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ON CRITERIA OF CONVERGENCE IN MEASURE OF A SEQUENCE OF FUNCTIONS

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Abstract. It is well known that the Lebesgue and F. Riesz theorems show an interrelation between the convergence in measure and the convergence almost everywhere of a sequence of functions; the first one is a sufficient and the second one is a necessary condition of convergence in measure of a sequence of functions.

In the present paper we formulate a theorem representing a necessary and sufficient condition of convergence in measure of a sequence of functions.

Let $(f_n(x))_{n=1}^{\infty}$ be a sequence of functions defined on a measurable set $E \subset [0,1]$ and μ be the Lebesgue linear measure.

Definition 1. *A sequence of functions $(f_n(x))_{n=1}^{\infty}$ is called convergent in measure to a function $f(x)$, if*

$$\lim_{n\to\infty} \mu\{x \in E : \ |f_n(x) - f(x)| \geq \delta\} = 0$$

for any $\delta > 0$.

The symbol $f_n(x) \xrightarrow{\mu} f(x)$ denotes the convergence in measure of a sequence $(f_n(x))_{n=1}^{\infty}$ to a function $f(x)$.

Lebesgue (see [1, p. 92]) and F. Riesz (see [1, p. 96]) established theorems representing relations between the convergence in measure and the convergence almost everywhere of a sequence of functions.

Namely, the following theorem holds.

Theorem A (Lebesgue)**.** *If $\lim\limits_{n\to\infty} f_n(x) = f(x)$ for a. e. $x \in E$, then $f_n(x) \xrightarrow{\mu} f(x)$.*

It is known that there exists a sequence of functions which is convergent in measure to zero and there exists no point at which this series converges to zero.

However, it should be noted that the following theorem holds.

Theorem B (F. Riesz)**.** *If $f_n(x) \xrightarrow{\mu} f(x)$, then there exists a sequence of natural numbers $n_k \uparrow \infty$ such that*

$$\lim_{k\to\infty} f_{n_k}(x) = f(x) \quad \text{for a. e.} \quad x \in E.$$

So, Theorem A is a sufficient condition of the convergence in measure of a sequence of functions and Theorem B is a necessary condition of the convergence in measure of one.

It holds the following theorem which is a necessary and sufficient condition of the convergence in measure of a sequence of functions.

Theorem 1. *$f_n(x) \xrightarrow{\mu} f(x)$ if and only if there exists a sequence of natural numbers $n_k \uparrow \infty$ such that if $(m_k)_{k=1}^{\infty}$ is any sequence of natural numbers such that $n_k \leq m_k < n_{k+1}$, then*

$$\lim_{k\to\infty} f_{m_k}(x) = f(x) \quad \text{a. e.} \quad x \in E.$$

The following Proposition also holds.

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Proposition 1. *Let $(f_n(x))_{n=1}^{\infty}$ be any sequence of measurable functions and $f(x)$ be any measurable function, then the following two conditions are equivalent to each other:*

(i) *there exists a sequence of natural numbers $n_k \uparrow \infty$ such that if $(m_k)_{k=1}^{\infty}$ is any sequence of natural numbers such that $n_k \le m_k < n_{k+1}$, then*

$$\lim_{k\to\infty} f_{m_k}(x) = f(x) \quad a.\ e. \quad x \in E.$$

(ii) *If $(p_k)_{k=1}^{\infty}$ is any sequence of natural numbers, then there exists a sequence of natural numbers $n_k \uparrow \infty$ such that $n_{k+1} - n_k \ge p_k$, and for any sequence of sets $(B_k)_{k=1}^{\infty}$ such that*

$$B_k \subset \{n:\ n_k \le n < n_{k+1}\}, \quad \operatorname{Card} B_k = p_k \ \text{ and } \ B = \bigcup_{k=1}^{\infty} B_k,$$

the following equality

$$\lim_{\substack{n\to\infty \\ n\in B}} f_n(x) = f(x), \quad a.\ e. \quad x \in E$$

holds.

The conjunction of Theorem 1 and Proposition 1 implies the following

Theorem 2. *Let $(f_n(x))_{n=1}^{\infty}$ be a sequence of measurable functions, then the following three conditions are equivalent to each other:*

α) $f_n(x) \xrightarrow{\mu} f(x)$;

β) *there exists a sequence of natural numbers $n_k \uparrow \infty$ such that if $(m_k)_{k=1}^{\infty}$ is any sequence of natural numbers such that $n_k \le m_k < n_{k+1}$, then*

$$\lim_{k\to\infty} f_{m_k}(x) = f(x) \quad a.\ e. \quad x \in E.$$

γ) *If $(p_k)_{k=1}^{\infty}$ is any sequence of natural numbers, then there exists a sequence of natural numbers $n_k \uparrow \infty$ such that $n_{k+1} - n_k \ge p_k$, and for any sequence of sets $(B_k)_{k=1}^{\infty}$ such that*

$$B_k \subset \{n:\ n_k \le n < n_{k+1}\}, \quad \operatorname{Card} B_k = p_k \ \text{ and } \ B = \bigcup_{k=1}^{\infty} B_k$$

the following equality

$$\lim_{\substack{n\to\infty \\ n\in B}} f_n(x) = f(x), \quad a.\ e. \quad x \in E$$

holds.

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