Transactions of A. Razmadze
Mathematical Institute
Vol. **173** (2019), 111–119

WAVE PROPAGATION THROUGH A SQUARE LATTICE WITH SOURCES ON LINE SEGMENTS

D. KAPANADZE

Abstract. We study the problems related to the propagation of time harmonic waves through a two-dimensional square lattice with sources on line segments. The discrete Helmholtz equation with the wave number $k \in (0, 2\sqrt{2})\backslash\{2\}$ and input data prescribed on finite rows/columns of lattice sites is investigated without passing to the complex wave number. Similarly to the continuum theory, we use the notion of radiating solution. The unique solvability result and the Green's representation formula are obtained with the help of difference potentials. Finally, we propose a method for numerical calculation. Efficiency of our approach is demonstrated in examples related to the propagation problems in the left-handed 2D inductor-capacitor metamaterial.

1. Introduction

Consider two-dimensional passive propagation media that can be used for signal processing and filtering. Assume that at the fine scale these media consist of a lattice of repeated single type cells. As an example of such propagation media, we can take a host microstrip line network periodically loaded with series capacitors and shunt inductors as is shown in Figure 1. This type of inductor-capacitor lattice is referred to a negative-refractive-index transmission-line (NRI-TL) metamaterial [6], or simply, left-handed 2D metamaterial. Suppose that monochromatic inputs are applied to finite rows/columns of lattice sites. Assume that the number of unit cells in this slab is large enough to make it prohibitively expensive to solve numerically for the voltage/current at every cell in the lattice until the system reaches steady state. As a simplifying strategy, it can be anticipated that the limiting case, when the lattice is effectively infinite, is more amenable to analysis and provides a good approximation of the steady-state output at an exterior boundary.

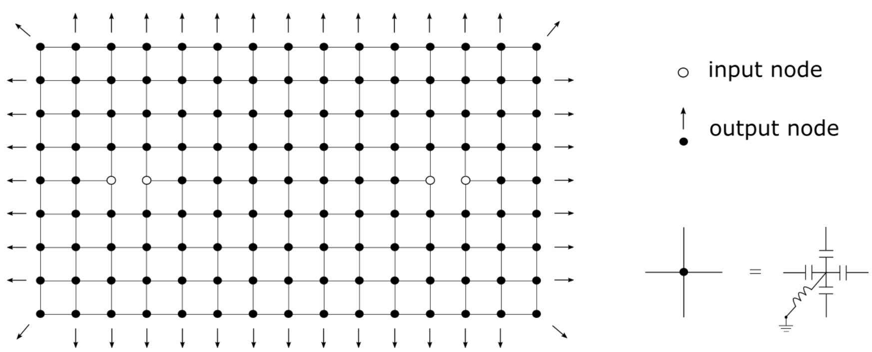


Figure 1. Left-handed 2D inductor-capacitor metamaterial with sources on two line segments. A host transmission-line is loaded periodically with series capacitors and shunt inductors.

The present paper examines the effect of finite sources on line segments in an infinite square lattice. Mathematical modeling of the propagation problem under consideration leads us to study an exterior problem for a discrete 2D Helmholtz equation with Dirichlet boundary conditions. Note that the same

2010 *Mathematics Subject Classification.* 78A45, 35J05, 39A14, 39A60, 74S20.
Key words and phrases. Discrete wave equation; Helmholtz equation; Crack/screen problems; Lattice model; Metamaterials.

category of problems can be originated by composite right/left-handed and dual-composite right/left-handed lattices [2,3,5,6,11,14] and mass spring lattices (cf., e.g., [18,19]).

It is well known that for the negative discrete Laplacian the spectrum is (absolutely continuous) $[0, 8]$, but there is an exceptional set $\{0, 4, 8\}$ in $[0,\ 8]$, where the limiting absorption principle fails [15]. Therefore for "admissible" wave numbers $k \in (0, 2\sqrt{2})\backslash\{2\}$, one can study the problem as the limit $k + \iota 0$ of the complex wave number. This method is applied by Sharma in [16–19], where diffraction problems on a square lattice are investigated by a finite crack and a rigid constraint. In this paper, we use the results obtained in [9] and carry out our investigation without passing to the complex wave number. We use the radiation conditions and asymptotic estimates from Shaban et al. [15], a Rellich-Vekua type theorem from Isozaki el al. [8], and asymptotic estimates of the lattice Green's function derived by Martin [12]. Further, the unique solvability result and the Green's representation formula are obtained with the help of difference potentials. For the numerical calculation, we apply the method developed by Berciu et al. [1] which allows us to calculate the lattice Green's functions without the need to perform integrals and appears to be much more effective than the recurrence relations due to Morita [13].

2. Basic Notations and Formulation of the Problem

Following the customary notation in mathematics, let $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ denote the set of integers, positive integers, real numbers, and complex numbers, respectively. We denote by $e_1 = (1, 0)$, $e_2 = (0, 1)$ the standard base of the square lattice $\mathbb{Z}^2$ $(= \mathbb{Z} \times \mathbb{Z})$.

