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PARTICULARITIES OF INTEGRATING THE PERIODIC FUNCTIONS IN THE PRESENCE OF THE TIMESCALE GRADIENTS AND TURBULENCE ISSUES

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Abstract. The paper demonstrates that with a strictly periodic nature of fluctuation of scalar parameters at a fixed point of a space, in the presence of timescale gradients at this point, the gradients of these parameters are not periodic functions, but they are complex, almost periodic in the form of a combination of regular and low-frequency (or periodic) fluctuations. This result indicates the link between the discrete spectrum of turbulence and the frequency gradients, as well as the need for considering a more general mode with regard for the timescale effect.

Introduction

In our studies, we are aimed at the fact that the main cause of turbulence problem is disregard of the timescales in integrating the Navier-Stokes equations. In particular, the work [1] has demonstrated that consideration of timescale gradients, when integrating the differential equations for conservation of mass and energy, allows one to get two scalar equations for the turbulent processes.

In this section, we show that taking into account the specifics of integration of periodic processes is even more important, since this offers rich possibilities to solve the turbulence problem.

To this end, in this section, we want to engage the reader's interest in the specifics of integration within a single cycle and within a large (or infinite) time interval. In fact, we consider the simplest problem which, to some extent, corresponds to the mean value theorem based on the Liouville-Arnold theorem.

Strictly periodic functions. Consider a function F, which changes periodically, with an interval τ_o, for an infinitely long period of time $\tau_\infty \to \infty$ (Figure 1). Assuming that all the cyclic processes are identical, we can say that there is a **strictly periodic process.** Moreover, there may also exist such cyclical fluctuations differing in form, during which the mean values of the parameter, in different cycles, are identical. In this case, we will have **processes that are strictly periodic by a mean value.** In this paper, the oscillatory processes of both natures will be referred to as **strictly periodic.**

Mean value of a function F, within a cycle, can be determined by integrating

$$\overline{F}_\tau = \frac{1}{\tau_O} \int\limits_o^{\tau_O} F dt.$$

Naturally, if the process is strictly periodic, then the average value of this parameter will not be changed during the integration within several cycles.

$$\overline{F_\tau} = \frac{1}{\tau_O} \int\limits_o^{\tau_O} F dt = \frac{1}{2\tau_O} \int\limits_0^{2\tau_O} F dt = \dots = \frac{1}{N\tau_o} \int\limits_O^{N\tau_O} F dt.$$

In this connection, averaging over the cycle, the integration limits should be observed exactly, since for low deviations from this condition, the average value of the parameter will be determined with a significant error.

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On the other hand, the average value of this parameter can also be determined within an interval of an infinitely long period of time [2]:

$$\overline{F}_{\infty} = \frac{1}{\tau_{\infty}} \int\limits_{o}^{\tau_{\infty}} F dt.$$

It is not difficult to show that the accuracy of determining the mean value increases with increasing τ_{∞}. Naturally, for the strictly periodic functions, the average values of the function within a cycle and a large interval are equal. At the same time, unlike averaging over a cycle, when averaging over a large interval, the period of integration τ_{∞} can be chosen in an arbitrary way, without observing the multiplicity condition.

The almost periodic or complex periodic functions. If the average value of the parameter changes insignificantly in each subsequent cycle, then it can be said that the function is not strictly periodic, almost periodic or complex.

In order to determine the nature of this function, it is necessary to compare the average within a cycle value with the mean value within a large time interval.

Naturally, if the mean value of the function in each cycle increases (or decreases) monotonely, then in an infinitely large interval, the difference over time $\delta(t) = \delta(\tau_{\infty})$ will increase infinitely, and we can say that in a large interval, the function is not periodic.

If the difference between the mean values has some limiting value $\delta(x, y, z)$, which is not a function of the averaging period, then we can say that we have a complex function, which is the sum of a strictly periodic function and some low-frequency periodic function.

