Transactions of A. Razmadze
Mathematical Institute
Vol. **173** (2019), 121–125

THE PROBLEMS OF A PUNCH IN THE LINEAR THEORY OF VISCO-ELASTICITY

G. KAPANADZE AND L. GOGOLAURI

Abstract. The problem of pressure of a rigid punch upon a viscous half-plane is considered. As is known, building and composition materials possess the property of visco-elasticity and its affect is reflected in the Hook's law. Unlike the elastic bond, the stresses for visco-elastic bodies are proportional to deformations and to their time derivatives. Investigations of different possible forms of visco-elastic correlations can be found in [1–5, 8–10].

The goal of the present work is to extend the well-known Kolosov–Muskhelishvili's method elaborated for the problem of pressure of a rigid punch in the case of the classical theory of plane elasticity to the theory of linear visco-elasticity based of the Kelvin–Vogt model [9].

1. Introduction

One of the models of the linear theory of visco-elasticity is the Kelvin–Vogt model which is characterized by the fact that stresses in the Hook's law are proportional both to deformations and to time derivatives, where the former describes the Hook's law and the latter the Newton law of viscosity.

Following the Kelvin–Vogt model [9], the Hook's law for visco-elasic bodies has the form

$$\begin{aligned} X_x &= \lambda\theta + 2\mu e_{xx} + \lambda^*\dot{\theta} + 2\mu^*\dot{e}_{xx}, \\ Y_y &= \lambda\theta + 2\mu e_{yy} + \lambda^*\dot{\theta} + 2\mu^*\dot{e}_{yy}, \\ X_y &= \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + \mu^*\left(\frac{\partial \dot{v}}{\partial x} + \frac{\partial \dot{u}}{\partial y}\right), \end{aligned} \tag{1}$$

where $\vartheta = e_{xx} + e_{yy} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$, X_x, Y_y, X_y, u, v, e_{xx}, e_{yy}, e_{xy} are the functions of variables x, y, t. Under t we mean the time parameter and the points in the expressions $\dot{\theta}, \dots, \dot{u}$ denote derivatives in time t; λ, μ and λ^*, μ^* are, respectively, elastic and visco-elastic constants.

In what follows, the use will be made of the known Kolosov–Muskhelishvili's formulas which may be referred to any solid bodies. The above-mentioned formulas are of the form [6]

$$\begin{aligned} X_x + Y_y &= 4\operatorname{Re}[\Phi(z,t)] = 4\operatorname{Re}[\varphi'(z,t)], \\ Y_y - X_x + 2iX_y &= 2[\overline{z}\varphi''(z,t) + \psi'(z,t)] = 2[\overline{z}\Phi'(z,t) + \Psi(z,t)], \end{aligned} \tag{2}$$

where $\Phi(z,t) = \varphi'(z,t)$; $\Psi(z,t) = \psi'(z,t)$. From (2) we have the formula

$$Y_y - iX_y = \Phi(z,t) + \overline{\Phi(z,t)} + z\overline{\Phi'(z,t)} + \overline{\Psi(z,t)}, \tag{3}$$

which will frequently be used in the sequel.

The principle vector (X,Y) of external forces applied to the boundary is assumed to be finite and the stresses and rotation vanish at infinity, hence for large $|z|$, we have

$$\Phi(z,t) = -\frac{X+iY}{2\pi z} + o\left(\frac{1}{z}\right); \quad \Psi(z,t) = \frac{X-iY}{2\pi z} + o\left(\frac{1}{z}\right).$$

From relations (1), in view of (2), for the function $\vartheta(z,t) = e_{xx} + e_{yy}$ we get the differential equation

$$\dot{\vartheta}(z,t) + k\vartheta(z,t) = \frac{2}{\lambda^* + \mu^*}\operatorname{Re}[\varphi'(z,t)], \quad \left(k = \frac{\lambda+\mu}{\lambda^*+\mu^*}\right),$$

2010 *Mathematics Subject Classification.* 74B05.