For any point $x = (x_1, x_2) \in \mathbb{Z}^2$, we define the 4-neighborhood F_x^0 as the set of points $\{(x_1 - 1, x_2), (x_1 + 1, x_2), (x_1, x_2 - 1), (x_1, x_2 + 1)\}$ and the neighborhood F_x as $F_x^0 \bigcup \{x\}$. We say that $R \subset \mathbb{Z}^2$ is a region if there exist disjoint nonempty subsets $\mathring{R}$ and ∂R of R such that

(a) $R = \mathring{R} \cup \partial R$,

(b) if $x \in \mathring{R}$ then $F_x \subset R$,

(c) if $x \in \partial R$ then there is at least one point $y \in F_x^0$ such that $y \in \mathring{R}$.

Clearly, the subsets $\mathring{R}$ and ∂R are not defined uniquely by R, but henceforth, for a given region R in $\mathbb{Z}^2$ it will always be assumed that $\mathring{R}$ and ∂R are also given and fixed. Next, we say that x is an interior (boundary) point of R if $x \in \mathring{R}$ ($x \in \partial R$). Further, a region $R \subset \mathbb{Z}^2$ is said to be connected if for any $y, z \in R$ there exists a sequence $x^{(1)}, \dots, x^{(n)} \in R$ with $x^{(1)} = y$ and $x^{(n)} = z$, such that for all $0 \le i \le n - 1$, $|x^{(i)} - x^{(i+1)}| = 1$. By definition, a region R with one interior point x is connected and coincides with F_x. Denote by S_N a region defined as a discrete square $([-N, N]^2 \cap \mathbb{Z}^2)\backslash\{(N, N), (-N, N), (-N, -N), (N, -N)\}$, $N \in \mathbb{N}$, where $\mathring{S}_N := [-N + 1, N - 1]^2 \cap \mathbb{Z}^2$ and $\partial S_N := S_N \backslash \mathring{S}_N$ will be fixed throughout the paper.

A boundary point $y \in \partial R$ is said to be

$$\begin{aligned} &\text{a left point if} && y + e_1 \in \mathring{R}, \\ &\text{a right point if} && y - e_1 \in \mathring{R}, \\ &\text{a top point if} && y - e_2 \in \mathring{R}, \\ &\text{a bottom point if} && y + e_2 \in \mathring{R}. \end{aligned}$$

The union of all left (right, top and bottom) points we denote by ∂R_l (∂R_r, ∂R_t and ∂R_b, respectively) and call it a side of the boundary ∂R. Note that a boundary point y can simultaneously be a left, right, top and bottom point. Thus, ∂R_l, ∂R_r, ∂R_t and ∂R_b may overlap each other. Clearly, ∂R is the union of its four sides, $\partial R = \partial R_l \bigcup \partial R_r \bigcup \partial R_t \bigcup \partial R_b$.

Let Γ_j, $j = 1, \dots, n$, $n \in \mathbb{N}$, be a finite row or column of lattice sites and consider a region $\mathring{\Omega} = \mathbb{Z}^2 \backslash \partial\Omega$, where $\partial\Omega = \cup_{j=1}^n \Gamma_j$. From now on, we assume that Γ_j are located so that $\mathring{\Omega}$ is connected and satisfies the cone condition, cf. [8,9]. Finally, we emphasize that $\Omega = \mathring{\Omega} \cup \partial\Omega$ and $\mathbb{Z}^2$ coincide as the sets.

Given the problem and assumptions described in Introduction, we suppose that there is an inductor connecting each node $x \in \mathring{\Omega}$ to a common ground plane, and there is a capacitor connecting each node

$x \in \mathring{\Omega}$ to its four nearest neighbors $(x_1 \pm 1, x_2 \pm 1)$ (cf., Figure 1). Assume that all inductances equal to L and all capacitances equal to C, where both L and C are positive constants. Then, Kirchhoff's laws of voltage and current (while suppressing the explicit dependence on time t) imply the following second-order equation for the voltage $U(x)$ across the inductor at the node x:

$$LC\frac{d^2}{dt^2}(\Delta_d U(x)) = U(x). \tag{1}$$

Here, Δ_d denotes the discrete Laplacian defined as follows:

$$\Delta_d U(x) = \sum_{i=1}^{2}(U(x+e_i) + U(x-e_i)) - 4U(x). \tag{2}$$

We specify that (1) holds for all $x \in \mathring{\Omega}$ and along the boundary $\partial\Omega$ we have the time-dependent boundary condition

$$U(y) = f(y)e^{-\iota\omega t}. \tag{3}$$

Here, ι denotes the imaginary unit, and $f : \partial\Omega \to \mathbb{C}$ is a given function. We assume that at time $t = 0$, $U(x)$ and all its derivatives are zero for all $x \in \mathring{\Omega}$. Then, as t increases, the boundary term causes wave to propagate into the lattice, and the system approaches steady state. At this point, the solution is given by $U(x) = u(x)e^{-\iota\omega t}$. Substituting this expression into (1) and (3), for the discrete Helmholtz equation in Ω, we obtain the following problem:

$$(\Delta_d + k^2)u(x) = 0, \qquad \text{in } \mathring{\Omega}, \tag{4a}$$

$$u(y) = f(y), \quad \text{on } \partial\Omega, \tag{4b}$$

where k and ω are related through the relation $k^2 = (\omega^2 LC)^{-1}$. It is well known that (1) admits plane wave solutions $U(x) = Ae^{\iota(-\xi_1 x_1 - \xi_2 x_2 - \omega t)}$, where $A \in \mathbb{C}$ is a constant, as long as the following dispersion relation

$$\omega^2 = \frac{1}{4LC(\sin^2\frac{\xi_1}{2} + \sin^2\frac{\xi_2}{2})},$$

is satisfied, where $(\xi_1, \xi_2) \in [-\pi, \pi]^2$ known as the first Brillouin zone [4]. Thus we have

$$k^2 = 4\left(\sin^2\frac{\xi_1}{2} + \sin^2\frac{\xi_2}{2}\right) \in [0, 8].$$

Clearly, other values of k are also a subject of investigation, but this case is rather straightforward and will not be considered here.

Recall that for the negative discrete Laplacian, the spectrum is (absolutely continuous) $[0, 8]$, but there is an exceptional set $\{0, 4, 8\}$ in $[0, 8]$, where the limiting absorption principle fails (cf., [15]). Consequently, we assume that $k \in (0, 2\sqrt{2})\backslash\{2\}$.

Thus, we are interested in studying the problem of the existence and uniqueness of a function $u : \Omega \to \mathbb{C}$ such that $u(x)$ satisfies the discrete Helmholtz equation (4a) with $k \in (0, 2\sqrt{2})\backslash\{2\}$ and the boundary condition (4b). From now on, we will refer to this problem as Problem $\mathcal{P}$.

3. Green's Representation Formula and Uniqueness Result

In this section we mainly recall the results from [9]. Let R be a region in $\mathbb{Z}^2$. As it was already mentioned above, $y \in \partial R$ may be a point of intersection of several sides of ∂R. However, in our arguments presented below, it will always be clear which side is needed to be considered. Under this condition, we define the discrete derivative in the outward normal direction

$$Tu(y) = u(y) - u(y - \nu_y), \quad y \in \partial R,$$

where ν_y is $-e_1$ (e_1, e_2 or $-e_2$) if y is an element of ∂R_l (∂R_r, ∂R_t or ∂R_b).

Let R be a finite region. Then we have a discrete analogues of the Green's first and second identities

$$\sum_{x\in\mathring{R}}(\nabla_d^+ u(x)\cdot\nabla_d^+ v(x) + \nabla_d^- u(x)\cdot\nabla_d^- v(x) + u(x)\Delta_d v(x)) = \sum_{y\in\partial R} u(y)Tv(y), \tag{5}$$

and

$$\sum_{x\in\mathring{R}}(u(x)\Delta_d v(x)-v(x)\Delta_d u(x))=\sum_{y\in\partial R}(u(y)Tv(y)-v(y)Tu(y)), \tag{6}$$

respectively. Here, ∇_d^+ and ∇_d^- are defined as follows:

$$\nabla_d^+ u(x):=\begin{pmatrix}u(x+e_1)-u(x)\\ u(x+e_2)-u(x)\end{pmatrix},$$

and

$$\nabla_d^- u(x):=\begin{pmatrix}u(x-e_1)-u(x)\\ u(x-e_2)-u(x)\end{pmatrix}.$$

The next step in deriving the Green's representation formula is the introduction of the Green's function. Denote by $\mathcal{G}(x-y)$ the Green's function for (4a) centered at the point x and evaluated at y. Then $\mathcal{G}(x-y)$ satisfies

$$(\Delta_d+k^2)\mathcal{G}(x-y)=\delta_{x,y}, \tag{7}$$

where $\delta_{x,y}$ is the Kronecker delta. The lattice Green's function $\mathcal{G}$ is quite well known (cf., e.g., [7, 10, 12, 20]) and can be written in the following form:

$$\mathcal{G}(x)=\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{e^{-\iota(x_1\xi_1+x_2\xi_2)}}{\sigma(\xi_1,\xi_2;k)}\mathrm{d}\xi_1\mathrm{d}\xi_2, \tag{8}$$

or, equivalently, as

$$\mathcal{G}(x)=\frac{1}{\pi^2}\int_{0}^{\pi}\int_{0}^{\pi}\frac{\cos(x_1\xi_1)\cos(x_2\xi_2)}{\sigma(\xi;k)}\mathrm{d}\xi_1\mathrm{d}\xi_2, \tag{9}$$

where

$$\begin{aligned}\sigma(\xi;k)&=k^2-4+2\cos\xi_1+2\cos\xi_2\\&=k^2-4\sin^2\frac{\xi_1}{2}-4\sin^2\frac{\xi_2}{2}\\&=k^2-8+4\cos^2\frac{\xi_1}{2}+4\cos^2\frac{\xi_2}{2}\\&=k^2-4+4\cos\frac{\xi_1+\xi_2}{2}\cos\frac{\xi_1-\xi_2}{2},\quad \xi=(\xi_1,\xi_2).\end{aligned} \tag{10}$$

Notice that if $k^2\in\mathbb{C}\backslash[0,8]$, then $\sigma\neq 0$ and, consequently, $\mathcal{G}(x)$ in (8) is well defined. In this case, $\mathcal{G}(x)$ decays exponentially when $|x|\to\infty$. For the cases $0<k^2<4$ and $4<k^2<8$, the expression (8) is understood as follows: we replace k^2 by $k^2+\iota\varepsilon$ with $0<\varepsilon\ll 1$, and let $\varepsilon\to 0$ at the end of the calculation. We use Koster's method to evaluate the integrals, cf., [12]. Finally, notice that $\mathcal{G}(x-y)=\mathcal{G}(y-x)$.

