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SEMISIMPLICITY SETS FOR CYCLIC ELEMENTS IN SIMPLE LIE ALGEBRAS

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Abstract. This paper is a continuation of the theory of cyclic elements in semisimple Lie algebras, developed by Elashvili, Kac and Vinberg. We classify semisimple cyclic elements in terms of various nonassociative algebra structures on certain subspaces of the corresponding Lie algebra. The importance of such classification stems from the fact that each such element gives rise to an integrable hierarchy of Hamiltonian PDE of Drinfeld-Sokolov type.

1. Introduction

Let us recall that for an element a of a Lie algebra $\mathfrak{g}$, the *adjoint action operator* $\operatorname{ad} a : \mathfrak{g} \to \mathfrak{g}$ is given by

$$(\operatorname{ad} a)x = [a, x], \qquad x \in \mathfrak{g}.$$

The element a is called *semisimple* if the operator $\operatorname{ad} a$ is diagonalizable, and *nilpotent* if $\operatorname{ad} a$ is a nilpotent operator, i. e. if $(\operatorname{ad} a)^n = 0$ for some n.

Consider a semisimple finite-dimensional Lie algebra $\mathfrak{g}$. To each nilpotent element $e \in \mathfrak{g}$ corresponds a *grading* of $\mathfrak{g}$, i. e. a direct sum decomposition

$$\mathfrak{g} = \bigoplus_{j=-d}^{d} \mathfrak{g}_j, \text{ where } \mathfrak{g}_{\pm d} \neq 0$$

of $\mathfrak{g}$ with the property $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$. It is obtained as follows: by the Morozov–Jacobson theorem, there exists an $\mathfrak{sl}(2)$-*triple* $\mathfrak{s} = (e, h, f)$ for e, i. e. another nilpotent element $f \in \mathfrak{g}$ such that $[e, f] = h$ is semisimple and satisfies $[h, e] = 2e$ and $[h, f] = -2f$. One then defines $\mathfrak{g}_j := \{x \in \mathfrak{g} \mid [h, x] = jx\}$. The positive integer d is called the *depth* of the nilpotent element e.

Elements of the form $e + F$ for $F \in \mathfrak{g}_{-d}$ are called *cyclic elements* for e. Classification of semisimple cyclic elements is interesting by (at least) two different reasons. First, such elements can be used to understand the structure of regular elements in Weyl groups [3, 7, 10]. Second, such elements give rise to integrable Hamiltonian hierarchies of partial differential equations [1, 2].

2. Singular Sets of Nilpotent Elements

For a nilpotent e in a simple Lie algebra $\mathfrak{g}$, we call

$$\mathbf{\Sigma}_{\mathfrak{g}}(e) = \{F \in \mathfrak{g}_{-d} \mid e + F \text{ is not semisimple}\}$$

the *singular set* of e.

Our task is the description of these sets.

Our work is continuation of [3], where systematic classification of nilpotent elements from the above point of view has been undertaken.

If there exists a semisimple cyclic element $e + F$, then the nilpotent $e \in \mathfrak{g}$ is said to be of *semisimple type.*

If $e + F$ is nilpotent for every $F \in \mathfrak{g}_{-d}$, then e is of *nilpotent type.*

If none of these is true for e, then it is of *mixed type.*

Thus for elements of mixed or nilpotent types, the set $\mathbf{\Sigma}_{\mathfrak{g}}(e)$ coincides with the whole of $\mathfrak{g}_{-d}$, and the interesting case for us is that of e of semisimple type.

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In [3] nilpotents of each of these types have been completely described. One of the central notions in that paper are the notion of a *reducing subalgebra* and of *rank* for a nilpotent e.

For an $\mathfrak{sl}(2)$-triple $\mathfrak{s}$ for e as above, let $Z(\mathfrak{s})$ denote the centralizer of $\mathfrak{s}$ under the action of the adjoint group G of $\mathfrak{g}$.

Definition 2.1. A semisimple subalgebra $\mathfrak{q} \subseteq \mathfrak{g}$ is called *reducing* for e, if it is normalized by $\mathfrak{s}$ and moreover $Z(\mathfrak{s})\mathfrak{q}_{-d}$ is Zariski dense in $\mathfrak{g}_{-d}$.

The *rank* $\operatorname{rk} e$ of e is the smallest possible dimension $\dim \mathfrak{q}_{-d}$ for any reducing subalgebras $\mathfrak{q}$ for e.

