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MAXIMAL AND CALDERÓN–ZYGMUND OPERATORS IN WEIGHTED GRAND VARIABLE EXPONENT LEBESGUE SPACES

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Abstract. The boundedness of maximal and Calderón–Zygmund operators is established in weighted grand variable exponent Lebesgue spaces with power-type weights. The same problem for commutators of Calderón–Zygmund operators is also studied. These spaces unify two non-standard function spaces, namely, grand Lebesgue and variable exponent Lebesgue spaces. The spaces and operators are defined, generally speaking, on quasi-metric measure spaces with doubling measure. Exponents of spaces satisfy log-Hölder continuity condition.

1. Introduction

In this note, the boundedeness of maximal and singular integral operators is derived in the weighted grand variable exponent Lebesgue spaces (GVELS, for short) with power weights. The same problem is studied also for commutators of singular integrals. The operators and spaces are defined on quasi-metric measure spaces with doubling measure. The spaces of functions under consideration $L^{p(\cdot),\theta}_w$ are non-reflexive, non-separable and non-rearrangement invariant. Generally speaking, GVELS unifies two non-standard function spaces: variable exponent Lebesgue spaces and grand Lebesgue spaces. The unweighted GVELSs were introduced in [11] (see also [13, Chapter 14]), where also the mapping properties of operators of Harmonic Analysis were established. Later, in [6], the authors introduced new scale of GVELSs and studied integral operators boundedness in those spaces. In [7], the boundedness of the operator

$$Mf(x) := \sup_{r>0} M_r f(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int\limits_{B_r(x)\cap\Omega} |f(y)|dy$$

in $L^{p(\cdot),\theta}_w(\Omega)$ with $w(x) = |x-x_0|^{\gamma}$ was proved. In particular, it was proved the following statement:

Theorem 1.1 ([7])**.** *Let Ω be a bounded open set in $\mathbb{R}^n$, $1 < p_-(\Omega) \le p_+(\Omega) < \infty$, $\theta > 0$, $x_0 \in \Omega$, and suppose that γ is a constant satisfying the condition $-n < \gamma < n(p(x_0)-1)$. Then the operator M is bounded in $L^{p(\cdot),\theta}_w(\Omega)$ with $w(x) = |x-x_0|^{\gamma}$.*

In the recent years it was realized that the classical function spaces are no longer appropriate spaces when we attempt to solve a number of contemporary problems arising naturally in: the non-linear elasticity theory, fluid mechanics, mathematical modelling of various physical phenomena, solvability problems of non-linear partial differential equations. It thus became necessary to introduce and study the spaces mentioned above from various viewpoints. One of such spaces is the variable exponent Lebesgue space (we refer, e.g., to the monographs [4], [3], [12] for the properties of variable exponent function spaces and integral operators in these spaces).

In this paper, we deal with the weighted grand variable exponent Lebesgue space $L^{p(\cdot),\theta}_w(X)$, with power-type weight $w(x) = d(x_0,x)^{\beta}$. This space for $\beta = 0$ has been introduced in [11] (see also [13, Ch. 14]). The norm in $L^{p(\cdot),\theta}_w(X)$ entrains those ones of the two variants of the Lebesgue spaces: variable exponent and grand Lebesgue spaces. This space is a Banach space. In [11], the authors also derived the boundedness of some operators of Harmonic Analysis in $L^{p(\cdot),\theta}$ defined on quasi-metric measure spaces with doubling measure.

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The grand Lebesgue spaces were introduced in the 90s of the past century by T. Iwaniec and C. Sbordone [10]. In the subsequent years, quite a number of problems of Harmonic Analysis and the theory of non-linear differential equations were studied in these spaces (see, e.g., the monographs [12], [13] and references cited therein).

2. Preliminaries

Let (X, d, μ) be a quasi-metric measure spaces. A quasi-metric d is a function $d : X \times X \to [0, \infty)$ which satisfies the following conditions:

(a) $d(x, y) = 0$ if and only if $x = y$.

(b) For all $x, y \in X$, $d(x, y) = d(y, x)$.

(c) There is a constant $\mathcal{K} > 0$ such that $d(x, y) \le \mathcal{K}(d(x, z) + d(z, y))$ for all $x, y, z \in X$.

Let

$$d_X := \operatorname{diam}(X) := \sup\{d(x, y) : x, y \in X\}$$

be the diameter of X. We denote by $B(x, r)$ the ball with center x and radius r, i.e., $B(x, r) := \{y \in X : d(x, y) < r\}$.

We assume that $\mu(X) < \infty$.