Theorem 1. *Let R be a finite region. Then for a given function $u:\mathring{R}\to\mathbb{C}$ and any point $x\in\mathring{R}$, we have a discrete Green's representation formula*

$$u(x)=\sum_{y\in\partial R}\big(u(y)T\mathcal{G}(x-y)-\mathcal{G}(x-y)Tu(y)\big)+\sum_{y\in\mathring{R}}\mathcal{G}(x-y)(\Delta_d+k^2)u(y).$$

In particular, if u is a solution to the discrete Helmholtz equation

$$(\Delta_d+k^2)u(x)=0\quad\text{in }\mathring{R},$$

then

$$u(x)=\sum_{y\in\partial R}\big(u(y)T\mathcal{G}(x-y)-\mathcal{G}(x-y)Tu(y)\big). \tag{11}$$

Further, we need to apply the notion of a radiation condition for the discrete Helmholtz operators. We emphasize that we are dealing with the case $k \in (0, 2\sqrt{2})\backslash\{2\}$, and, therefore, it is natural to require an extra condition at infinity (cf., [15]). We say that $u : \Omega \to \mathbb{C}$ satisfies the radiation condition at infinity, if

$$\begin{cases} u(x) = O(|x|^{-\frac{1}{2}}) \\ u(x+e_j) = -e^{\iota\xi_j^*(\alpha,k)}u(x) + O(|x|^{-\frac{3}{2}}), \quad j = 1, 2, \end{cases} \tag{12}$$

with the remaining term decaying uniformly in all directions $x/|x|$, where $x \in \Omega$ is characterized as $x_1 = |x|\cos\alpha$, $x_2 = |x|\sin\alpha$, $0 \le \alpha < 2\pi$. Here, $\xi_j^*(\alpha, k)$ is the jth coordinate of the point $\xi^*(\alpha, k)$ defined as follows:

$$\begin{aligned} \xi_1^*(\alpha,k) &= 2\,\mathrm{sgn}\big((\pi/2-\alpha)(3\pi/2-\alpha)\big)\arcsin\left(\frac{k}{2}\cos\theta^*\right), \\ \xi_2^*(\alpha,k) &= 2\,\mathrm{sgn}\big(\pi-\alpha\big)\arcsin\left(\frac{k}{2}\sin\theta^*\right), \end{aligned} \tag{13}$$

for $0 < k^2 < 4$. Here,

$$\theta^* = \theta^*(\alpha,k) := \begin{cases} \arctan\sqrt{-\lambda+\sqrt{\lambda^2+\tan^2\alpha}}, & \text{if} \quad \alpha \ne \frac{\pi}{2}, \frac{3\pi}{2}, \\ \frac{\pi}{2}, & \text{if} \quad \alpha = \frac{\pi}{2}, \frac{3\pi}{2}, \end{cases} \tag{14}$$

where

$$\lambda = \lambda(\alpha,k) := \frac{2(1-\tan^2\alpha)}{4-k^2}.$$

In the second case $4 < k^2 < 8$, we have

$$\begin{aligned} \xi_1^*(\alpha,k) &= 2\,\mathrm{sgn}\big((\pi/2-\alpha)(3\pi/2-\alpha)\big)\arccos\left(\frac{\sqrt{8-k^2}}{2}\cos\theta^*\right), \\ \xi_2^*(\alpha,k) &= 2\,\mathrm{sgn}\big(\pi-\alpha\big)\arccos\left(\frac{\sqrt{8-k^2}}{2}\sin\theta^*\right), \end{aligned} \tag{15}$$

where θ^* is defined as in (14) with

$$\lambda = \lambda(\alpha,k) := \frac{2(1-\tan^2\alpha)}{k^2-4}.$$

Definition 2. Let $k \in (0, 2\sqrt{2})\backslash\{2\}$. A solution u to the discrete Helmholtz equation (4a) is called radiating if it satisfies the radiation condition (12).

Notice that the second condition in (12) can be written in the following forms

$$u(x) - u(x+e_j) = \left(1 + e^{\iota\xi_j^*(\alpha,k)}\right)u(x) + O(|x|^{-\frac{3}{2}}), \quad |x| \to \infty, \tag{16}$$

and

$$u(x+e_j) - u(x) = \left(1 + e^{-\iota\xi_j^*(\alpha,k)}\right)u(x+e_j) + O(|x|^{-\frac{3}{2}}), \quad |x| \to \infty. \tag{17}$$

Notice also that $u(x) = O(|x|^{-\frac{1}{2}})$ implies

$$\frac{1}{N}\sum_{x\in S_{2N}\backslash S_N} |u(x)|^2 < const < \infty$$

for any integer $N \ge N_0$.