A nilpotent $e \in \mathfrak{g}$ is called *irreducible* if it does not admit any reducing subalgebras different from $\mathfrak{g}$.

Note that in particular for an irreducible nilpotent e we have $\operatorname{rk} e = \dim \mathfrak{g}_{-d}$.

For us, reducing subalgebras are crucial as they enable us to give description of the singular set $\mathbf{\Sigma}_{\mathfrak{g}}(e)$ in terms of $\mathbf{\Sigma}_{\mathfrak{q}}(e)$, for smallest possible reducing subalgebras $\mathfrak{q}$ for e. When e is of semisimple type, it is irreducible in such $\mathfrak{q}$, so in a sense all kinds of singular sets can be described in terms of singular sets for irreducible nilpotents. It turns out that there are very few cases of irreducible nilpotents. These are as follows ($k \geqslant 1$):

Table 1: Irreducible nilpotents of semisimple type

#	$\mathfrak{g}$	orbit	depth	rank	$Z(\mathfrak{s})\vert\mathfrak{g}_{-d}$
1_k	$\mathfrak{sl}(2k+1)$	$[2k+1]$	$4k$	1	1
2_k	$\mathfrak{sp}(2k)$	$[2k]$	$4k-2$	1	1
3_k	$\mathfrak{so}(2k+1)$	$[2k+1]$	$4k-2$	1	1
4_k	$\mathfrak{so}(4k+4)$	$[2k+3, 2k+1]$	$4k+2$	2	1
5	G_2	G_2	10	1	1
6	F_4	F_4	22	1	1
7	F_4	$\mathrm{F}_4(a_2)$	10	2	π_2
8	E_6	$\mathrm{E}_6(a_1)$	16	1	1
9	E_7	E_7	34	1	1
10	E_7	$\mathrm{E}_7(a_1)$	26	1	1
11	E_7	$\mathrm{E}_7(a_5)$	10	3	π_3
12	E_8	E_8	58	1	1
13	E_8	$\mathrm{E}_8(a_1)$	46	1	1
14	E_8	$\mathrm{E}_8(a_2)$	38	1	1
15	E_8	$\mathrm{E}_8(a_4)$	28	1	1
16	E_8	$\mathrm{E}_8(a_5)$	22	2	π_2
17	E_8	$\mathrm{E}_8(a_6)$	18	2	σ_2
18	E_8	$\mathrm{E}_8(a_7)$	10	4	σ_4

The last column of the table shows that in all these cases the group $Z(\mathfrak{s})|\mathfrak{g}_{-d}$ is finite: here π_n, resp. σ_{n-1} denotes the permutation representation, resp. the $n-1$-dimensional irreducible subrepresentation, of the symmetric group S_n.

Thus for irreducible e, only ranks $\leqslant 4$ occur, and in these cases it turns out that we have

Theorem 2.2. *For an irreducible nilpotent e with $\operatorname{rk} e = r$, the singular set $\mathbf{\Sigma}_{\mathfrak{g}}(e)$ is a union of exactly $\frac{r(r+1)}{2}$ distinct $r-1$-dimensional linear subspaces of $\mathfrak{g}_{-d}$.*

Moreover for any given irreducible e the singular set $\mathbf{\Sigma}_{\mathfrak{g}}(e)$ can be explicitly determined.

Example 2.3. Let $\mathfrak{g}$ be the simple Lie algebra of type D_{2k}, i. e. isomorphic to the algebra $\mathfrak{so}(4k)$ of $4k \times 4k$ skew-symmetric matrices with the Lie bracket given by commutators of matrices. Let e be a nilpotent corresponding to the partition $(2k+1, 2k-1)$, i. e. the one which acts on the standard $4k$-dimensional representation of $\mathfrak{g}$ via the matrix with two Jordan blocks, of sizes $2k+1$ and $2k-1$.