Key words and phrases. The Kelvin–Voigt model; Kolosov–Muskhelishvili formulas; The problem of punch; The boundary value problem of linear conjugation.

a solution of which under zero initial conditions (i.e., for $\vartheta(z;0)=0$) has the form

$$\vartheta(z,t)=\frac{2}{\lambda^*+\mu^*}\int_0^t \operatorname{Re}[\varphi'(z,\tau)]e^{k(\tau-t)}d\tau. \tag{4}$$

Analogously, from (1) and (2), for the function $\gamma(z,t)=e_{xx}-e_{yy}$ we obtain the differential equation

$$\dot\gamma(z,t)+m\gamma(z,t)=-\frac{1}{\mu^*}\operatorname{Re}[\overline{z}\varphi''(z,t)+\psi'(z,t)], \quad \left(m=\frac{\mu}{\mu^*}\right),$$

a solution of which under zero initial conditions has the form

$$\gamma(z,t)=-\frac{1}{\mu^*}\int_0^t \operatorname{Re}[\overline{z}\varphi''(z,\tau)+\psi'(z,\tau)]e^{m(\tau-t)}d\tau. \tag{5}$$

Thus in view of (4) and (5), with respect to e_{xx} and e_{yy}, we have a system which solution is represented as follows:

$$\begin{aligned} e_{xx}=&\frac{1}{2\mu^*}\int_0^t \operatorname{Re}\left[æ^*\varphi'(z,\tau)e^{k(\tau-t)}-\left(\overline{z}\varphi''(z,\tau)+\psi'(z,\tau)\right)e^{m(\tau-t)}\right]d\tau,\\ e_{yy}=&\frac{1}{2\mu^*}\int_0^t \operatorname{Re}\left[æ^*\varphi'(z,\tau)e^{k(\tau-t)}+\left(\overline{z}\varphi''(z,\tau)+\psi'(z,\tau)\right)e^{m(\tau-t)}\right]d\tau, \end{aligned} \tag{6}$$

where

$$æ^*=\frac{2\mu^*}{\lambda^*+\mu^*}.$$

Taking into account equalities $dx=dz$, $dx=d\overline{z}$, $dy=-idz$, $dy=id\overline{z}$, and integrating (6), we obtain the formula

$$2\mu^*(u+iv)=\int_0^t \left[æ^*\varphi(z,\tau)e^{k(\tau-t)}+\left(\varphi(z,\tau)-z\overline{\varphi'(z,\tau)}-\overline{\psi(z,\tau)}\right)e^{m(\tau-t)}\right]d\tau+2\mu^*(u_0+iv_0), \tag{7}$$

where $u_0=u(z,0)$, $v_0=v(z,0)$.

Formula (7) is an analogue of Kolosov–Muskhelishvili's formula for the second basic problem of the plane theory of elasticity (see [6]) in the case of a visco-elastic isotropic body.

From formula (7), differentiating with respect to x, we get

$$\begin{aligned} 2\mu^*v'(x,y,t)=&\operatorname{Im}\left[\int_0^t æ^*e^{k(\tau-t)}\Phi(z,\tau)d\tau\right]\\ &+\operatorname{Im}\left[\int_0^t e^{m(\tau-t)}\left(\Phi(z,\tau)-\overline{\Phi(z,\tau)}-z\overline{\Phi'(z,\tau)}-\overline{\Psi(z,\tau)}\right)d\tau\right]+2\mu^*v_0'(x,y,0). \end{aligned} \tag{8}$$

Statement of the Problem. Let a visco-elastic body occupy the lower half-plane S^-. By L we denote the boundary of that domain (i.e., the Ox-axis) and assume that a segment $L'=[-1;1]$ enters in contact with a punch having a given base shape and the punch is pressed into the half-plane with a given force directed vertically downward. Assume also that the punch displacement is translational in a normal direction with respect to the boundary, in the absence of friction. Under the given assumptions, tangential stress is zero and the boundary conditions have the form

$$\begin{gathered} X_y^-(x,0,t)=0, \quad x\in L; \quad Y_y^-(x,0,t)=0, \quad x\in L''=L-L';\\ v^-(x,0,t)=f(x,t), \quad x\in L', \end{gathered} \tag{9}$$

where $f(x,0)=f(x)$ is the given function defining the shape of the punch base before it is pressed into the half-plane.

In the sequel, the expression $v^-(x,0,t)$ will be written as $v^-(x,0,t) = v^-(x,t)$ and so we will do for other similar expressions.

Assume that external forces acting on the punch have a resultant

$$X = 0, \qquad Y = -N_0 = -\int_{-1}^{1} N(x,t)dx,$$

where $N(x,t)$ is a normal stress at the point $x \in L'$.

Our problem is to define elastic equilibrium of the domain S^- and normal stress $P(x,t)$ acting under the punch.

