\oplus R_{A,\lambda}^{M(g_1(\cdot),q_{n-1})}\Big[A(x_{n-1}) + \frac{\lambda}{\rho_1}(w_1 - Q_1(f_1(x_{n-1}), v_{n-1}))\Big]\Big]. \tag{4.16}$$

By using the same argument as in Theorem 4.1, for (4.7), we have

$$\|x_{n+1} \oplus x_n\| = \|x_{n+1} - x_n\|$$
$$\leq (1-\pi)\|x_n - x_{n-1}\| + \pi\Big[\lambda_C \frac{\mu_1(\lambda_A\rho_1 + \lambda\alpha_1\lambda_{f_1})}{\rho_1}\|x_n - x_{n-1}\|$$
$$+ \lambda_C \frac{(1+\frac{1}{n+1})(\mu_1\lambda\alpha_2\varrho_2 + \rho_1\delta_2\sigma_2)}{\rho_1}\|y_n - y_{n-1}\|\Big]$$
$$\leq \Big[(1-\pi) + \pi\frac{\lambda_C\mu_1(\lambda_A\rho_1 + \lambda\alpha_1\lambda_{f_1})}{\rho_1}\Big]\|x_n - x_{n-1}\|$$
$$+ \pi\Big[\frac{(1+\frac{1}{n+1})\lambda_C(\mu_1\lambda\alpha_2\varrho_2 + \rho_1\delta_2\sigma_2)}{\rho_1}\Big]\|y_n - y_{n-1}\|. \tag{4.17}$$

Similarly,

$$0 \leq y_{n+1} \oplus y_n$$
$$= \Big((1-\pi)y_n + \pi R_{A,\lambda}^{N(p_n,g_2(\cdot))}\Big[A(y_n) + \frac{\lambda}{\rho_2}(w_2 - Q_2(u_n, f_2(y_n)))\Big]\Big)$$
$$\oplus \Big((1-\pi)y_{n-1} + \pi R_{A,\lambda}^{N(p_{n-1},g_2(\cdot))}\Big[A(y_{n-1}) + \frac{\lambda}{\rho_2}(w_2 - Q_2(u_{n-1}, f_2(y_{n-1})))\Big]\Big)$$
$$= (1-\pi)(y_n \oplus y_{n-1}) + \pi\Big(R_{A,\lambda}^{N(p_n,g_2(\cdot))}\Big[A(y_n) + \frac{\lambda}{\rho_2}(w_2 - Q_2(u_n, f_2(y_n)))\Big]$$
$$\oplus R_{A,\lambda}^{N(p_{n-1},g_2(\cdot))}\Big[A(y_{n-1}) + \frac{\lambda}{\rho_2}(w_2 - Q_2(u_{n-1}, f_2(y_{n-1})))\Big]\Big). \tag{4.18}$$

Importing the same logic as in Theorem 4.1 for (4.9), we have

$$\|y_{n+1} \oplus y_{n-1}\| = \|y_{n+1} - y_{n-1}\| \leq \Big[(1-\pi) + \pi\frac{\lambda_C\mu_2(\lambda_A\rho_2 + \lambda\beta_2\lambda_{f_2})}{\rho_2}\Big]\|y_n - y_{n-1}\|$$
$$+ \pi\frac{\lambda_C\big(1+\frac{1}{n+1}\big)(\mu_2\lambda\beta_1\varrho_1 + \delta_1\sigma_1\rho_2)}{\rho_2}\|x_n - x_{n-1}\|. \tag{4.19}$$

From (4.17) and (4.19), we have

$$\|x_{n+1} - x_{n-1}\| + \|y_{n+1} - y_{n-1}\| \leq \Big[(1-\pi) + \frac{\pi\lambda_C\mu_1(\lambda_A\rho_1 + \lambda\alpha_1\lambda_{f_1})}{\rho_1}\Big]\|x_n - x_{n-1}\|$$
$$+ \Big[\frac{\pi\big(1+\frac{1}{n+1}\big)\lambda_C(\mu_1\lambda\alpha_2\varrho_2 + \rho_1\delta_2\sigma_2)}{\rho_1}\Big]\|y_n - y_{n-1}\|$$
$$+ \Big[(1-\pi) + \pi\frac{\lambda_C\mu_2(\lambda_A\rho_2 + \lambda\beta_2\lambda_{f_2})}{\rho_2}\Big]\|y_n - y_{n-1}\|$$
$$+ \pi\frac{\lambda_C\big(1+\frac{1}{n+1}\big)(\mu_2\lambda\beta_1\varrho_1 + \delta_1\sigma_1\rho_2)}{\rho_2}\|x_n - x_{n-1}\|$$
$$\leq (1-\pi)(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|)$$
$$+ \pi\Big[\frac{\lambda_C\mu_1(\lambda_A\rho_1 + \lambda\alpha_1\lambda_{f_1})}{\rho_1} + \frac{\lambda_C\big(1+\frac{1}{n+1}\big)(\mu_2\lambda\beta_1\varrho_1 + \delta_1\sigma_1\rho_2)}{\rho_2}\Big]\|x_n - x_{n-1}\|$$
$$+ \pi\Big[\frac{\lambda_C\big(1+\frac{1}{n+1}\big)(\mu_1\lambda\alpha_2\varrho_2 + \rho_1\delta_2\sigma_2)}{\rho_1} + \frac{\lambda_C\mu_2(\lambda_A\rho_2 + \lambda\beta_2\lambda_{f_2})}{\rho_2}\Big]\|y_n - y_{n-1}\|$$