For the fixed point $x \in \mathbb{Z}^2$ and any point $y \in \partial S_N$, the radiation conditions (12) imply

$$\sum_{y\in\partial S_N} \big(u(y)T\mathcal{G}(x-y) - \mathcal{G}(x-y)Tu(y)\big) \to 0, \quad N \to \infty.$$

Indeed, for instance, since $\alpha(y-x)$ tends to $\alpha=\alpha(y)$ as $|y|\to\infty$, therefore for a sufficiently large N, we have

$$\begin{aligned} u(y)\big(\mathcal{G}(x-y)-\mathcal{G}(x-(y+e_1))\big)-\mathcal{G}(x-y)(u(y)-u(y+e_1)) \\ = u(y)\left[(e^{\iota\xi_j^*(\alpha,k)}+1)\mathcal{G}(x-y)+O(N^{-\frac{3}{2}})\right] \\ -\mathcal{G}(x-y)\left[(e^{\iota\xi_j^*(\alpha,k)}+1)u(y)+O(N^{-\frac{3}{2}})\right] \\ = u(y)\cdot O(N^{-\frac{3}{2}})+\mathcal{G}(x-y)\cdot O(N^{-\frac{3}{2}})=O(N^{-2}). \end{aligned}$$

Consequently, Theorem 1 applied for $\Omega\cap S_N$, where $N\in\mathbb{N}$ is sufficiently large, and then passing to the limit $N\to\infty$, yields the following Green's formula for a radiating solution u to the discrete Helmholtz equation (4a)

$$u(x)=\sum_{y\in\partial\Omega}(u(y)T\mathcal{G}(x-y)-\mathcal{G}(x-y)Tu(y))=\sum_{j=1}^{n}\sum_{y\in\Gamma_j}(u(y)T\mathcal{G}(x-y)-\mathcal{G}(x-y)Tu(y)). \quad (18)$$

Using the results obtained in [12, 15] and (18), we can conclude that every radiating solution u to the discrete Helmholtz equation (4a) has the following asymptotic expansion:

$$u(x)=-\frac{e^{\iota\mu(\alpha,k)|x|}}{|x|^{\frac{1}{2}}}\left\{u_\infty(\hat{x})+O\left(\frac{1}{|x|}\right)\right\},\quad |x|\to\infty, \quad (19)$$

where $\mu(\alpha,k):=\xi^*(\alpha,k)\cdot\hat{x}$, $\hat{x}:=x/|x|$, and $\xi^*(\alpha,k)=(\xi_1^*(\alpha,k),\xi_2^*(\alpha,k))$ (cf., (13) and (15)). Here, the function $u_\infty(\hat{x})$, known as the far field pattern of u, can be expressed with the help of formula (13) from [15].

Finally, let us formulate the following uniqueness theorem.

Theorem 3 (cf. [9])**.** *The problem $\mathcal{P}$ has at most one radiating solution.*

4. Difference Potentials and Existence of a Solution

For any function $\varphi:\partial R\to\mathbb{C}$, we define difference single-layer and double-layer potentials as follows:

$$V\varphi(x)=\sum_{y\in\partial R}\mathcal{G}(x-y)\varphi(y),\quad \text{for all } x\in\mathbb{Z}^2, \quad (20)$$

and

$$W\varphi(x)=\sum_{y\in\partial R}\big(T\mathcal{G}(x-y)+\delta_{x,y}\big)\varphi(y),\quad \text{for all } x\in\mathbb{Z}^2, \quad (21)$$

respectively. Since, $\delta_{x,y}=0$ for every $x\in\mathring{R}$ and $y\in\partial R$, the equation (18) can be written as

$$u(x)=Wu(x)-V(Tu)(x),\quad x\in\mathring{\Omega}.$$

The role of the summand $\delta_{x,y}$ is clarified by the following result.

Lemma 4 (cf. [9])**.** *For every $x\in\mathring{R}$, we have*

$$(\Delta_d+k^2)V\varphi(x)=0,\quad \text{and}\quad (\Delta_d+k^2)W\varphi(x)=0.$$

As a consequence of Lemma 4,

$$V\varphi(x)=\sum_{y\in\partial\Omega}\mathcal{G}(x-y)\varphi(y),\quad x\in\mathring{\Omega},$$

and

$$W\varphi(x)=\sum_{y\in\partial\Omega}\left(T\mathcal{G}(x-y)+\delta_{x,y}\right)\varphi(y),\quad x\in\mathring{\Omega},$$

are radiating solutions to the equation (4a) for any function $\varphi:\partial\Omega\to\mathbb{C}$.

