Transactions of A. Razmadze
Mathematical Institute
Vol. **173** (2019), 169–172

ON CANTOR'S Λ FUNCTIONALS AND THE RECONSTRUCTION OF COEFFICIENTS OF MULTIPLE FUNCTION SERIES

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Abstract. In the present paper, the Λ summable single and multiple function series are considered. The notion of a sequence of Cantor's Λ functionals, which represents formulas of reconstruction of coefficients of a single function series and is also a generalization of Fourier formulas for calculation of coefficients of an orthonormal function series is introduced.

A theorem representing a possibility of reconstruction of coefficients of a multiple function series via iterated application of Cantor's Λ functionals is formulated.

Introduction

Let Λ be an infinite matrix, $\Phi = (\varphi_i(t))_{i=0}^{\infty}$ be a system of finite and measurable functions defined on $[0,1]$ and

$$\sum_{i=0}^{\infty} a_i \varphi_i(t) \tag{1}$$

be a series with respect to Φ which is Λ summable to a function $f(t)$.

In the present paper, the problem of reconstruction of coefficients of series (1) in terms of $f(t)$ is considered.

A necessary condition of solvability of the above-mentioned problem is the validity of the following uniqueness theorem:

If a series (1) is summable to zero on $[0,1]$, then every coefficient of this series is equal to zero.

In a particular case, when Λ summability coincides with the convergence of the series and Φ is a trigonometric system, the following Cantor's fundamental theorem [1] concerning the convergence of trigonometric series holds.

Theorem A (Cantor)**.** *If a trigonometric series converges everywhere to zero, then every coefficient of this series is equal to zero.*

The following Vallée-Poussin's theorem [2] generalizing Theorem A holds, as well.

Theorem B (Vallée-Poussin)**.** *If a trigonometric series converges to a finite integrable function f everywhere, except possibly at countably many points, then it is the Fourier series of f.*

(The validity of Theorem A and Theorem B for multiple trigonometric series in Pringsheim's convergence sense is proved in [3]).

It is known that there exists a trigonometric series such that the sum of this series is a Lebesgue nonintegrable function. The following series

$$\sum_{k=2}^{\infty} \frac{\sin 2\pi k t}{\ln k}$$

is an example of the above-mentioned one. So, Vallée-Poussin's theorem and Fourier-Lebesgue's formulas cannot be applied to reconstruct the coefficients of such a series.

On the other hand, the uniqueness of coefficients of an everywhere convergent trigonometric series follows from Cantor's uniqueness theorem.

2010 *Mathematics Subject Classification.* 42B05, 42B08.

Key words and phrases. Functional series; d-fold functional series; Λ summability, Reconstruction of coefficients.

The above-mentioned circumstance causes the necessity of a generalization of the Lebesgue integral such that any everywhere convergent trigonometric series is the Fourier series of its sum in the generalized integral sense. This problem has been solved by Denjoy.

The Marcinkiewicz–Zygmund integral and the James integral are also examples of such a generalization of the Lebesgue integral (see [6]).

The presented work consists of two sections.

In Section 1, single series with respect to system Φ which are summable by some Λ-method are considered. For such series, the notion of the sequence of Cantor's Λ functionals is introduced. The notion of a sequence of Cantor's functionals for convergent function series was introduced in [4] and [5].

It should be noted that Fourier formulas for coefficients of a single trigonometric series in every above mentioned generalized integral sense represents a particular example of the notion of the sequence of Cantor's Λ functionals.

In Section 2, the theorem asserting the possibility of reconstruction of coefficients of multiple function series by iterated application of Cantor's Λ functionals is presented.

1. Single Series. Cantor's Λ Functionals

Let $\Phi = (\varphi_i(t))_{i=0}^{\infty}$ be a system of finite and Lebesgue measurable functions defined on $[0,1]$. Also, $\Lambda = \|\lambda_{p,i}\|$ be a matrix of numbers, where $p = 0,1,2,\dots$ and $i = 0,1,2,\dots$.

With respect to Φ, we consider a series

$$\sum_{i=0}^{\infty} a_i \varphi_i(t).$$

Denote by $\sigma_p(t)$ the p-th Λ mean of this series, so,

$$\sigma_p(t) = \sum_{i=0}^{\infty} \lambda_{p,i} a_i \varphi_i(t),$$

which means that for every integer $p \ge 0$, the series

$$\sum_{i=0}^{\infty} \lambda_{p,i} a_i \varphi_i(t)$$

converges for any $t \in [0,1]$.

