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ON MEIR–KEELER CONTRACTION IN BRANCIARI b-METRIC SPACES

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Abstract. In this paper we consider Meir–Keeler type results in the context of Branciari b-metric spaces. Our results generalize, improve and complement several ones in the existing literature.

1. Introduction and Preliminaries

In the paper [14] the authors introduced the concept of $b_v(s)$-metric space as follows.

Definition 1.1 ([14]). Let X be a set, let d be a function from $X \times X$ into $[0, \infty)$ and let $v \in \mathbf{N}$. Then (X, d) is said to be a $b_v(s)$-metric space if for all $x, y \in X$ and for all distinct points $u_1, u_2, \dots, u_v \in X$, each of them different from x and y the following hold:
$(B1)$ $d(x, y) = 0$ if and only if $x = y$;
$(B2)$ $d(x, y) = d(y, x)$;
$(B_v3(s))$ there exists a real number $s \geq 1$ such that

$$d(x, y) \leq s[d(x, u_1) + d(u_1, u_2) + \cdots + d(u_v, y)].$$

Note that:

- $b_1(1)$-metric space is usual metric space,
- $b_1(s)$-metric space is b-metric space with coefficient s of Czerwik [3,4],
- $b_2(1)$-metric space is rectangular metric space or Branciari metric space [2],
- $b_2(s)$-metric space is rectangular b-metric space with coefficient s of George et al [8] or Branciari b-metric space [9],
- $b_v(1)$-metric space is v-generalized metric space of Branciari [2],
- Let (X, d_K) be a N-polygonal K-metric space over an ordered Banach space $(V, ||\cdot||, K)$ (see [7]) such that K is a closed normal cone with normal constant λ and the function $D : X \times X \to [0, \infty)$ defined by $D(x, y) = ||d_K(x, y)||$. Then (X, D) is $b_N(\lambda)$-metric space.

Definition 1.2 ([14]). Let (X, d) be a $b_v(s)$-metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then

(a) The sequence $\{x_n\}$ is said to be convergent in (X, d) and converges to x, if for every $\varepsilon > 0$ there exists $n_0 \in \mathbf{N}$ such that $d(x_n, x) < \varepsilon$ for all $n > n_0$ and this fact is represented by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(b) The sequence $\{x_n\}$ is said to be Cauchy sequence in (X, d) if for every $\varepsilon > 0$ there exists $n_0 \in \mathbf{N}$ such that $d(x_n, x_{n+p}) < \varepsilon$ for all $n > n_0, p > 0$.

(c) (X, d) is said to be a complete $b_v(s)$-metric space if every Cauchy sequence in X converges to some $x \in X$.

Definition 1.3 ([11]). Let (X, d) be a metric space. A mapping $T : X \to X$ is called Meir–Keeler contraction if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\epsilon \leq d(x, y) < \epsilon + \delta \Rightarrow d(Tx, Ty) < \epsilon \text{ for all } x, y \in X.$$

Definition 1.4 ([16]). A mapping $T : X \to X$ is called α-admissible if for all $x, y \in X$ we have

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1,$$

where $\alpha : X \times X \to [0, \infty)$ is a given function. A function α is transitive if, given $x, y, z \in X$,

$$\alpha(x, y) \geq 1, \alpha(y, z) \geq 1 \Rightarrow \alpha(x, z) \geq 1.$$

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Lemma 1.1 ([1]). *Let $T : X \to X$ be an α-admissible mapping and let $\{x_n\}$ be a Picard sequence of T based on a point $x_0 \in X$. If x_0 satisfies $\alpha(x_0, Tx_0) \geq 1$, then $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbf{N}$. Additionally, if α is transitive, then $\alpha(x_n, x_m) \geq 1$ for all $n, m \in \mathbf{N}$ such that $n < m$.*

One generalization on Meir–Keeler mappings was given by Gülyaz et al in the paper [9].

Definition 1.5 ([9]). Let (X, d) be a Branciari b-metric space with a constant $s \geq 1$. Let $T : X \to X$ be an α-admissible mapping. If for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\epsilon \leq M(x, y) < \epsilon + \delta \text{ implies } \alpha(x, y)d(Tx, Ty) < \frac{\epsilon}{s}, \tag{1}$$

where

$$M(x, y) = \max\{d(x, y), d(Tx, x), d(Ty, y)\}$$

for all $x, y \in X$, then T is called generalized α-Meir–Keeler contraction.