Now we are ready to find a solution to Problem $\mathcal{P}$. In order to reduce a number of numerical computations, we do the following: if y_i is a point of intersection of several sides, we choose and fix only one side of the boundary. Let m be a number of points of $\partial\Omega$. Then $\partial\Omega$ can be represented as a sequence of points $y_1, y_2, ..., y_m$, such that $y_i = y_j$, if and only if $i = j$, for all $1 \le i, j \le m$. Thus, in (20) we will have only one summand related to the boundary point y_i, and we denote the corresponding difference potentials by $\widetilde{V}$. Further, from the given function f on $\partial\Omega$, we form a vector $F = (f_1, \dots, f_m)^\top$, $f_i := f(y_i)$. Similarly, for an unknown function φ, we write $\Phi = (\varphi_1, \dots, \varphi_m)^\top$, $\varphi_i := \varphi(y_i)$.

We look for a solution to Problem $\mathcal{P}$ in the form

$$u(x) = \widetilde{V}\varphi(x) = \sum_{i=1}^{m} \mathcal{G}(x - y_i)\varphi_i, \quad x \in \mathring{\Omega}. \tag{22}$$

As in the proof of Lemma 4, we can easily show that u is a radiating solution to Eq. (4a), and we only need to satisfy the boundary conditions (4b). Then (4b) implies the following linear system of boundary equations

$$\mathcal{H}\Phi = F, \tag{23}$$

where

$$\mathcal{H} = \begin{pmatrix} \mathcal{G}(y_1 - y_1) & \mathcal{G}(y_1 - y_2) & \mathcal{G}(y_1 - y_3) & \dots & \mathcal{G}(y_1 - y_m) \\ \mathcal{G}(y_2 - y_1) & \mathcal{G}(y_2 - y_2) & \mathcal{G}(y_2 - y_3) & \dots & \mathcal{G}(y_2 - y_m) \\ \dots & \dots & \dots & \dots & \dots \\ \mathcal{G}(y_m - y_1) & \mathcal{G}(y_m - y_2) & \mathcal{G}(y_m - y_3) & \dots & \mathcal{G}(y_m - y_m) \end{pmatrix}.$$

Lemma 5. *The linear system of boundary equations* (23) *is uniquely solvable.*

Proof. It is sufficient to proof that the homogeneous system

$$\mathcal{H}\Phi = 0 \tag{24}$$

has only the trivial solution. Let $\Phi^* = (\varphi_1^*, \dots, \varphi_m^*)^\top$ be a nonzero solution to (24). Then

$$u(x) = \widetilde{V}\varphi^*(x)$$

is a radiating solution to the homogeneous Problem $\mathcal{P}$. Therefore, due to Theorem 3, we have $u \equiv 0$ in Ω. Since $\Omega = \mathbb{Z}^2$, at any boundary point $y_i \in \partial\Omega$, we have

$$0 = (\Delta_d + k^2)u(y_i) = (\Delta_d + k^2)\widetilde{V}\varphi^*(y_i) = \sum_{i=1}^{m} \delta_{y_i,y}\varphi^*(y) = \varphi^*(y_i) = \varphi_i^*,$$

for all $1 \le i \le m$. □

Due to a direct combination of the results obtained above, we have now the main conclusions of the present work.

Theorem 6. *Problem $\mathcal{P}$ has a unique radiating solution which is represented as* (22)*, where Φ is a unique solution to the system of linear equations* (23)*.*

5. Numerical Results and Illustrative Examples

The main task in numerically evaluating (23) is to compute the lattice Green's function. For this purpose, we apply the method developed in [1]. Using the eightfold symmetry, we need only to compute the lattice Green's function $\mathcal{G}(i, j)$ with $i \ge j \ge 0$. Following [1], let us introduce the vectors $\mathcal{V}_{2p} = (\mathcal{G}(2p, 0), \mathcal{G}(2p-1, 1), \dots, \mathcal{G}(p, p))^\top$, and $\mathcal{V}_{2p+1} = (\mathcal{G}(2p+1, 0), \mathcal{G}(2p, 1), \dots, \mathcal{G}(p+1, p))^\top$ that collect all distinct Green's functions $\mathcal{G}(i, j)$ with "Manhattan distances" $|i| + |j|$ of $2p$ and $2p + 1$, respectively. For any Manhattan distance larger than 1, the equation

$$(\Delta_d + k^2)\mathcal{G}(x) = \delta_{x,0}$$

can be written in the matrix form $\mathcal{V}_n = \alpha_n(k)\mathcal{V}_{n-1} + \beta_n(k)\mathcal{V}_{n+1}$, where $\alpha_n(k)$ and $\beta_n(k)$ are easy to identify sparse matrices (cf., "Appendix A"). Notice that only the dimensions of these matrices depend on n, their elements are just multiples of $\frac{1}{4-k^2}$ and do not depend on n. It is shown in [1] that for any $n \ge 1$, we have