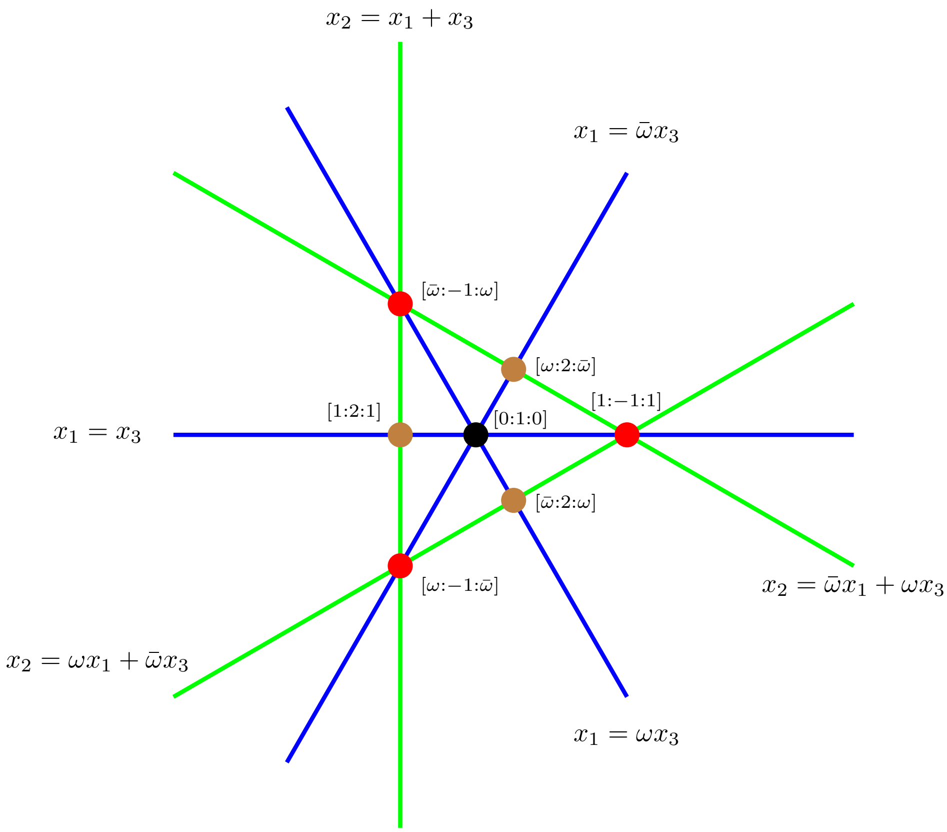
For this e we have $\dim \mathfrak{g}_{-d} = 2$, and in the root vector basis (F_1, F_2) of $\mathfrak{g}_{-d}$ the set $\boldsymbol{\Sigma}_{\mathfrak{g}}(e)$ consists of three lines given by scalar multiples of F_2, $F_1 + F_2$ and $F_1 - F_2$ respectively.

Example 2.4. Let $\mathfrak{g}$ be of type E_7 and

$$e := e_{\substack{100000\\0}} + e_{\substack{000111\\0}} + e_{\substack{000110\\1}} + e_{\substack{001100\\1}} + e_{\substack{001110\\0}} + e_{\substack{011100\\0}} + e_{\substack{011111\\1}}$$

be the nilpotent with label $\mathrm{E}_7(a_5)$. Here a letter e with subscripts denotes a root vector corresponding to a root given by the linear combination of simple roots with coefficients indicated in the subscript.

In this case $\dim \mathfrak{g}_{-d} = 3$ and the image of the singular set $\boldsymbol{\Sigma}_{\mathfrak{g}}(e)$ in the projective plane is as follows:

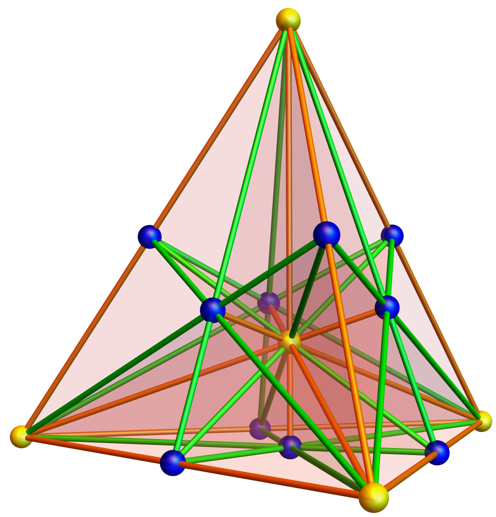


where ω is the third root of 1. That is, a cyclic element

$$c := e + x_1 f_{\substack{123421\\2}} + x_2 f_{\substack{123431\\2}} + x_3 f_{\substack{123432\\2}}$$

fails to be semisimple if and only if the corresponding point $[x_1 : x_2 : x_3]$ of the projective plane lies in the indicated set.