Definition 1. *We say that a set $A \subset [0,1]$ belongs to a class $U(\Phi,\Lambda)$ if $\lim\limits_{p\to\infty} \sigma_p(t) = 0$, when $t \in A$ implies that $a_i = 0$ for every integer $i \ge 0$.*

Definition 2. *We say that a finite function $f(t)$ belongs to a class $J(A,\Phi,\Lambda)$ if $A \in U(\Phi,\Lambda)$ and there exists a series* (1) *such that*

$$\lim_{p\to\infty} \sigma_p(t) = f(t), \quad \textit{when} \quad t \in A. \tag{2}$$

Let us note that if $f \in J(A,\Phi,\Lambda)$, then $A \in U(\Phi,\Lambda)$ implies the uniqueness of coefficients of any series (1) which satisfies (2).

So, for every integer $i \ge 0$, there exists a functional $G_i^{A,\Phi,\Lambda}(\cdot)$ defined on $J(A,\Phi,\Lambda)$ such that

$$a_i = G_i^{A,\Phi,\Lambda}(f).$$

Definition 3. *Let $i \ge 0$ be any fixed and integer number and $G_i^{A,\Phi,\Lambda}(\cdot)$ be a functional defined on $J(A,\Phi,\Lambda)$. We say that $G_i^{A,\Phi,\Lambda}(\cdot)$ is a Cantor's Λ functional if for any series* (1) *satisfying* (2) *the following equality*

$$a_i = G_i^{A,\Phi,\Lambda}(f)$$

holds; we say that the sequence of functionals $\left(G_i^{A,\Phi,\Lambda}(\cdot)\right)_{i=0}^{\infty}$ is a sequence of Cantor's Λ functionals.

The following propositions hold.

Proposition 1. *If $A \in U(\Phi,\Lambda)$, then $J(A,\Phi,\Lambda)$ is a linear space.*

Proposition 2. *If* $\left(G_i^{A,\Phi,\Lambda}(\cdot)\right)_{i=0}^{\infty}$ *is a sequence of Canor's* Λ *functionals defined on* $J(A,\Phi,\Lambda)$*, then* $G_i^{A,\Phi,\Lambda}(\cdot)$ *is a linear functional for every integer* $i \geq 0$.

Thus the sequence of Cantor's Λ functionals represents the general form of formulas to calculate the coefficients of Λ summable any function series in the sense that any such type formula is a particular case of a Cantor's Λ functional.

2. Multiple Series. Coefficients Reconstruction

Let us formulate notation and definitions we use below.

Let $d \geq 2$ be a positive integer. By R^d we denote a d-dimensional Euclidean space and by Z_0^d the set of all points with nonnegative integer coordinates in R^d. The symbols $m = (m_1, \dots, m_d)$ and $n = (n_1, \dots, n_d)$ will stand for the points of Z_0^d. We write $m \to \infty$ if $m_j \to \infty$ for every integer j satisfying $1 \leq j \leq d$ independently of one another. Also, we write $m \geq 0$ if $m_j \geq 0$ $(1 \leq j \leq d)$. By $E_1 \times \dots \times E_d$ we denote the Cartesian product of sets $E_j \subset [0,1]$, where $1 \leq j \leq d$. We use $x = (x_1, \dots, x_d)$ to denote the points of a unit cube $[0,1]^d$. The linear Lebesgue measure is denoted by μ.

Definition 4. *Let* $\delta \in (0,1]$*. We call a system* $\Phi = \{\varphi_i(t)\}_{i=0}^{\infty}$ *a* δ *linear independence system if any finite part of this system is lineary independent on any set* $A \subset [0,1]$*, where* $\mu A > 1 - \delta$.

If $\delta = 1$*, then* Φ *is called an essentially lineary independence system.*

Let us note that the trigonometric system

$$T = \{1, \cos 2\pi i t, \sin 2\pi i t\}_{i=1}^{\infty}, \quad t \in [0,1]$$

is an essentially lineary independence system.

Let

$$\Phi^{(j)} = \left\{\varphi_{n_j}^{(j)}(x_j)\right\}_{n_j=0}^{\infty}, \quad x_j \in [0,1]$$

for every integer j satisfying inequalities $1 \leq j \leq d$ be a system of measurable and finite functions defined on $[0,1]$.