We assume also that the following condition holds for (X, d, μ): there are positive constants n, C_1 and C_2 such that for all $x \in X$ and $0 < r < d_X$ the inequality

$$C_1 r^n \le \mu\big(B(x, r)\big) \le C_2 r^n. \tag{1}$$

It is easy to check that condition (1) guarantees that (X, d, μ) is a space of homogeneous type (SHT, for short), i.e., measure μ satisfies the doubling condition ($\mu \in DC(X)$): there is a constant $D_\mu > 0$ such that

$$\mu B(x, 2r) \le D_\mu \mu B(x, r)$$

for every $x \in X$ and $r > 0$.

As examples of SHT are regular curves (see the definition below), domains in $\mathbb{R}^n$ with the so-called $\mathcal{A}$ condition, nilpotent Lie groups with Haar measure (homogeneous groups), etc. (see, e.g., [2], [5]).

We denote by $P(X)$ a family of all real-valued μ- measurable functions p on X such that

$$1 < p_- \le p_+ < \infty,$$

where $p_- := p_-(X) := \inf_X p(\cdot)$, $p_+ := p_+(X) := \sup_X p(\cdot)$.

Let w be a weight function on X, i.e., w be $\mu-$ a.e. locally integrable function on X. The Lebesgue space with a variable exponent $p(\cdot)$ with a weight function w denoted by $L^{p(\cdot)}_w(X)$ (or by $L^{p(x)}_w(X)$) is the class of all μ- measurable functions f on X for which

$$S_{p,w}(f) := \int_X |f(x)|^{p(x)} w(x) d\mu(x) < \infty.$$

The norm in $L^{p(\cdot)}_w(X)$ is defined as follows:

$$\|f\|_{L^{p(\cdot)}_w(X)} = \inf\Big\{\lambda > 0 : S_{p,w}(f/\lambda) \le 1\Big\}.$$

If $\beta = 0$, then we denote $L^{p(\cdot)}_\beta(X)$ by $L^{p(\cdot)}(X)$.

The space $L^{p(\cdot),\theta}_w(X)$ is a Banach space. The closure of $L^{p(\cdot)}_w(X)$ in $L^{p(\cdot),\theta}_w(X)$ consists of those $f \in L^{p(\cdot),\theta}_w(X)$ for which $\lim\limits_{c\to 0} \varepsilon^{\frac{\theta}{p_- - c}} \|f(\cdot)\|_{L^{p(\cdot)-c}_w(X)} = 0$.

Let $\theta > 0$. For a weight function w, we denote (see [11] in an unweighted case) by $L^{p(\cdot),\theta}_w(X)$ the class of all measurable functions $f : X \mapsto \mathbb{R}$ for which the norm

$$\|f\|_{L^{p(\cdot),\theta}_w(X)} := \sup_{0<c<p_- -1} c^{\frac{\theta}{p_- - c}} \|f\|_{L^{p(x)-c}_w(X)}$$

is finite. In this definition, c is a constant.

If $w(x) = d(x_0, x)^\beta$, where x_0 is a fixed point in X, then we use the symbol $L^{p(\cdot),\theta}_\beta(X)$ for $L^{p(\cdot),\theta}_w(X)$.

Let $\mathcal{P}^{\log}_{\mu}(X)$ be a class of those exponents p belonging to $P(X)$ that satisfy the following log-Hölder condition: there exists a positive constant a such that for all $x, y \in X$ with $d(x,y) \le 1/2$,

$$|p(x) - p(y)| \le \frac{a}{-\ln\big(d(x,y)\big)}.$$

3. Main Results

We denote by M the maximal operator defined on X:

$$Mf(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int\limits_B |f(y)| d\mu(y),$$

where f is a locally integrable function on X.

Our result regarding the maximal operator reads as follows:

Theorem 3.1. *Let $p \in P(X) \cap \mathcal{P}^{\log}_{\mu}(X)$, $\theta > 0$ and let x_0 be a point in X. Suppose that $-n < \beta < n(p(x_0) - 1)$, where n is from* (1)*. Then the Hardy–Littlewood maximal operator M is bounded in $L^{p(\cdot),\theta}_{\beta}(X)$.*

Let K be the Calderón-Zygmund operator defined on a SHT, i.e., K satisfies the following conditions (see, e.g., [1], [2]):

(i) K is linear and bounded in $L^{p_c}(X)$ for every $p_c \in (1, \infty)$;

(ii) there is a measurable function $k : X \times X \mapsto \mathbb{R}$ such that for every $f \in D(X)$,

$$Kf(x) = \int\limits_X k(x,y) f(y) d\mu(y),$$

for a.e. $x \notin \operatorname{supp} f$, where $D(X)$ is the class of bounded functions with compact supports defined on X.