Solution of the Problem. Passing in (8) to the limit as $z \to x \in L'$ ($z \in S^-$) and taking into account (3) and (9), we have

$$\operatorname{Im}\left[æ^* e^{-kt}\int_0^t \Phi^-(x,\tau)e^{k\tau}d\tau + 2e^{-mt}\int_0^t \Phi^-(x,\tau)e^{m\tau}d\tau\right] = 2\mu^* v'^-(x,t) - 2\mu^* v_0'^-(x,0). \tag{10}$$

Following N. I. Muskhelishvili (see [6]), we extend the function $\Phi(z,t)$ to the upper half-plane (S^+) in such a way that its values continue analytically the values of $\Phi(z,t)$ in the lower half-plane through the unloaded sections (i.e., on the section L'').

In our case, proceeding from the boundary conditions and formula (3), we define $\Phi(z,t)$ in S^+ as follows:

$$\Phi(z,t) = -\Phi_*(z,t) - z\Phi'_*(z,t) - \Psi_*(z,t), \qquad z \in S^+, \tag{11}$$

where $\Phi_*(z,t) = \overline{\Phi(\overline{z},t)}$; $\Psi_*(z,t) = \overline{\Psi(\overline{z},t)},$. From (11), we have

$$\Phi_*(z,t) = -\Phi(z,t) - z\Phi'(z,t) - \Psi(z,t).$$

The obtained in a such a way piecewise-holomorphic function we again denote by $\Phi(z,t)$. Then for finding the function $\Psi(z,t)$ by means of $\Phi(z,t)$, we obtain the following correlation

$$\Psi(z,t) = -\Phi(z,t) - \Phi_*(z,t) - z\Phi'(z,t),$$

and thus the stress and displacement components are expressed by one piecewise-holomorphic function $\Phi(z,t)$. Substituting the obtained value of $\Psi(z,t)$ into (3), we find that

$$Y_y - iX_y = \Phi(z,t) - \Phi(\overline{z},t) + (z-\overline{z})\overline{\Phi'(z,t)}.$$

On the basis of the above formula, we have

$$Y_y^-(x,t) - i\,X_y^-(x,t) = \Phi^-(x,t) - \Phi^+(x,t), \quad x \in L', \tag{12}$$

or passing to the complex-conjugate value and taking into account the equalities $\overline{\Phi^-(x,t)} = \Phi_*^+(x,t)$; $\overline{\Phi^+(x,t)} = \Phi_*^-(x,t)$, we obtain

$$Y_y^-(x,t) + i\,X_y^-(x,t) = \Phi_*^+(x,t) - \Phi_*^-(x,t). \tag{13}$$

Subtracting (18) and (13), in view of the fact that $X_y^-(x,t) = 0$, $x \in L$, we obtain

$$\Phi^-(z,t) + \Phi_*^-(z,t) = \Phi^+(z,t) + \Phi_*^+(z,t).$$

This implies that the function $\Phi(z,t) + \Phi_*(z,t)$ is holomorphic on the whole plane, and since it vanishes at infinity, we have the equality

$$\Phi_*(z,t) = -\Phi(z,t). \tag{14}$$

We get back now to equality (10). On the basis of (14), formula (10) can be written in the form

$$e^{-kt}\int_0^t æ^* e^{k\tau}\Omega(x,\tau)d\tau + 2e^{-mt}\int_0^t e^{m\tau}\Omega(x,\tau)d\tau = 4i\mu^* f_1(x,t), \tag{15}$$

where

$$\Omega(x,t) = \Phi^+(x,t) + \Phi^-; \quad f_1(x,t) = 4i\mu^*[f'(x,t) - f'(x)]. \tag{16}$$

Differentiating (15) with respect to t, we obtain

$$-ke^{-kt}\int_0^t æ^* e^{k\tau}\Omega(x,\tau)d\tau-2me^{-mt}\int_0^t e^{m\tau}\Omega(x,\tau)d\tau+(æ^*+2)\Omega(x,t)=\dot f_1(x,t). \tag{17}$$