$$= (1-\pi)(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|) + \pi(\Omega_1^n \|x_n - x_{n-1}\| + \Omega_2^n \|y_n - y_{n-1}\|)$$
$$= (1-\pi)(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|) + \pi\Omega^n(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|), \tag{4.20}$$

where $\Omega^n = \max\{\Omega_1^n, \Omega_2^n\}$ and

$$\Omega_1^n = \frac{\lambda_C}{\rho_1\rho_2}\Big[\mu_1\rho_2(\lambda_A\rho_1 + \lambda\alpha_1\lambda_{f_1}) + \Big(1 + \frac{1}{n+1}\Big)\rho_1(\mu_2\lambda\beta_1\varrho_1 + \delta_1\sigma_1\rho_2)\Big],$$
$$\Omega_2^n = \frac{\lambda_C}{\rho_1\rho_2}\Big[\Big(1 + \frac{1}{n+1}\Big)\rho_2(\mu_1\lambda\alpha_2\varrho_2 + \rho_1\delta_2\sigma_2) + \rho_1\mu_2(\lambda_A\rho_2 + \lambda\beta_2\lambda_{f_2})\Big].$$

Let $\Omega_1^n \longrightarrow \Omega_1$ and $\Omega_2^n \longrightarrow \Omega_2$ whenever $n \longrightarrow \infty$, therefore $\Omega^n \longrightarrow \Omega$ as $n \longrightarrow \infty$. Then condition (4.2) implies $\Omega < 1$ and so $\Omega_n < 1$ for sufficiently large n. By (4.20), for sufficient n, we have

$$\begin{aligned}\|x_{n+1} - x_{n-1}\| + \|y_{n+1} - y_{n-1}\| &\le (1-\pi)(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|)\\ &\quad + \pi\Omega(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|)\\ &\le (1-\pi+\pi\Omega)(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|)\\ &\le \varsigma(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|).\end{aligned} \tag{4.21}$$

where $\varsigma = 1 - \pi + \pi\Omega$ and $\Omega = \max\{\Omega_1, \Omega_2\}$, and

$$\Omega_1 = \frac{\lambda_C}{\rho_1\rho_2}[\rho_2\mu_1(\lambda_A\rho_1 + \lambda\alpha_1\lambda_{f_1}) + \rho_1(\mu_2\lambda\beta_1\varrho_1 + \delta_1\sigma_1\rho_2)],$$
$$\Omega_2 = \frac{\lambda_C}{\rho_1\rho_2}[\rho_2(\mu_1\lambda\alpha_2\varrho_2 + \rho_1\delta_2\sigma_2) + \rho_1\mu_2(\lambda_A\rho_2 + \lambda\beta_2\lambda_{f_2})].$$

By (4.2), we have $\Omega < 1$. So there exists $\Omega^0 < 1$ such that for sufficiently large n, $\Omega^n < \Omega^0$ and

$$\|x_{n+1} - x_{n-1}\| + \|y_{n+1} - y_{n-1}\| \le \varsigma^0(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|), \tag{4.22}$$

where $\varsigma^0 = 1 - \pi + \pi\Omega^0 < 1$.
It follow that $\{x_n\}$ is a Cauchy sequence in X. Since X is a complete space, there exists $x \in X$ such that $x_n \longrightarrow x$ as $n \longrightarrow \infty$. From (4.22), $\{y_n\}$ is also a Cauchy sequence in X and $y_n \longrightarrow y$ as $n \longrightarrow \infty$. Condition (4.15) and the $\mathfrak{D}$-Lipschitz continuity of T_1, T_2, F_1, F_2 imply that $\{u_n\}, \{v_n\}, \{p_n\}$ and $\{q_n\}$ are all the Cauchy sequences. Let $u_n \longrightarrow u$, $v_n \longrightarrow v$, $p_n \longrightarrow p$ and $q_n \longrightarrow q$, *respectively*. By (4.15), we have