Definition 1.6 ([9]). A Branciari b-metric space (X, d) is called α-regular if for any sequence $\{x_n\}$ such that $\lim d(x_n, x) = 0$ and satisfying $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbf{N}$, we have $\alpha(x_n, x) \geq 1$ for all $n \in \mathbf{N}$.

We note that Gülyaz et al in the paper [9] define Brancari b-metric spaces, but this class of space has already been defined by George et al in the paper [8] and others called them rectangular b-metric spaces. Also in the paper [9] Gülyaz et al prove Lemma 2. 5 (see [9, p. 5449]).

Lemma 1.2 (Lemma 2. 5. in [9]). *Let (X, d) be a Branciari b-metric space with a constant $s \geq 1$. Let $\{x_n\}$ be a sequence in X satisfying*

1. *$x_m \neq x_n$ for all $m \neq n, m, n \in \mathbf{N}$,*
2. *$d(x_n, x_{n+1}) \leq \frac{1}{s} d(x_{n-1}, x_n)$, for all $n \in \mathbf{N}$,*
3. *$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0$. Then $\{x_n\}$ is a Cauchy sequence in (X, d).*

Unfortunately, the Lemma 1.2 is not correct, as shown in the following example.

Example 1.1. Put $X = \mathbf{R}$, $d(x, y) = |x - y|, x, y \in X$ and $x_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. Then (X, d) is Branciari b-metric space with coefficient $s = 1$ and sequence $\{x_n\}$ fulfills the conditions of Lemma 1.2 but not the Cauchy sequence.

Of course, then main result in the [9] is not correct, because its proof is needed by Lemma 2. 5. Here we prove the new version of Lemma 2. 5. in [9], also we show that continuity of function T is not necessary. Also, note that condition (1) follows the following condition

$$\alpha(x, y)d(Tx, Ty) \leq \lambda M(x, y),$$

for all $x, y \in X$, where $\lambda \in (0, \frac{1}{s})$. In addition, the authors in [9] use that is the next result.

Proposition 1.1 (Proposition 1.6. in [9]). *Let $\{x_n\}$ be a Cauchy sequence in a Branciari metric space (X, d) such that $\lim d(x_n, x) = 0$, where $x \in X$. Then $\lim d(x_n, y) = d(x, y)$, for all $y \in X$. In particular, the sequence $\{x_n\}$ does not converge to y if $y \neq x$.*

For proof of the main result in [9] (Theorem 2.6) authors used that the Proposition 1.1 is valid if replace Branciari metric space by a Branciari b-metric space.

Unfortunately, Proposition 1.1 is not true in Branciari b-metric space (see Example 1.7. in [8]).

2. Main Results

Lemma 2.1. *Let (X, d) be a complete $b_2(s)$-metric space and let $\{x_n\}$ be a sequence in X such that x_n $(n \geq 0)$ are all different. Suppose that exists $\lambda \in [0, \frac{1}{\sqrt{s}})$ such that*

(1) *$d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n)$,*
(2) *$d(x_n, x_{n+2}) \leq \lambda d(x_{n-1}, x_{n+1})$,*

for all $n \geq 1$. Then $\{x_n\}$ is a convergent sequence in (X, d). Additionally, if d is continuous, then for x^* for which $x^* = \lim x_n$ the next estimate holds

$$d(x_n, x^*) \leq \frac{2s\lambda^n}{1 - s\lambda^2} d(x_0, x_1) + 3\lambda^n [d(x_0, x_1) + d(x_0, x_2)]. \tag{2}$$

Proof. First, we note that from conditions 1 and 2 we follow

$$d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1), \tag{3}$$

and

$$d(x_n, x_{n+2}) \leq \lambda^n d(x_0, x_2), \tag{4}$$

for all $n \geq 1$.
Let $n, m \in \mathbf{N}$ and $m > n$.