(iii) the kernels k and k^* (here, $k^*(x,y) := k(y,x)$) satisfy the following pointwise Hörmander's condition: there are positive constants C, β and $A > 1$ such that

$$|k(x_0, y) - k(x,y)| \le C \frac{d(x_0,x)^{\beta - n}}{d(x_0,y)^{\beta}}$$

holds for every $x_0 \in X$, $r > 0$, $x \in B(x_0, r)$, $y \in X \setminus B(x_0, Ay)$;

(iv) there is a positive constant C such that for all $x, y \in X$,

$$|k(x,y)| \le \frac{C}{d(x,y)^n}.$$

We say that a weight w belongs to the Muckenhoupt class $A_{p_c}(X)$, where p_c is a constant such that $1 < p_c < \infty$, if

$$[w]_{A_{p_c}(X)} := \sup_B \bigg(\frac{1}{\mu B} \int\limits_B w d\mu \bigg) \bigg(\frac{1}{\mu B} \int\limits_B w^{1-(p_c)'} d\mu \bigg)^{p_c - 1} < \infty, \quad (p_c)' = \frac{p_c}{p_c - 1},$$

where the supremum is taken with respect to all balls B in X.

The operator K (see, e.g., [14] and references therein) is bounded in $L^{p_c}_w(X)$ for $1 < p_c < \infty$ and $w \in A_{p_c}(X)$. Moreover, the following estimate

$$\|K\|_{L^{p_c}_w(X)} \le c_0([w]_{A_{p_c}(X)})$$

holds, where $c_0([w]_{A_{p_c}(X)})$ is a constant depending on $[w]_{A_{p_c}(X)}$ so that the mapping $x \mapsto c_0(x)$ is non-decreasing.

Theorem 3.2. *Let the conditions of Theorem* 3.1 *be satisfied. Then there is a positive constant c depending only on p such that the following inequality*

$$\|Kf\|_{L^{p(\cdot),\theta}_{\beta}(X)} \le c \|f\|_{L^{p(\cdot),\theta}_{\beta}(X)}, \quad f \in D(X),$$

holds.

Let K be a Calderón-Zygmund operator. By K_b^m we denote a higher order commutator on X. For a function b on X,

$$K_b^1 f = bKf - K(bf)$$

is well defined for $b \in BMO(X)$ and $f \in D(X)$ (see [14]), where $BMO(X)$ is the well-known class of functions b defined on X for which

$$\|b\|_{BMO} = \sup_{\substack{x\in X\\ 0<r<d_X}} \frac{1}{\mu B(x,r)} \int\limits_{B(x,r)} |f(y) - f_{B(x,r)}| d\mu(y) < \infty.$$

Further, let m be an integer $m \geq 2$. Then by the definition,

$$K_b^m f(x) = \int\limits_X (b(x) - b(y))^m k(x,y) f(y) d\mu(y).$$

Theorem 3.3. *Let the conditions of Theorem* 3.1 *be satisfied and let* $b \in BMO(X)$*. Then there is a positive constant* c *such that for all* $f \in D(X)$*, the inequality*

$$\|K_b^m f\|_{L_\beta^{p(\cdot),\theta}(X)} \leq c\|f\|_{L_\beta^{p(\cdot),\theta}(X)}$$

holds.

As an application, we can obtain weighted estimates for maximal and singular integral operators defined on a rectifiable curve Γ.

Let

$$\Gamma(z,r) := \Gamma \cap D(z,r), \quad z \in \Gamma, \quad r > 0,$$

where $D(z,r)$ is a disc on the complex plane with center z and radius r.

We say that the curve Γ is regular (Carleson) if there is a positive constant C such that for all $z \in \Gamma$ and $r > 0$,

$$\nu\big(\Gamma(z,r)\big) \leq Cr,$$

where ν is the arc-length measure on γ.

This condition guarantees that Γ with the Euclidean distance and ν is a SHT, where the balls are $\Gamma(z,r)$.

Observe that

$$C_1 r \leq \nu\big(\Gamma(z,r)\big) \leq C_2 r, \quad 0 < r < d_\Gamma,$$

for some positive constants C_1 and C_2.

Suppose that by M_Γ and K_Γ are denoted the Hardy-Littlewood maximal operator and the Cauchy singular integrals on Γ defined, respectively, as follows:

$$M_\Gamma f(z) = \sup_{r>0} \frac{1}{\Gamma(z,r)} \int\limits_{\Gamma(z,r)} |f(t)| d\nu(t), \quad z \in \Gamma, \quad f \in L_{\mathrm{loc}}(\Gamma),$$

$$S_\Gamma f(z) = (p.v.) \int\limits_\Gamma \frac{f(t))}{t-z} d\nu(t), \quad z \in \Gamma, \quad f \in D(\Gamma).$$

As a consequence of Theorems 3.1–3.3, we have

Theorem 3.4. *Let* $p \in P(\Gamma) \cap \mathcal{P}_\mu^{\log}(\Gamma)$*,* $\theta > 0$ *and let* z_0 *be a point on* Γ*. Suppose that* Γ *is a regular curve and* $-1 < \beta < p(z_0) - 1$*. Then the Hardy–Littlewood maximal operator* M_Γ *is bounded in* $L_\beta^{p(\cdot),\theta}(\Gamma)$*.*

Theorem 3.5. *Let the conditions of Theorem* 3.4 *be satisfied. Then there is a positive constant* c *depending only on* p *such that the following inequality*

$$\|S_\Gamma f\|_{L_\beta^{p(\cdot),\theta}(\Gamma)} \leq c\|f\|_{L_\beta^{p(\cdot),\theta}(\Gamma)}, \quad f \in D(\Gamma),$$

holds.