$$\begin{aligned}d(u, T_1(u)) &\le \|u - u_n\| + d(u_n, T(u))\\ &\le \|u - u_n\| + \mathfrak{D}(T_1(u_n), T_1(u))\\ &\le \|u - u_n\| + \varrho_1\|u_n - u\| \longrightarrow 0, \text{ as } n \longrightarrow \infty,\end{aligned} \tag{4.23}$$

and so $u \in T_1(x)$. Similarly, we can show that $v \in T_2(y), p \in F_1(x)$ and $q \in F_2(y)$. By (4.15), we have

$$x_{n+1} = (1-\pi)x_n + \pi R_{A,\lambda}^{M(g_1(\cdot), q_n)}\Big[A(x_n) + \frac{\lambda}{\rho_1}(w_1 - Q_1(f_1(x_n), v_n))\Big],$$
$$y_{n+1} = (1-\pi)y_n + \pi R_{A,\lambda}^{N(p_n, g_2(\cdot))}\Big[A(y_n) + \frac{\lambda}{\rho_2}(w_2 - Q_2(u_n, f_2(y_n)))\Big].$$

By Lemma 2.16 and the assumptions in Theorem 4.1, letting $n \longrightarrow \infty$ in the above equations, we can obtain

$$x = (1-\pi)x + \pi R_{A,\lambda}^{M(g_1(\cdot), q)}\Big[A(x) + \frac{\lambda}{\rho_1}(w_1 - Q_1(f_1(x), v))\Big],$$
$$y = (1-\pi)y + \pi R_{A,\lambda}^{N(p, g_2(\cdot))}\Big[A(y) + \frac{\lambda}{\rho_2}(w_2 - Q_2(u, f_2(y)))\Big].$$

By Lemma 3.1, $\{(x, y, u, v, p, q)\}$ is a solution of system (3.1). This completes the proof. □

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References

1. M. K. Ahmad, Salahuddin, Resolvent equation technique for generalized nonlinear variational inclusions. *Adv. Nonlinear Var. Inequal.* **5** (2002), no. 1, 91–98.
2. X. P. Ding, H. R. Feng, Algorithm for solving a new class of generalized nonlinear implicit quasi-variational inclusions in Banach spaces. *Appl. Math. Comput.* **208** (2009), no. 2, 547–555.
3. X. P. Ding, Salahuddin, A system of general nonlinear variational inclusions in Banach spaces. *Appl. Math. Mech. (English Ed.)* **36** (2015), no. 12, 1663–1672.
4. Y. P, Du, Fixed points of increasing operators in ordered Banach spaces and applications. *Appl. Anal.* **38** (1990), no. 1-2, 1–20.
5. B. S. Lee, Salahuddin, Fuzzy general nonlinear ordered random variational inequalities in ordered Banach spaces. *East Asian Math. J.* **32** (2016), no. 5, 685–700.
6. H. G. Li, Approximation solution for generalized nonlinear ordered variational inequality and ordered equation in ordered Banach space. *Nonlinear Anal. Forum*, **13** (2008), no. 2, 205–214.
7. H. G. Li, Approximation solution for a new class of general nonlinear ordered variational inequalities and ordered equations in ordered Banach spaces. *Nonlinear Anal. Forum* **14** (2009), 89–97.
8. H. G. Li, A nonlinear inclusion problem involving (α, λ)-NODM set-valued mappings in ordered Hilbert space. *Appl. Math. Lett.* **25** (2012), no. 10, 1384–1388.
9. H. G. Li, D. Qiu, M. M. Jin, GNM ordered variational inequality system with ordered Lipschitz continuous mappings in an ordered Banach space. *J. Inequal. Appl.* **2013**, 2013:514, 11 pp.
10. H. G. Li, X. B. Pan, Z. Y. Deng, C. Y. Wang, Solving GNOVI frameworks involving (γ_G, λ)-weak-GRD set-valued mappings in positive Hilbert spaces. *Fixed Point Theory Appl.* **2014**, 2014:146, 9 pp.
11. H. G. Li, L. P. Li, M. M. Jin, A class of nonlinear mixed ordered inclusion problems for ordered (α_a, λ)-ANODM set-valued mappings with strong comparison mapping *A*. *Fixed Point Theory and Appl.* **2014**, no. 1, (2014):79, 9 pp.
12. Salahuddin, Solvability for a system of generalized nonlinear ordered variational inclusions in ordered Banach spaces. *Korean J. Math.* **25** (2017), no. 3, 359–377.
13. H. H. Schaefer, *Banach Lattices and Positive Operators.* Springer-Verlag, Berlin, 1974.
14. R. U. Verma, Projection methods, algorithms, and a new system of nonlinear variational inequalities. *Comput. Math. Appl.* **41** (2001), no. 7-8, 1025–1031.

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