1. Case: $m - n = 2k$ for any $k \in \mathbf{N}$.

From condition $(B_2 3(s))$ we have

$$\begin{aligned} d(x_n, x_m) &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)] \\ &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] \\ &+ s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4})] \\ &+ s^3[d(x_{n+4}, x_{n+5}) + d(x_{n+5}, x_{n+6})] \\ &\vdots \\ &+ s^{k-2}[d(x_{n+2k-6}, x_{n+2k-5}) + d(x_{n+2k-5}, x_{n+2k-4})] \\ &+ s^{k-1}[d(x_{n+2k-4}, x_{n+2k-3}) + d(x_{n+2k-3}, x_{n+2k-2})] \\ &+ s^{k-1} d(x_{n+2k-2}, x_{n+2k}) \end{aligned}$$

From conditions (3) and (4) we obtain

$$\begin{aligned} d(x_n, x_m) &\leq s\lambda^n(1+\lambda)d(x_0, x_1) \\ &+ s^2\lambda^{n+2}(1+\lambda)d(x_0, x_1) \\ &+ s^3\lambda^{n+4}(1+\lambda)d(x_0, x_1) \\ &\vdots \\ &+ s^k\lambda^{n+2k-2}(1+\lambda)d(x_0, x_1) \\ &+ s^k\lambda^{n+2k-2}d(x_0, x_2). \end{aligned}$$

So,

$$\begin{aligned} d(x_n, x_m) &\leq s\lambda^n(1+\lambda)d(x_0, x_1)[1 + s\lambda^2 + \cdots + (s\lambda^2)^{k-1}] \\ &+ (s\lambda^2)^{k-1}\lambda^n d(x_0, x_2). \end{aligned}$$

How is it $0 \leq s\lambda^2 < 1$, we obtain

$$d(x_n, x_m) \leq \frac{s\lambda^n(1+\lambda)d(x_0, x_1)}{1 - s\lambda^2} + \lambda^n d(x_0, x_2). \tag{5}$$

Now from (5), we conclude that $\{x_n\}$ is Cauchy.

2. Case: $m - n = 2k + 1$ for any $k \in \mathbf{N}$. Similar to the previous case from condition $B_2 3(s)$ we have

$$\begin{aligned} d(x_n, x_m) &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)] \\ &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] \\ &+ s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4})] \\ &+ s^3[d(x_{n+4}, x_{n+5}) + d(x_{n+5}, x_{n+6})] \\ &\vdots \\ &+ s^{k-2}[d(x_{n+2k-6}, x_{n+2k-5}) + d(x_{n+2k-5}, x_{n+2k-4})] \\ &+ s^{k-1}[d(x_{n+2k-4}, x_{n+2k-3}) + d(x_{n+2k-3}, x_{n+2k-2})] \\ &+ s^k[d(x_{n+2k-2}, x_{n+2k-1}) + d(x_{n+2k-1}, x_{n+2k}) \\ &+ d(x_{n+2k}, x_{n+2k+1})], \end{aligned}$$

and from here again using the inequalities (3) and (4) we get

$$\begin{aligned} d(x_n, x_m) &\le s\lambda^n(1+\lambda)d(x_0,x_1) \\ &+ s^2\lambda^{n+2}(1+\lambda)d(x_0,x_1) \\ &+ s^3\lambda^{n+4}(1+\lambda)d(x_0,x_1) \\ &\vdots \\ &+ s^k\lambda^{n+2k-2}(1+\lambda)d(x_0,x_1) \\ &+ s^k\lambda^{n+2k-2}(1+\lambda+\lambda^2)d(x_0,x_1). \end{aligned}$$

So we have

$$\begin{aligned} d(x_n, x_m) &\le \frac{s\lambda^n(1+\lambda)d(x_0,x_1)}{1-s\lambda^2} \\ &+ s\lambda^n(s\lambda^2)^{k-1}(1+\lambda+\lambda^2)d(x_0,x_1) \end{aligned}$$

$$d(x_n, x_m) \le \frac{s\lambda^n(1+\lambda)d(x_0,x_1)}{1-s\lambda^2} + \lambda^n(1+\lambda+\lambda^2)d(x_0,x_1). \tag{6}$$

So, $\{x_n\}$ is Cauchy. The estimate (2) follows from (5) and (6) when we let us m run infinitely. □

Lemma 2.2. *Let $T : X \to X$ be an α-admissible mapping and let $\{x_n\}$ be a Picard sequence of T based on a point $x_0 \in X$. If α is transitive, x_0 satisfies $\alpha(x_0, Tx_0) \ge 1$ and*

$$\alpha(x,y)d(Tx,Ty) \le \lambda d(x,y), \tag{7}$$

for all $x, y \in X$, where $\lambda \in (0,1)$, then it is

$$d(x_{m+k}, x_{n+k}) \le \lambda^k d(x_m, x_n),$$

for all $m, n, k \in \mathbf{N}, n < m$.