So, we have:

A) For the nonperiodic functions $\overline{F_{\infty}} = \overline{F_{\tau}} + \delta(\tau_{\infty})$,

B) For the strictly periodic functions $\overline{F_{\infty}} = \overline{F_{\tau}} = \overline{F}$,

C) For the complex periodic functions $\overline{F_{\infty}} = \overline{F_{\tau}} + \delta(x, y, z)$, (1)

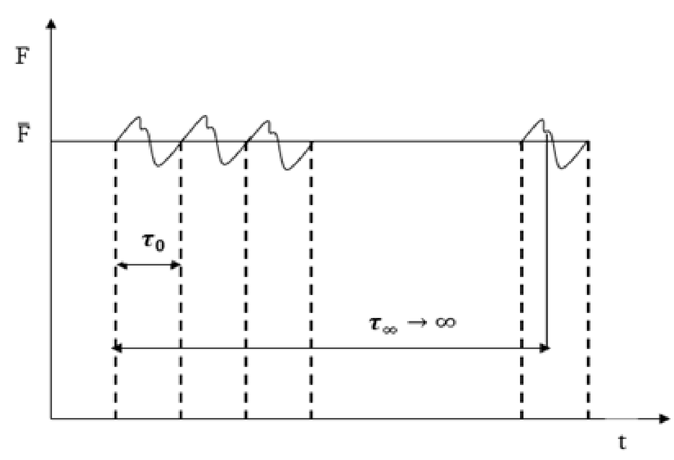

FIGURE 1. A strictly periodic function.

Thus, when we have the strictly periodic functions, the average values with respect to a cycle and to a long period are identical for them. On the other hand, there are various requirements to the limits of integration. As we will see below, this difference provides very interesting information necessary to solve the turbulence problem.

Let us consider, in a continuous medium space, a function ∇F from a strictly periodic function F, for which the fair condition is $\overline{F_\infty} = \overline{F_\tau} = \overline{F}$, and let us show that in the presence of the timescale gradients, ∇F is a complex periodic function.

Let us determine the average value of a function ∇F within a large period of time which can be set as the same everywhere inside the space $\tau_\infty = \text{const}$. Accordingly, at any point in this space, the average value with respect to a large time period of a function ∇F can be determined from the equation

$$\overline{\nabla F}_\infty = \frac{1}{\tau_\infty}\int\limits_0^{\tau_\infty}(\nabla F)dt = \frac{1}{\tau_\infty}\nabla\int\limits_0^{\tau_\infty}F dt = \frac{1}{\tau_\infty}\nabla(\tau_\infty\overline{F_\infty}). \tag{2}$$

In this connection, if the limit of integration is the same for the entire space, τ_∞ can be taken outside the differential sign. Therefore, we have

$$\overline{\nabla F}_\infty = \nabla\overline{F}_\infty = \nabla\overline{F}. \tag{3}$$

Thus, when averaging the Navier-Stokes equation [3–6], the averaging sign is introduced inside the sign ∇, it is good to bear in mind that the averaging takes place within an infinitely long period of time. Of course, this is true, but, at the same time, the cycle integration gives also the quite legitimate, different results, and this difference bears serious information for the problem of turbulence.

Let us determine the mean value of a function F with respect to a cycle

$$\overline{\nabla F}_\tau = \frac{1}{\tau_o}\int\limits_0^{\tau_0}(\nabla F)dt = \frac{1}{\tau_0}\nabla\int\limits_0^{\tau_o}F dt = \frac{1}{\tau_o}\nabla(\tau_o\overline{F}),$$

or

$$\overline{\nabla F}_\tau = \nabla\overline{F} + \frac{\overline{F}}{\tau_o}\nabla\tau_o = \nabla\overline{F} + \overline{F}\,\mathrm{grad}(\ln\tau_0). \tag{4}$$

Comparison of (2) and (4) leads to the conclusion

$$\overline{\nabla F}_\tau = \overline{\nabla F}_\infty + \overline{F}\,\mathrm{grad}(\ln\tau_0).$$

Accordingly, in the general case, when $\overline{F}\,\mathrm{grad}(\ln\tau_0) \neq 0$, we have

$$\overline{\nabla F}_\tau \neq \overline{\nabla F}_\infty.$$

Thus, **if at a fixed point in space the periodic function F varies strictly periodically, and at this point there are the frequency gradients, then the function ∇F is, generally, nonperiodic, or complex periodic**.