Let us denote

$$S^m_{\Gamma,b} f(z) = (p.v.) \int\limits_{\Gamma} \frac{(b(z)-b(t))^m}{z-t} f(t)d\nu(t), \quad z \in \Gamma.$$

Theorem 3.6. *Let the conditions of Theorem* 3.4 *be satisfied and let $b \in BMO(\Gamma)$. Then there is a positive constant c such that for all $f \in D(\Gamma)$, the inequality*

$$\|S^m_{\Gamma,b} f\|_{L^{p(\cdot),\theta}_{\beta}(\Gamma)} \le c \|f\|_{L^{p(\cdot),\theta}_{\beta}(\Gamma)}$$

is fulfilled.

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References

1. H. Aimar, Singular integrals and approximate identities on spaces of homogeneous type. *Trans. Amer. Math. Soc.* **292** (1985), no. 1, 135–153.
2. R. R. Coifman, G. Weiss, *Analyse Harmonique Non-Commutative sur Certains Espaces Homoǵenes.* (French) Lecture Notes in Mathematics, 242. Springer-Verlag, Berlin-New York, 1971.
3. D. V. Cruz-Uribe, A. Fiorenza, *Variable Lebesgue Spaces. Foundations and Harmonic Analysis.* Applied and Numerical Harmonic Analysis. Birkhäuser/Springer, Heidelberg, 2013.
4. L. Diening, P. Harjulehto, P. Hästö, M. Ružička, *Lebesgue and Sobolev Spaces with Variable Exponents.* Lecture Notes in Mathematics, 2017. Springer, Heidelberg, 2011.
5. D. E. Edmunds, V. Kokilashvili, A. Meskhi, *Bounded and Compact Integral Operators.* Mathematics and its Applications, 543. Kluwer Academic Publishers, Dordrecht, 2002.
6. D. E. Edmunds, V. Kokilashvili, A. Meskhi, Sobolev-type inequalities for potentials in grand variable exponent Lebesgue spaces. *Math. Nachr.*, 2019, 15 pp., DOI: 10.1002/mana.201800239.
7. A. Fiorenza, V. Kokilashvili, A. Meskhi, Hardy-Littlewood maximal operator in weighted grand variable exponent Lebesgue space. *Mediterr. J. Math.* **14** (2017), no. 3, Art. 118, 20 pp.
8. L. Greco, T. Iwaniec, C. Sbordone, Inverting the p-harmonic operator. *Manuscripta Math.* **92** (1997), no. 2, 249–258.
9. P. Harjulehto, P. Hästö, M. Pere, Variable exponent Lebesgue spaces on metric spaces: the Hardy-Littlewood maximal operator. *Real Anal. Exchange* **30** (2004/05), no. 1, 87–103.
10. T. Iwaniec, C. Sbordone, On the integrability of the Jacobian under minimal hypotheses. *Arch. Rational Mech. Anal.* **119** (1992), no. 2, 129–143.
11. V. Kokilashvili, A. Meskhi, Maximal and Calderón-Zygmund operators in grand variable exponent Lebesgue spaces. *Georgian Math. J.* **21** (2014), no. 4, 447–461.
12. V. Kokilashvili, A. Meskhi, H. Rafeiro, S. Samko, *Integral Operators in Non-standard Function Spaces. vol.* 1. *Variable exponent Lebesgue and amalgam spaces.* Operator Theory: Advances and Applications, 248. Birkhäuser/Springer, [Cham], 2016.
13. V. Kokilashvili, A. Meskhi, H. Rafeiro, S. Samko, *Integral Operators in Non-standard Function Spaces. vol.* 2. *Variable exponent Hölder, Morrey-Campanato and grand spaces.* Operator Theory: Advances and Applications, 249. Birkhäuser/Springer, [Cham], 2016.
14. G. Pradolini, O. Salinas, Commutators of singular integrals on spaces of homogeneous type. *Czechoslovak Math. J.* **57 (132)** (2007), no. 1, 75–93.
15. J. O. Strömberg, A. Torchinsky, *Weighted Hardy Spaces.* Lecture Notes in Mathematics, 1381. Springer-Verlag, Berlin, 1989.

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