Proof. Using Lemma 1.1 we get

$$\alpha(x_m, x_n) \ge 1 \text{ for all } n < m.$$

From condition (7) follows

$$\begin{aligned} d(x_{m+k}, x_{n+k}) &\le \frac{\lambda}{\alpha(x_{m+k-1}, x_{n+k-1})} d(x_{m+k-1}, x_{n+k-1}) \\ &\le \lambda d(x_{m+k-1}, x_{n+k-1}) \\ &\vdots \\ &\le \lambda^k d(x_m, x_n). \end{aligned}$$

□

Lemma 2.3. *Let (X, d) be a $b_v(s)$-metric space, $T : X \to X$ be a mapping and let $\{x_n\}$ be a sequence in X such that $x_0 \in X$ and $x_{n+1} = Tx_n$. If there exists $\lambda \in [0,1)$ and such that*

$$d(x_n, x_{n+1}) \le \lambda d(x_{n-1}, x_n) \text{ for all } n \ge 1, \tag{8}$$

then T has a fixed point or $x_n \ne x_m$ for all $n \ne m$.

Proof. If $x_n = x_{n+1}$ then x_n is fixed point of T and proof is hold. So, suppose that $x_n \ne x_{n+1}$ for all $n \ge 0$. Then $x_n \ne x_{n+k}$ for all $n \ge 0, k \ge 1$. Namely, if $x_n = x_{n+k}$ for some $n \ge 0$ and $k \ge 1$ we have that $Tx_n = Tx_{n+k}$ and $x_{n+1} = x_{n+k+1}$. Then (8) implies that

$$d(x_{n+1}, x_n) = d(x_{n+k+1}, x_{n+k}) \le \lambda^k d(x_{n+1}, x_n) < d(x_{n+1}, x_n)$$

is a contradiction. Thus we assume that $x_n \ne x_m$ for all distinct $n, m \in \mathbf{N}$. □

Theorem 2.1. *Let (X,d) be a complete α-regular $b_2(s)$-metric space and $T : X \to X$ be a α-admissible such that T satisfies the conditions*

$$\alpha(x,y)d(Tx,Ty) \le \lambda d(x,y),$$

for all $x,y \in X$, where $\lambda \in (0,1)$. If $\alpha(x_0,Tx_0) \ge 1$ for some $x_0 \in X$ and α transitive then T has a fixed point in X.

Proof. Let $\lambda \in [0,1)$. Since $\lim_{n\to\infty} \lambda^n = 0$, there exists a natural number N such that

$$0 < \lambda^k \cdot s < 1, \tag{9}$$

for all $k \ge N$.

Let $x_0 \in X$ such that $\alpha(x_0,Tx_0) \ge 1$. From Lemma 1.1 we have that

$$\alpha(x_n,x_{n+1}) \ge 1 \text{ for all } n \in \mathbf{N}.$$

Define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \ge 0$. If $x_n = x_{n+1}$ then x_n is fixed point of T and proof is hold. So, suppose that $x_n \ne x_{n+1}$ for all $n \ge 0$. Then $x_n \ne x_m$ for all $n < m$. Since, (X,d) is $b_2(s)$-metric space, from condition $(B_2(s))$ we have

$$d(x_m,x_n) \le s[d(x_m,x_{m+k}) + d(x_{m+k},x_{n+k}) + d(x_{n+k},x_n)].$$

Using Lemma 2.2 we get

$$\begin{aligned} d(x_m,x_n) &\le s[\lambda^m d(x_0,x_k) + \lambda^k d(x_m,x_n) + \lambda^n d(x_0,x_k)] \\ (1-s\lambda^k)d(x_m,x_n) &\le s(\lambda^m+\lambda^n)d(x_0,x_k). \end{aligned}$$

