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THE WEIGHTED RIGHT FOCAL BOUNDARY VALUE PROBLEM FOR SECOND ORDER SINGULAR IN THE TIME VARIABLE FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. Sufficient conditions are found for the solvability of the boundary value problem

$$u''(t) = f(t, u(\tau(t))),$$

$$\lim_{t\to a} \frac{u(t)}{(t-a)^\alpha} = 0, \quad \lim_{t\to b} u'(t) = 0$$

in the case where the function f has singularities of arbitrary order in the time variable at the point $t = a$ as well as at the points of the interval $]a, b]$.

On a finite open interval $]a, b[$, we consider the differential equation

$$u''(t) = f(t, u(\tau(t))) \tag{1}$$

with the boundary conditions

$$\lim_{t\to a} \frac{u(t)}{(t-a)^\alpha} = 0, \quad \lim_{t\to b} u'(t) = 0, \tag{2}$$

where $f : I \times \mathbb{R} \to \mathbb{R}$ is a measurable in the first argument and continuous in the second argument function,

$$I \subset]a, b[, \quad \operatorname{mes} I = b - a,$$

and $\alpha \in [0, 1[$.

Suppose

$$f^*(t, r) = \max\{|f(t, x)| : \ |x| \le r\} \ \text{ for } \ t \in I, \ \ r \ge 0.$$

If

$$\int_a^b f^*(t, r)dt < +\infty \quad \text{for } \ r > 0,$$

then problem $(1), (2)$ is said to be regular. If

$$\int_a^b f^*(t, r)dt = +\infty \ \text{ for some } \ r > 0, \tag{3}$$

then this problem is said to be singular in the time variable.

Unimprovable sufficient conditions for the solvability and unique solvability of problem $(1), (2)$ in the case, where $\alpha = 0$, $\tau(t) \equiv t$, and the function f has a singularity of arbitrary order in the time variable at the point $t = a$, are contained in [1, 3–8].

For $\alpha = 0$ and $\tau(t) \not\equiv t$, the singular problem $(1), (2)$ is also studied under the assumption that the function f has a non-integrable singularity in the time variable only at the point $t = a$ (see, [2, 9–12]). Therefore, the papers [2, 9–12] concern only the case where along with (3) the condition

$$\int_t^b f^*(s, r)ds < +\infty \ \text{ for } \ a < t < b, \ \ r \ge 0 \tag{4}$$

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is satisfied.

In contrast to the results of [2,9–12], theorems proven by us on the solvability and unique solvability of problem $(1),(2)$ cover the case where condition (4) is violated , i.e., the case where the function f has a non-integrable singularity in the time variable at the points of the interval $]a,b]$. In particular, it is assumed that there exist points $t_i \in]a,b[$ $(i=1,\dots,n)$ such that for an arbitrarily small $\varepsilon>0$ and for any $x\neq 0$ and $\lambda>0$, the conditions

$$\int\limits_{t_i-\varepsilon}^{t_i+\varepsilon} |t-t_i|^{\lambda}|f(t,x)|dt=+\infty \ \ (i=1,\dots,n), \quad \int\limits_{b-\varepsilon}^{b}(b-t)^{\lambda}|f(t,x)|dt=+\infty \tag{5}$$

hold.

Introduce the function

$$\chi(t)=\begin{cases}1 & \text{if } \ t=\tau(t),\\ 0 & \text{if } \ t\neq\tau(t).\end{cases}$$

We investigate the solvability of problem $(1),(2)$ in the case where

$$\int\limits_{t}^{b} f^{*}\big(s,(\tau(s)-a)^{\alpha}r\big)ds<+\infty \ \text{ for } \ a<t<b, \ \ r\geq 0, \tag{6}$$

and on the set $I\times\mathbb{R}$ the inequality

$$\chi(t)f(t,x)\,\mathrm{sgn}(x)-(1-\chi(t))|f(t,x)|\geq -g(t)|x|-h(t) \tag{7}$$

is satisfied, where g and $h: I\to[0,+\infty[$ are measurable functions.