Example 2.5. As a last example, let $\mathfrak{g}$ be of type E_8 and e a nilpotent with label $\mathrm{E}_8(a_7)$. Then $\dim \mathfrak{g}_{-d} = 4$, and $\boldsymbol{\Sigma}_{\mathfrak{g}}(e)$ is the union of 10 3-dimensional subspaces. For a particular choice of e, these subspaces are given by the equations $x_1 + x_2 + x_3 + x_4 = 0$, $x_i + x_j = x_k + x_4$ and $x_i \pm x_j = 0$, with $\{i, j, k\} = \{1, 2, 3\}$. Projectivization of these subspaces gives a configuration of ten planes in the projective 3-space that looks as follows:



The authors are indebted to Noam Elkies for finding a particularly tractable parametrization of this configuration which helped them to identify it [4].

In the remaining (non-irreducible) cases, the space $\mathfrak{g}_{-d}$ contains a copy of one of the above singular sets $\mathbf{\Sigma}_{\mathfrak{q}}(e)$ for some reducing subalgebra $\mathfrak{q} \subseteq \mathfrak{g}$, in which e becomes irreducible, while the whole singular set $\mathbf{\Sigma}_{\mathfrak{g}}(e)$ in $\mathfrak{g}_{-d}$ can be described in terms of the image of $\mathbf{\Sigma}_{\mathfrak{q}}(e)$ under the action of $Z_{\mathfrak{g}}(\mathfrak{s})$ on $\mathfrak{g}_{-d}$.

3. "Explanation" of Singular Sets by Algebra Structures

It is also possible to understand more conceptually why exactly are the singular sets of the form that we found. For this aim, we equip the space $\mathfrak{g}_{-d}$ with additional structure.

First,

$$(x, y)_e := \left\langle (\operatorname{ad} e)^d x, y \right\rangle$$

where $\langle \cdot, \cdot \rangle$ is the Killing form on $\mathfrak{g}$, defines a nondegenerate $Z(\mathfrak{s})$-invariant symmetric bilinear form on $\mathfrak{g}_{-d}$.

Second, when d is even (which happens precisely if e is not of nilpotent type), the formula

$$x \underset{e}{\star} y := [(\operatorname{ad} e)^{\frac{d}{2}} x, y]$$

defines a bilinear operation on $\mathfrak{g}_{-d}$. This operation is skew-commutative if $d/2$ is even and commutative if $d/2$ is odd. Moreover the above symmetric bilinear form is invariant for this operation, in the sense that

$$(x \underset{e}{\star} y, z)_e = (x, y \underset{e}{\star} z)_e$$

holds for any $x, y, z \in \mathfrak{g}_{-d}$.

It then turns out that

Theorem 3.1. *The singular set $\mathbf{\Sigma}_{\mathfrak{g}}(e)$ is the union of all proper $\underset{e}{\star}$-subalgebras of the algebra $(\mathfrak{g}_{-d}, \underset{e}{\star})$.*

In each of the cases then, the algebra $(\mathfrak{g}_{-d}, \underset{e}{\star})$ can be identified with interesting well known algebras. First, when d is divisible by 4 then it turns out that this algebra is a direct sum of an abelian algebra (with zero multiplication), some simple Lie algebras, and a simple 7-dimensional Maltsev algebra of imaginary octonions under the commutator operation $[x, y] = xy - yx$ [8, 9].

In the case $d = 2$ it has been known for a long time that using the operation $\underset{e}{\star}$ one may obtain all simple Jordan algebras. It turns out that more generally, for even d not divisible by 4 the algebra $(\mathfrak{g}_{-d}, \underset{e}{\star})$ is Jordan provided e is of semisimple type and not irreducible. Whereas if e is irreducible, then one obtains one of the family of commutative algebras $\mathscr{C}_\lambda(n)$ given by the basis $p_1, \dots, p_n$ with the multiplication table

$$p_i^2 = p_i, \qquad p_i p_j = \lambda(p_i + p_j),\ i \neq j.$$

These algebras are known in the literature related to representations of sporadic finite simple groups, to Hessian algebras appearing in differential geometry, and to the theory of vertex algebras [5, 6].

For example, in 2.3 we get the algebra $\mathscr{C}_{1-k}(2)$, in 2.4 it is $\mathscr{C}_{-\frac{1}{3}}(3)$, and in 2.5 it is $\mathscr{C}_{-\frac{1}{3}}(4)$.