From this, together with (9), we obtain

$$d(x_m,x_n) \le \frac{s(\lambda^m+\lambda^n)}{1-s\lambda^k} d(x_0,x_k).$$

Thus $\{x_n\}$ is a Cauchy sequence in X. By completeness of (X,d) there exists $x^* \in X$ such that

$$\lim_{n\to\infty} x_n = x^*.$$

Now we obtain that x^* is a fixed point of T. Namely, for any $n \in \mathbf{N}$ we have

$$\begin{aligned} d(x^*,Tx^*) &\le s[d(x^*,x_n) + d(x_n,x_{n+1}) + d(x_{n+1},Tx^*)] \\ &= s[d(x^*,x_n) + d(x_n,x_{n+1}) + d(Tx_n,Tx^*)] \\ &\le s\Big[d(x^*,x_n) + d(x_n,x_{n+1}) + \frac{\lambda d(x_n,x^*)}{\alpha(x_n,x^*)}\Big] \\ &\le s[d(x^*,x_n) + d(x_n,x_{n+1}) + \lambda d(x_n,x^*)]. \end{aligned}$$

Since, $\lim_{n\to\infty} d(x^*,x_n) = 0$ and $\lim_{n\to\infty} d(x_n,x_{n+1}) = 0$, we have $d(x^*,Tx^*) = 0$ i. e., $Tx^* = x^*$. □

Remark 2.1. We note that the previous Theorem is an improvement in the results in [13] (Theorem 2.1).

In the next Theorem we do not assume that the function α is transitive.

Theorem 2.2. *Let (X,d) be a complete α-regular $b_2(s)$-metric space and $T : X \to X$ be a α-admissible such that T satisfies the conditions*

$$\alpha(x,y)d(Tx,Ty) \le \lambda M(x,y), \tag{10}$$

for all $x,y \in X$, where $\lambda \in (0,\frac{1}{s})$. If $\min\{\alpha(x_0,Tx_0), \alpha(x_0,T^2x_0)\} \ge 1$ for some $x_0 \in X$, then T has a fixed point in X.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$ and $x_{n+1} = Tx_n, n = 1, 2, \ldots$. Since T is a α-admissible, from Lemma 1.1 we obtain

$$\alpha(x_n, x_{n+1}) \geq 1 \text{ for all } n \in \mathbf{N}. \tag{11}$$

Similarly, from $\alpha(x_0, T^2x_0) \geq 1$ follows

$$\alpha(x_n, x_{n+2}) \geq 1 \text{ for all } n \in \mathbf{N}. \tag{12}$$

From conditions (10) and (11) we have

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &= d(Tx_n, Tx_{n+1}) \\ &\leq \alpha(x_n, x_{n+1})d(Tx_n, Tx_{n+1}) \\ &\leq \lambda M(x_n, x_{n+1}), \end{aligned}$$

since

$$M(x_n, x_{n+1}) = \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}$$

and

$$d(x_{n+1}, x_{n+2}) \leq \lambda d(x_{n+1}, x_{n+2})$$

not possible, we conclude that it is

$$d(x_{n+1}, x_{n+2}) \leq \lambda d(x_n, x_{n+1}), \tag{13}$$

so, we obtain

$$d(x_{n+1}, x_{n+2}) \leq \lambda^n d(x_1, x_0).$$

Similarly, from conditions (10) and (12) we obtain

$$\begin{aligned} d(x_n, x_{n+2}) &= d(Tx_{n-1}, Tx_{n+1}) \\ &\leq \alpha(x_{n-1}, x_{n+1})d(Tx_{n-1}, Tx_{n+1}) \\ &\leq \lambda M(x_{n-1}, x_{n+1}), \end{aligned}$$

since

$$M(x_{n-1}, x_{n+1}) = \max\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n), d(x_{n+1}, x_{n+2})\}$$

and

$$d(x_{n+1}, x_{n+2}) \leq \lambda^2 d(x_{n-1}, x_n),$$

we conclude that it is

$$d(x_n, x_{n+2}) \leq \lambda \max\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n)\}. \tag{14}$$