On the basis of the law of conservation of the integral flows of mass and energy, the work [1] demonstrated that the fair conditions look as follows:

$$V\,\mathrm{grad}(\tau_0) = 0, \tag{5}$$
$$e_{\sum}\,\mathrm{grad}(\tau_0) = 0. \tag{6}$$

Accordingly, for the vectors of the flows of mass $g = \rho W$ and energy $e_{\sum}$, we have

$$\overline{\nabla\rho W}_\tau = \overline{\nabla\rho W}_\infty,$$
$$\overline{\nabla e_{\sum}}_\tau = \overline{\nabla e_{\sum}}_\infty.$$

This means that the functions $\nabla\rho W$ and $\nabla e_{\sum}$ are strictly periodic.
Moreover, in the general case, for the parameters of the turbulent flow of continuous medium,

$$\overline{F}\,\mathrm{grad}(\tau_0) \neq 0.$$

The obtained result gives very interesting information for the theory of turbulence.

Result 1. Let us consider an arbitrary scalar parameter S (pressure, density, temperature, etc.) to be a periodic function. If the fluctuation of a scalar parameter is strictly periodic (or strictly periodic with respect to a mean value) and the condition (3) is satisfied for it, then in this case, the following expressions

$$\overline{\nabla S}_\infty = \nabla \overline{S},$$
$$\overline{\nabla S}_\tau = \nabla \overline{S} + \overline{S}\,\mathrm{grad}(\tau_0) \tag{7}$$

are fair.

In the case of the timescale gradients, the last term in (7) cannot be zero. Therefore, we have an instance

$$\overline{\nabla S}_\infty = \overline{\nabla S}_\tau + \delta(x, y, z),$$

corresponding to condition (1). The obtained expression allows us for conclude **that for a periodic nature of fluctuation of the scalar parameters at a fixed point of the turbulent flow of a continuous medium, in the presence of the timescale gradients at this point, the scalar parameter gradients are the complex periodic functions, which indicates the appearance of the low-frequency variations of gradients under the influence of the timescale gradients.**

Result 2. If the pressure, as a scalar parameter, varies strictly periodically at different points of the flow, then the pressure gradient, in the general case, may be of the nature of a complex periodic function.

Moreover, the pressure gradient is the main factor of the velocity change. Accordingly, the low-frequency variations of the pressure gradients are able to generate the additional velocity changes or the accompanying fluctuations that distort the reality of the regular periodic pulsations.

For this reason, **the velocity vector in a turbulent flow should be sought as the sum of a stationary function and the independent pulsations of at least two types (traditional and low-frequency, or the slow ones).**

$$W = V + \upsilon + u, \tag{8}$$

where υ reflects the regular pulsations, but u corresponds to periodic fluctuations, or to the low-frequency pulsations, which manifest themselves during a large time interval. These low-frequency fluctuations cannot be correlated with the regular pulsations, but they can create an additional tensor $\tau(\overline{u, u})$.

Let us consider the Navier-Stockes equation

$$\frac{\partial \rho W}{\partial t} + \nabla I_{\sum} = 0, \tag{9}$$
$$I_{\sum} = \rho\tau(W, W) + PI + (2/3)\mu\,\mathrm{div}(W)I - 2\mu D(W) = 0.$$

Integration of (9) within a large time interval, taking into account (8) results in

$$\rho(V\nabla)V = -\,\mathrm{div}[\rho\tau(\overline{\upsilon, \upsilon}) + \rho\tau(\overline{u, u}) + PI + (2/3)\mu\,\mathrm{div}(V)I - 2\mu D(V)]. \tag{10}$$

The obtained expression differs from the traditional Reynolds equation by the presence of an additional tensor of the accompanying pulsations $\tau(\overline{u, u})$.