The algebras $\mathscr{C}_\lambda(n)$ are mostly not Jordan, but satisfy quartic identities

$$\langle a, b, c\rangle\, d - \langle a, d, c\rangle\, b = (ab)(cd) - (ad)(bc)$$

and

$$\langle a, bd, c\rangle + \langle b, cd, a\rangle + \langle c, ad, b\rangle = 0,$$

where $\langle x, y, z\rangle$ denotes the associator $(xy)z - x(yz)$. The latter identity can be also written in the form

$$[\mathrm{L}_a, \mathrm{L}_b]\, \mathrm{L}_c + [\mathrm{L}_b, \mathrm{L}_c]\, \mathrm{L}_a + [\mathrm{L}_c, \mathrm{L}_a]\, \mathrm{L}_b = 0,$$

where L_x denotes the multiplication operator, $\mathrm{L}_x(y) = xy$. Note the close resemblance with the Jordan identity, which is equivalent to

$$\langle ab, d, c\rangle + \langle bc, d, a\rangle + \langle ca, d, b\rangle = 0,$$

or in terms of the multiplication operators,

$$[\mathrm{L}_{ab}, \mathrm{L}_c] + [\mathrm{L}_{bc}, \mathrm{L}_a] + [\mathrm{L}_{ca}, \mathrm{L}_b] = 0.$$

These algebras then give an explanation of the particular form of singular sets $\mathbf{\Sigma}_{\mathfrak{g}}(e)$, in view of 3.1. Indeed, all algebras $\mathscr{C}_\lambda(n)$ that occur in our case contain exactly $2^n - 1$ nonzero idempotents, and all of their subalgebras are spanned by linearly independent subsets of idempotents. For example, the algebra $\mathscr{C}_{1-k}(2)$ from 2.3 has exactly three nonzero idempotents, and its subalgebras are the one-dimensional ones spanned by one of them, which explains why in 2.2 exactly three 1-dimensional subspaces occur when $\dim \mathfrak{g}_{-d} = 2$. Similarly, when $\dim \mathfrak{g}_{-d} = 3$, every maximal proper subalgebra of $\mathscr{C}_{-\frac{1}{3}}(3)$ is spanned by two linearly independent idempotents and contains three of them, which gives six distinct subalgebras, while for $\dim \mathfrak{g}_{-d} = 4$, any three linearly independent idempotents of $\mathscr{C}_{-\frac{1}{3}}(4)$ span a 3-dimensional subalgebra that contains seven of the idempotents, which gives ten 3-dimensional subalgebras in total, according to the projectivized picture in 2.5 where points correspond to 1-dimensional subalgebras, lines to 2-dimensional subalgebras and planes to 3-dimensional subalgebras.

References

1. A. De Sole, V. G. Kac, D. Valeri, Classical W-algebras and generalized Drinfeld-Sokolov bi-Hamiltonian systems within the theory of Poisson vertex algebras. *Comm. Math. Phys.* **323** (2013), no. 2, 663–711.
2. V. G. Drinfeld, V. V. Sokolov, Lie algebras and equations of Korteweg-de Vries type. *J. Sov. Math.* **30** (1985), 1975–2084.
3. A. G. Elashvili, V. G. Kac, E. B. Vinberg, Cyclic elements in semisimple Lie algebras. *Transform. Groups* **18** (2013), no. 1, 97–130.
4. N. Elkies (https://mathoverflow.net/users/14830/noam-d-elkies), Seeking a more symmetric realization of a configuration of 10 planes, 25 lines and 15 points in projective space, Math-Overow. URL: https://mathoverflow.net/q/313198/ (version: 2018-10-19).
5. D. Fox (https://mathoverflow.net/users/9471/dan-fox), Have you ever seen this bizarre commutative algebra? Mathoverflow. URL: https://mathoverflow.net/q/319405/ (version: 2018-12-24).
6. K. Harada, On a commutative nonassociative algebra associated with a doubly transitive group. *J. Algebra* **91** (1984), no. 1, 192–206.
7. B. Kostant, The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group. *Amer. J. Math.* **81** (1959), no. 4, 973–1032.
8. E. N. Kuz'min, Mal'tsev algebras and their representations. *Algebra and Logic* **7** (1968), no. 4, 233–244.
9. A. A. Sagle, Simple Malcev algebras overfields of characteristic zero. *Pacific J. Math.* **12** (1962), 1057–1078.

10. T. A. Springer, Regular elements of finite reflection groups. *Invent. Math.* **25** (1974), no. 2, 159–198.

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