From conditions (13) and (14) we obtain

$$d(x_n, x_{n+2}) \leq \lambda^n \max\{d(x_1, x_0), d(x_0, x_2)\}. \tag{15}$$

From (13) and (15) and Lemma 2.1 we conclude that $\{x_n\}$ is Cauchy, so it converges to a limit $x^* \in X$. How is (X, d) α-regular $b_2(s)$-metric space, from (11) we get that $\alpha(x_n, x^*) \geq 1$ for all $n \in \mathbf{N}$. From Lemma 2.3 we conclude that $x_n \neq x_m$ for all $n \neq m$. Now we obtain that x^* is the fixed point of T. Namely, for any $n \in \mathbf{N}$ we have

$$\begin{aligned} d(x^*, Tx^*) &\leq s[d(x^*, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, Tx^*)] \\ &= s[d(x^*, x_n) + d(x_n, x_{n+1}) + d(Tx_n, Tx^*)] \\ &\leq s\Big[d(x^*, x_n) + d(x_n, x_{n+1}) + \frac{\lambda M(x_n, x^*)}{\alpha(x_n, x^*)}\Big] \\ &\leq s[d(x^*, x_n) + d(x_n, x_{n+1}) \\ &+ \lambda \max\{d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, Tx^*)\}]. \end{aligned}$$

Since, $\{x_n\}$ converges to x^* and $\lambda < \frac{1}{s}$, we have $Tx^* = x^*$. □

Remark 2.2. We note that in the previous Theorem 2.2, for the proof of the convergence of the sequence $\{x_n\}$, a sufficient condition is that it is $\lambda \in (0, \frac{1}{\sqrt{s}})$. Also, if $M(x,y) = d(x,y)$, we get that

$$\begin{aligned} d(x^*, Tx^*) &\leq s[d(x^*, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, Tx^*)] \\ &= s[d(x^*, x_n) + d(x_n, x_{n+1}) + d(Tx_n, Tx^*)] \\ &\leq s[d(x^*, x_n) + d(x_n, x_{n+1}) + \frac{\lambda d(x_n, x^*)}{\alpha(x_n, x^*)}] \\ &\leq s[d(x^*, x_n) + d(x_n, x_{n+1}) + \lambda d(x_n, x^*)]. \end{aligned}$$

So, $Tx^* = x^*$.

Thus, the following result follows from the Theorem 2.2 and Remark 2.2.

Theorem 2.3. *Let (X, d) be a complete α-regular $b_2(s)$-metric space and $T : X \to X$ be a α-admissible such that T satisfies the conditions*

$$\alpha(x, y)d(Tx, Ty) \leq \lambda d(x, y),$$

for all $x, y \in X$, where $\lambda \in (0, \frac{1}{\sqrt{s}})$. If $\min\{\alpha(x_0, Tx_0), \alpha(x_0, T^2x_0)\} \geq 1$ for some $x_0 \in X$, then T has a fixed point in X.

Remark 2.3. If $\alpha(x, y) = 1$, for all $x, y \in X$ then T has unique fixed point. Let y^* be another fixed point of T. Then it follows from (8) that $d(x^*, y^*) = d(Tx^*, Ty^*) \leq \lambda d(x^*, y^*) < d(x^*, y^*)$, is a contradiction. Therefore, we must have $d(x^*, y^*) = 0$, i.e., $x^* = y^*$.

We note that from Theorem 2.3 we obtain the following result (Theorem 2.1. in [8]).

Theorem 2.4 ([8]). *Let (X, d) be a complete rectangular b-metric space with coefficient $s > 1$ and $T : X \to X$ be a mapping satisfying:*

$$d(Tx, Ty) \leq \lambda d(x, y)$$

for all $x, y \in X$, where $\lambda \in [0, \frac{1}{s}]$. Then T has a unique fixed point.

Remark 2.4. As $\frac{1}{s} < \frac{1}{\sqrt{s}}, (s > 1)$, using the Lemma 2.1, the following results can be improved Theorem 2.1. in [6], Theorem 2. 1. in [5], Theorem 1. in [15], Theorem 2.1. in [18].

The following result is known for $b_1(s)$-metric space (see R. Miculescu and A. Mihail [12, Lemma 2.2] and T. Suzuki [17, Lemma 6]).

Lemma 2.4 ([12, 17]). *Every sequence $(x_n)_{n \in \mathbf{N}}$ of elements from a b-metric space (X, d, s), having the property that there exists $\gamma \in [0, 1)$ such that*

$$d(x_{n+1}, x_n) \leq \gamma d(x_n, x_{n-1}),$$

for every $n \in \mathbf{N}$, is Cauchy.

It is therefore natural to ask the following question.
Question. Does the conclusion of Lemma 2.1 hold if $\frac{1}{\sqrt{s}}$ is replaced by 1?

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