When investigating the uniqueness of a solution of problem $(1),(2)$, we assume that the function f on the set $I\times\mathbb{R}$ instead of condition (7) satisfies the one-sided Lipschitz condition

$$\chi(t)\big[f(t,x)-f(t,y)\big]\,\mathrm{sgn}(x-y)-(1-\chi(t))|f(t,x)-f(t,y)|\geq -g(t)|x-y|. \tag{8}$$

Theorem 1. *If along with* (6) *and* (7) *the conditions*

$$\int\limits_{a}^{b}(t-a)^{1-\alpha}(\tau(t)-a)^{\alpha}g(t)dt<1 \tag{9}$$

and

$$\int\limits_{a}^{b}(t-a)^{1-\alpha}h(t)dt<+\infty \tag{10}$$

hold, then problem $(1),(2)$ *has at least one solution.*

Theorem 2. *If along with* (6) *and* (8) *conditions* (9) *and* (10) *are satisfied, where* $h(t)=|f(t,0)|$, *then problem* $(1),(2)$ *has one and only one solution.*

Remark 1. Inequality (9) in Theorems 1 and 2 cannot be replaced by the inequality

$$\int\limits_{a}^{b}(t-a)^{1-\alpha}(\tau(t)-a)^{\alpha}g(t)dt\leq 1+\varepsilon,$$

no matter how small $\varepsilon>0$ would be. However, the question of whether it is possible to replace (9) by the nonstrict inequality

$$\int\limits_{a}^{b}(t-a)^{1-\alpha}(\tau(t)-a)^{\alpha}g(t)dt\leq 1$$

remains open.

Example 1. Suppose $\alpha \in]0,1[$, $a < a_0 < b$, m and n are natural numbers, $t_i \in]a,b[$ $(i = 1,\dots,n)$, $t_{n+1} = b$,

$$\tau(t) = t, \quad f(t,x) = \exp\Big(\frac{1+|x|}{t-a}\Big)x^{2m-1} + q(t) \ \text{ for } t \in]a,a_0[, \ x \in \mathbb{R},$$

$$\tau(t) = a + (b-a)\exp\bigg(-\sum_{i=1}^{n+1}\frac{1}{|t-t_i|}\bigg), \quad f(t,x) = \ell(t-a)^{\alpha-1}(\tau(t)-a)^{-\alpha}, \quad 0 < \ell(b-a) < 1$$
$$\text{for } \ t \in]a_0,b[\setminus\{t_1,\dots,t_n\}, \ \ x \in \mathbb{R},$$

$q :]a,b[\to \mathbb{R}$ is a measurable function such that

$$\int_a^b (t-a)^{1-\alpha}|q(t)|dt < +\infty. \tag{11}$$

Then by Theorem 2, problem (1), (2) has one and only one solution. On the other hand, in this case the function f satisfies conditions (5) for an arbitrarily small $\varepsilon > 0$ and for any $x \neq 0$ and $\lambda > 0$. It is also evident that the function f has a singularity of arbitrary order in the time variable at the point $t = a$ as well.

The particular case of equation (1) is the linear differential equation

$$u''(t) = p(t)u(\tau(t)) + q(t), \tag{12}$$

where p and $q :]a,b[\to \mathbb{R}$ are measurable functions.

Put

$$p_-(t) = \frac{|p(t)| - p(t)}{2}.$$

From Theorem 2 we have the following statement.

Corollary 1. *If*

$$\int_a^b \chi(t)(t-a)p_-(t)dt + \int_a^b (1-\chi(t))(t-a)^{1-\alpha}(\tau(t)-a)^{\alpha}|p(t)|dt < 1,$$

and the function q satisfies condition (11), *then problem* (12), (2) *has one and only one solution.*

It is easy to see that under the conditions of Corollary 1 the function p may have singularities of arbitrary order at the points of the interval $]a,b]$.

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