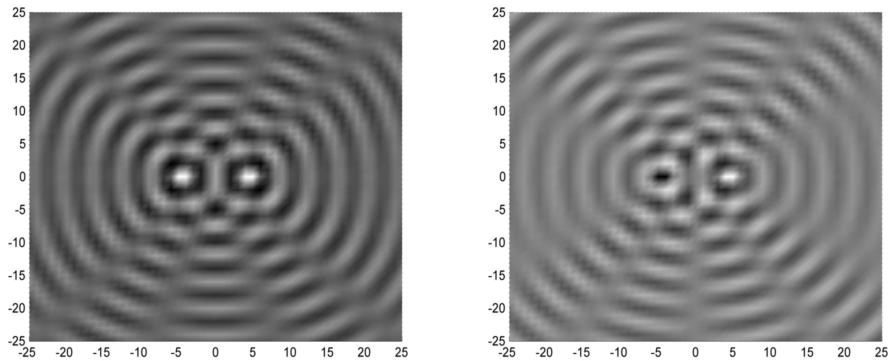
$$\mathcal{V}_n = A_n(k)\mathcal{V}_{n-1},$$

where the matrices $A_n(k)$ are defined by the following recurrence formula:

$$A_n(k) = [1 - \beta_n(k) A_{n+1}]^{-1} \alpha_n(k).$$

They can be computed starting from the sufficiently large N with $A_{N+1}(k) = 0$. However, we may have a better "initial guess" than $A_{N+1}(k) = 0$ (cf., "Appendix A").

Since $A_n(k)$ are known, we have $\mathcal{V}_n = A_n(k) \dots A_1(k)\mathcal{V}_0$, where $\mathcal{V}_0 = \mathcal{G}(0,0)$. In particular, $\mathcal{V}_1 = \mathcal{G}(1,0) = A_1(k)\mathcal{G}(0,0)$ which, together with $(k^2-4)\mathcal{G}(0,0) + 4\mathcal{G}(1,0) = 1$, gives $\mathcal{G}(0,0) = 1/[k^2 - 4 - 4A_1(k)]$. This completes the calculation of Green's function by using elementary operations and no integrals. Notice also one more important advantage of this method. The $A_n(k)$ matrices are calculated coming down from asymptotically large Manhattan distances. As they are propagated towards smaller Manhattan distances, it will definitely give us the physical solution.



(A) The density plot of Re u. Symmetric mode.

(B) The density plot of Re u. Skew-symmetric mode.

Figure 2. In the density plots, darker shade represents the lower values.

Finally, we demonstrate our theoretical and numerical approaches to Problem $\mathcal{P}$, when $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 = \{(-5,0),(-4,0)\}$ and $\Gamma_2 = \{(4,0),(5,0)\}$, cf. Figure 1. We consider two examples. In the first case of the symmetric mode, we take for simplicity $f(y) = 1$ on both line segments, while in the second case of the skew-symmetric mode, we take $f(y) = -1$ on Γ_1 and $f(y) = 1$ on Γ_2.

The vector $\Phi = (\varphi_1, \dots, \varphi_4)^\top$ is a unique solution to (23), where

$$\mathcal{H} = \begin{pmatrix} \mathcal{G}(0,0) & \mathcal{G}(1,0) & \mathcal{G}(9,0) & \mathcal{G}(10,0) \\ \mathcal{G}(1,0) & \mathcal{G}(0,0) & \mathcal{G}(8,0) & \mathcal{G}(9,0) \\ \mathcal{G}(9,0) & \mathcal{G}(8,0) & \mathcal{G}(0,0) & \mathcal{G}(1,0) \\ \mathcal{G}(10,0) & \mathcal{G}(9,0) & \mathcal{G}(1,0) & \mathcal{G}(0,0) \end{pmatrix}$$

is a symmetric matrix. In order to solve the obtained system of linear equations (23) and then find the solution u, we have developed MATLAB code that uses the efficient method described above to compute Green's functions. As a technical aside, these data were obtained in several minutes on a regular desktop. Some results of numerical evaluations are plotted in Figure 2, where (a) and (b) show the density plots of $\Re\mathrm{e}\, u$ for the first and second examples, respectively, when $k = \sqrt{2}$. Some key features of numerical solution can be immediately observed. Namely, the symmetry of $\Re\mathrm{e}\, u$ and the interference of waves.

Appendix A: Sparse Matrices

The sparse matrices $\alpha_n(k)$ and $\beta_n(k)$ are defined as follows: if $n = 2p$, then $\alpha_{2p}(k)$ is a $(p+1) \times p$ matrix such that $\alpha_{2p}(k)|_{i,i} = \frac{1}{4-k^2}$, $i = \overline{1,p}$, $\alpha_{2p}(k)|_{i,i-1} = \frac{1}{4-k^2}$, $i = \overline{2,p}$, while $\alpha_{2p}(k)|_{p+1,p} = \frac{2}{4-k^2}$, and all other matrix elements are zero. The $\beta_{2p}(k)$ is a $(p+1) \times (p+1)$ matrix such that $\beta_{2p}(k)|_{i,i} =$

$\frac{1}{4-k^2}$, $i=\overline{1,p}$, $\beta_{2p}(k)|_{i,i+1}=\frac{1}{4-k^2}$, $i=\overline{2,p}$, while $\beta_{2p}(k)|_{p+1,p+1}=\beta_{2p}(k)|_{1,2}\frac{2}{4-k^2}$, and all other matrix elements are zero.