Let $\overline{\Phi} = \{\Phi_n(x)\}$ be a d-fold system such that

$$\Phi_n(x) = \prod_{j=1}^{d} \varphi_{n_j}^{(j)}(x_j),$$

where $n \in Z_0^d$, $x \in [0,1]^d$ and $x_j \in [0,1]$ and let

$$\Lambda^{(j)} = \left\|\lambda_{m_j,n_j}^{(j)}\right\|, \quad \text{where} \quad m_j = 0,1,2,\dots \ \text{and} \ n_j = 0,1,2,\dots$$

be a matrix for every integer j satisfying $1 \leq j \leq d$.

Let $\overline{\Lambda} = \left\|\lambda_{m,n}\right\|$ be a d-fold matrix, where $m \in Z_0^d$, $n \in Z_0^d$ and

$$\lambda_{m,n} = \prod_{j=1}^{d} \lambda_{m_j,n_j}^{(j)}.$$

Everywhere below, we mean that for every integer j satisfying $1 \leq j \leq d-1$ a system $\Phi^{(j)} = \left\{\varphi_{n_j}^{(j)}(x_j)\right\}_{n_j=0}^{\infty}$ is a δ_j lineary independence system, where $\delta_j \in (0,1]$ and $\Lambda^{(j)} = \left\|\lambda_{m_j,n_j}\right\|$ is a matrix with finite rows, that is,

$$\lambda_{m_j,n_j} = 0 \ \text{ if } \ n_j \geq \gamma_j(m_j).$$

Consider the d-multiple series with respect to $\overline{\Phi}$:

$$\sum_{n=0}^{\infty} a_n \Phi_n(x) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_d=0}^{\infty} a_{n_1}, \dots, n_d \prod_{j=1}^{d} \varphi_{n_j}^{(j)}(x_j). \tag{3}$$

Denote by $\sigma_m(x)$ the m-th Λ mean of series (3), so

$$\sigma_m(x) = \sum_{n_1=0}^{\gamma_1(m_1)} \cdots \sum_{n_{d-1}=0}^{\gamma_{d-1}(m_{d-1})} \sum_{n_d=0}^{\infty} \prod_{j=1}^{d} \lambda_{m_j,n_j}^{(j)} a_{n_1,\dots,n_d} \prod_{j=1}^{d} \varphi_{n_j}^{(j)}(x_j).$$

Let $E_j \in U\left(\Phi^{(j)}, \Lambda^{(j)}\right)$ for every integer j satisfying $1 \le j \le d$ and

$$\lim_{m\to\infty} \sigma_m(x) = f(x), \quad if \quad x \in E_1 \times E_2 \times \cdots \times E_d.$$

Let $n \in Z_0^d$ and j be fixed and $G_{n_j}^{(j)}(f(t))$ stands for $G_{n_j}^{E_j,\Phi^{(j)},\Lambda^{(j)}}(f(t))$, where $t \in E_j$.
Consider

$$G_{n_j}^{(j)}(F(x_j, x_{j+1}, \dots, x_d)). \tag{4}$$

In such a case we mean that only $x_j \in E_j$ is a variable and a point $(x_{j+1}, \dots, x_d) \in E_{j+1} \times \cdots \times E_d$ is a fixed one, and

$$F(x_j, x_{j+1}, \dots, x_d) \in J(E_j, \Phi^{(j)}, \Lambda^{(j)}).$$

Thus, (4) depends on $(x_{j+1}, \dots, x_d)$ and does not depend on x_j.
The following theorem holds.

Theorem 1. *Let $E_j \in U\left(\Phi^{(j)}, \Lambda^{(j)}\right)$ for every integer j satisfying $1 \le j \le d$ and $\Phi^{(j)}$ be a δ_j lineary independence system for every integer j satisfying $1 \le j \le d-1$, where $\mu E_j > 1 - \delta_j$ and $\Lambda^{(j)}$ is the matrix with finite rows.*

If $f(x)$ is a finite function defined on $E_1 \times \cdots \times E_{d-1} \times E_d$ and $\lim_{m\to\infty} \sigma_m(x) = f(x)$ when $x \in E_1 \times \cdots \times E_{d-1} \times E_d$, then

$$a_{n_1,\dots,n_d} = G_{n_d}^{(d)}\left(\cdots\left(G_{n_1}^{(1)}(f(x_1, \dots, x_d))\right)\right),$$

for every $(n_1, \dots, n_d) \in Z_0^d$.

Acknowledgement

This work was supported by Shota Rustaveli National Science Foundation of Georgia, Grant Number: DI-18-118.

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