And now, we can integrate this equation within a single cycle. The probability of the occurrence of low-frequency fluctuations within a single pulsation is low. Therefore, the flow velocity in a short interval τ_0 can be considered to be a sum of the mean velocity and regular pulsations

$$W = V + \upsilon.$$

In such conditions, integration of (9) results in

$$\nabla \overline{I_{\sum\tau}} = -\frac{\overline{I_{\sum\tau}}}{\tau_o}\,\mathrm{grad}\,\tau_0, \tag{11}$$
$$\overline{I_{\sum\tau}} = \rho\tau(V, V) + \rho\tau(\overline{\upsilon, \upsilon}) + PI + \overline{\sigma_\tau},$$

where the mean value of the strain tensor has the form

$$\overline{\sigma_\tau} = (2/3)\mu\,\mathrm{div}(V)I - 2\mu\overline{D(V)_\tau}. \tag{12}$$

When determining the last term of the equation (12), we take into account the fact that the time-average value of the strain tensor from the vector F is determined from the equation:

$$\overline{D(F)} = (1 - \ln\tau_o)D(\overline{F}) + D(\overline{F}\ln\tau_o).$$

At the same time, the following expression

$$D(C\overline{F}) = CD(F) + \frac{1}{2}[\tau(F, \operatorname{grad} C) + \tau(\operatorname{grad} C, F)]$$

is true. Therefore, we have

$$\overline{D(V)} = (1 - \ln\tau_o)D(V) + D(V\ln\tau_o) = D(V) - \frac{1}{2}[\tau(V, A) + \tau(A, V)].$$

In the response to the latter, we have

$$\overline{I_{\sum_\tau}} = \rho\tau(V, V) + \rho\tau(\overline{v, v}) + PI + (2/3)\mu\operatorname{div}(V)I - 2\mu D(V) + \mu\tau(V, A) + \mu\tau(A, V). \quad (13)$$

Putting (13) into (11), the equation of motion becomes

$$\rho(V\nabla)V = -\operatorname{div}[\rho\tau(\overline{v, v}) + PI + (2/3)\mu\operatorname{div}(V)I - 2\mu D(V) + \mu\tau(V, A) + \mu\tau(A, V)]$$
$$+[\rho\tau(V, V) + \rho\tau(\overline{v, v}) + PI + (2/3)\mu\operatorname{div}(V)I - 2\mu D(V) + \mu\tau(V, A) + \mu\tau(A, V)]A. \quad (14)$$

By comparing (14) and (10), for a tensor of the accompanying pulsations, we obtain the following equation:

$$\operatorname{div}[\rho\tau(\overline{u, u})] = \operatorname{div}\{\mu[\tau(V, A) + \tau(A, V)]\} - [\rho\tau(\overline{v, v}) + PI$$
$$+(2/3)\mu\operatorname{div}(V)I - 2\mu D(V)]A - \mu A^2 V.$$

As we can see, the influence of this tensor depends on the timescale gradient, and it disappears in the absence of these gradients. To determine the vector $A = -\operatorname{grad}\ln\tau_0$, we already have two scalar equations – (5) and (6).

Thus, taking into account the time gradients makes it necessary to apply a more general model. Moreover, as we will see later, this insignificant complication simplifies the solution of this very complex problem of turbulence.

Conclusion

The presence of the timescale gradients is the reason for the appearance of additional low-frequency oscillations or fluctuations in the velocity of the turbulent flow of a continuous medium, highlighting the need to consider a more general model that not only takes into account the mentioned features of the flow, but also provides the additional equations for solving the turbulence problem.

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