Multiplying (15) by m and summing with (17), we get

$$æ^*(m-k)\int_0^t e^{k\tau}\Omega(x,\tau)d\tau+(æ^*+2)e^{kt}\Omega(x,t)=\dot f_2(x,t),$$

where

$$f_2(x,t)=e^{kt}[f_1(x,t)+mf_1(x,t)] \tag{18}$$

from which after differentiation with respect to t, we obtain the following equation:

$$\dot\Omega(x,t)+n\,\Omega(x,t)=-\frac{\dot f_2(x,t)}{æ^*+2}e^{-kt}, \tag{19}$$

where

$$n=\frac{\mathrm{m}æ^*+2k}{æ^*+2}.$$

Substituting $t=0$ into (17), we have

$$\Omega(x,0)=\frac{\dot f_1(x,0)}{æ^*+2}. \tag{20}$$

A solution of differential equation (19) under the initial condition (20) takes the form

$$\Omega(x,t)=e^{-nt}\left[\Omega(x,0)+\int_0^t\frac{e^{(n-k)\tau}\dot f_2(x,\tau)}{æ^*+2}d\tau\right].$$

Thus, on the basis of (16), for the function $\Phi(z,t)$ we obtain the boundary value problem of linear conjugation

$$\Phi^+(x,t)+\Phi^-(x,t)=F(x,t), \tag{21}$$

where

$$F(x,t)=e^{-nt}\left[\Omega(0,\omega)+\frac{1}{æ^*+2}\int_0^t e^{(n-k)\tau}\dot f_2(x,\tau)d\tau\right]. \tag{22}$$

The vanishing at infinity solution of problem (21) of the class h_0 (i.e., unbounded at the ends of the segment L') has the form (see [7])

$$\Phi(z,t)=-\frac{1}{2\pi\chi_0(z)}\int_{-1}^1\frac{\chi_0^+(\sigma)F(\sigma,t)}{\sigma-z}d\sigma+\frac{C_0}{\chi_0(z)},$$

where $\chi_0(z)=\sqrt{(1-z)(1+z)}=-i\sqrt{(z-1)(z+1)}$, $\chi_0^+(\sigma)$ is the positive value of the function $\chi_0(z)$ on the left-hand side (i.e., from S^+) of the segment L'.

Taking into account behaviour of the function $\Phi(z,t)\chi_0(z)$ at infinity, for the constant C_0 we obtain the formula

$$C_0=\frac{N_0}{2\pi}.$$

For the normal stress $P(x,t)$ under the punch, on the basis of (18) and (14), we get

$$P(x,t)=2\,\mathrm{Re}\,\Phi^+(x,t),$$

or taking into account that from (22) follows

$$\mathrm{Re}\,\Phi^+(x,t)=\frac{1}{2\pi i}\frac{1}{\chi_0^+(x)}\int_{-1}^1\frac{\chi_0^+(\sigma)F(\sigma,t)}{\sigma-x}d\sigma+\frac{N_0}{2\pi\chi_0^+(x)},$$

we will have

$$P(x,t) = -\frac{1}{\pi i \chi_0^+(x)} \int_{-1}^{1} \frac{\chi_0^+(\sigma) F(\sigma,t)}{\sigma - x} + \frac{N_0}{\pi \chi_0^+(x)},$$

where $F(\sigma,t)$ is defined by formula (22).

References

1. R. D. Bantsuri, N. N. Shavlakadze, Solutions of integro-differential equations related to contact problems of viscoelasticity. *Georgian Math. J.* **21** (2014), no. 4, 393–405.
2. D. R. Bland, *The Theory of Linear Viscoelasticity.* Pergamon press, Oxford, 1960.
3. R. M. Christensen, *Theory of Viscoelasticity.* (Russian) Mir, Moscow, 1974.
4. M. Fabririo, A. Morro, *Mathematical Problems in Linear Viscoelasticity.* SIAM Studies in Applied Mathematics, 12. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992.
5. L. A. Galin, *Contact Problems in the Theory of Elasticity and Viscoelasticity.* (Russian) Nauka, Moscow, 1980.
6. N. I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity.* (Russian) Nauka, Moscow, 1966.
7. N. I. Muskhelishvili, *Singular Integral Equations.* (Russian) Nauka, Moscow, 1968.
8. A. C. Pipkin, *Lecture on Viscoelasticity Theory.* New York, Springer Verlang, 1971.
9. Yu. N. Rabotnov, *Elements of Continuum Mechanics of Materials with Memory.* (Russian) Nauka, Moscow, 1977.
10. N. N. Shavlakadze, The effective solution of two-dimensional integro-differential equations and their applications in the theory of viscoelasticity. *ZAMM Z. Angew. Math. Mech.* **95** (2015), no. 12, 1548–1557.

A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 6 Tamarashvili Str., Tbilisi 0177, Georgia

E-mail address: kapanadze.49@mail.ru

E-mail address: lida@rmi.ge