If $n=2p+1$, then $\alpha_{2p+1}(k)$ is a $(p+1)\times(p+1)$ matrix such that $\alpha_{2p+1}(k)|_{i,i}=\frac{1}{4-k^2}$, $i=\overline{1,p+1}$, $\alpha_{2p+1}(k)|_{i,i-1}=\frac{1}{4-k^2}$, $i=\overline{2,p}$, and all other matrix elements are zero. The $\beta_{2p+1}(k)$ is a $(p+1)\times(p+2)$ matrix such that $\beta_{2p+1}(k)|_{i,i}=\frac{1}{4-k^2}$, $i=\overline{1,p+1}$, $\beta_{2p+1}(k)|_{i,i+1}=\frac{1}{4-k^2}$, $i=\overline{2,p+1}$, while $\beta_{2p+1}(k)|_{1,2}=\frac{2}{4-k^2}$, and all other matrix elements are zero.

Let N be a sufficiently large number with $N+1=2p$, $p\in\mathbb{Z}$ and let $\lambda=\lambda(k)$ be a root of the polynomial $2\lambda^2+(k^2-4)\lambda+2=0$ such that $|\lambda|<1$. Then $A_{N+1}(k)$ is a $(p+1)\times p$ matrix, where $A_{N+1}(k)|_{i,i}=A_{N+1}(k)|_{i,i-1}=\frac{1}{4-k^2-2\lambda}$, $i=\overline{2,p}$, while $A_{N+1}(k)|_{1,1}=\frac{1}{4-k^2-3\lambda}$, $A_{N+1}(k)|_{p+1,p}=\frac{2}{4-k^2-2\lambda}$, and all other matrix elements are zero.

References

1. M. Berciu, A. M. Cook, Efficient computation of lattice Green's functions for models with nearest-neighbour hopping. *Europhys Lett.* **92** (2010), no. 4, 40003.
2. H. S. Bhat, B. Osting, Discrete diffraction in two-dimensional transmission line metamaterials. *Microw. Opt. Technol. Lett.* **52** (2010), no. 3, 721–725.
3. H. S. Bhat, B. Osting, 2-D inductor-capacitor lattice synthesis. *IEEE T. Comput. Aid. D.* **30** (2011), 1483–1492.
4. L. Brillouin, Wave Propagation in Periodic Structures. *Dover Publications, 2nd ed*, 2003.
5. C. Caloz, Dual composite right/left-handed (D-CRLH) transmission line metamaterial. *IEEE Microw. Wirel. Compon. Lett.* **16** (2006), no. 11, 585–587.
6. C. Caloz, T. Itoh, *Electromagnetic Metamaterials: Transmission Line Theory and Microwave Applications the Engineering Approach.* John Wiley & Sons, Inc. Hoboken, New Jersey, 2005.
7. E. N. Economou, *Green's Functions in Quantum Physics.* Third edition. Springer Series in Solid-State Sciences, 7. Springer-Verlag, Berlin, 2006.
8. H. Isozaki, H. Morioka, A Rellich type theorem for discrete Schrödinger operators. *Inverse Probl. Imaging* **8** (2014), no. 2, 475–489.
9. D. Kapanadze, Exterior diffraction problems for two-dimensional square lattice. *Z. Angew. Math. Phys.* **69** (2018), no. 5, Art. 123.
10. S. Katsura, S. Inawashiro, Lattice Green's functions for the rectangular and the square lattices at arbitrary points. *J. Mathematical Phys.* **12** (1971), 1622–1630.
11. R. Marqués, F. Martín, M. Sorolla, *Metamaterials with Negative Parameters: Theory, Design, and Microwave Applications.* John Wiley & Sons, Inc. Hoboken, New Jersey, 2008.
12. P. A. Martin, Discrete scattering theory: Green's function for a square lattice. *Wave Motion* **43** (2006), 619–629.
13. T. Morita, Useful procedure for computing the lattice Green's function - square, tetragonal, and bcc lattices. *J. Math. Phys.* **12** (1971), 1744–1747.
14. A. Sanada, C. Caloz, T. Itoh, Characteristics of the composite right/left-handed transmission lines. *IEEE Microw. Wirel. Compon. Lett.* **14** (2004), no. 2, 68–70.
15. W. Shaban, B. Vainberg, Radiation conditions for the difference Schrödinger operators. *Appl. Anal.* **80** (2001), no. 3-4, 525–556.
16. B. L. Sharma, Diffraction of waves on square lattice by semi-infinite crack. *SIAM J. Appl. Math.* **75** (2015), no. 3, 1171–1192.
17. B. L. Sharma, Diffraction of waves on square lattice by semi-infinite rigid constraint. *Wave Motion* **59** (2015), 52–68.
18. B. L. Sharma, Near-tip field for diffraction on square lattice by crack. *SIAM J. Appl. Math.* **75** (2015), no. 4, 1915–1940.
19. B. L. Sharma, Near-tip field for diffraction on square lattice by rigid constraint. *Z. Angew. Math. Phys.* **66** (2015), no. 5, 2719–2740.
20. A. Zemla, On the fundamental solutions for the difference Helmholtz operator. *SIAM J. Numer. Anal.* **32** (1995), no. 2, 560–570.

A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 6 Tamarashvili Str., Tbilisi 0177, Georgia

Free University of Tbilisi, 240 David Aghmashenebeli Alley, 0159, Tbilisi, Georgia
E-mail address: david.kapanadze@gmail.com