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EIGENOSCILLATIONS AND STABILITY OF ORTHOTROPIC SHELLS, CLOSE TO CYLINDRICAL ONES, WITH AN ELASTIC FILLER AND UNDER THE ACTION OF MERIDIONAL FORCES, NORMAL PRESSURE AND TEMPERATURE

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Abstract. Eigenoscillations and stability of closed orthotropic shells of revolution, close by their form to cylindrical ones, with an elastic filler and under the action of meridional forces, external pressure and temperature are investigated. The shells of positive and negative Gaussian curvature are studied. Formulas and universal curves of dependence of the least frequency on orthotropy parameters, meridional loading, external pressure, temperature, rigidity of an elastic filler, as well as on the amplitude of shell deviation from the cylinder, are obtained. Critical values of outer effects are defined.

We study eigenoscillations and stability of closed orthotropic shells of revolution, close by their forms to cylindrical ones, with an elastic filler and under the action of meridional forces uniformly distributed over the end-walls of the shell, external pressure and temperature. We consider a light filler for which the influence of tangential stresses on the contact surface and the inertia forces may be neglected. The shell is considered to be thin and elastic. Temperature in a shell body is uniformly distributed. An elastic filler is modelled by the Winkler's base, its extension by heating is not taken into account. We investigate the shells of middle length whose form of midsurface generatrix is expressed by a parabolic function. We consider the shells of positive and negative Gaussian curvature. The boundary conditions on the end-walls correspond to a free support admitting certain radial displacement in the initial state. Formulas and universal curves of dependance of the least frequency on the orthotropy parameters, meridional loading, external pressure, temperature, rigidity of the elastic filler, as well as on the deviation amplitude of the shell from the cylinder are obtained. It is shown that the elastic orthotropy parameters affect significantly the least frequency and the corresponding form of the waveformation. A degree of influence of orthotropy parameters under separate and joint action of the above-mentioned outer factors on the lower frequencies is revealed. Critical values of outer effects are defined.

We consider the shell whose middle surface is formed by the rotation of a square parabola around the z-axis of the rectangular system of coordinates x, y, z with the origin in the middle of the segment of the axis of rotation. It is assumed that the cross-section radius R of the middle surface is defined by the equality $R = r + \delta_0\big[1 - \xi^2(r/\ell)^2\big]$, where r is the end-wall section radius, δ_0 is the maximal deviation from the cylindrical form (for $\delta_0 > 0$, the shell is convex, and for $\delta_0 < 0$, it is concave), $L = 2\ell$ is the shell length, $\xi = z/r$. We consider the shells of middle length [9], and it is assumed that

$$(\delta_0/r)^2 \ll 1, \quad (\delta_0/\ell)^2 \ll 1. \tag{1}$$

As the basic equations of oscillations we have taken those of the theory of shallow orthotropic shells [8]. For the shells of middle length, the forms of oscillations that correspond to the least frequencies have weak variability in longitudinal direction as compared with the circumferential one, therefore the correlation

$$\frac{\partial^2 f}{\partial \xi^2} \ll \frac{\partial^2 f}{\partial \varphi^2} \quad (f = w, \psi) \tag{2}$$

is valid, where w and ψ are, respectively, the functions of radial displacement and stress. As a result, the system of equations for the shells under consideration is reduced to the following resolving equation

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(due to the adopted assumption, temperature terms are equal to zero [7]):

$$\varepsilon\,\frac{\partial^8 w}{\partial\varphi^8}+\frac{E_1}{E_2}\left(\frac{\partial^4 w}{\partial\xi^4}+4\,\delta\,\frac{\partial^4 w}{\partial\xi^2\,\partial\varphi^2}+4\,\delta^2\,\frac{\partial^4 w}{\partial\varphi^4}-t_1^0\,\frac{\partial^6 w}{\partial\xi^2\,\partial\varphi^4}\right)$$
$$-\,t_2^0\,\frac{\partial^6 w}{\partial\varphi^6}-2\,s^0\,\frac{\partial^6 w}{\partial\xi\,\partial\varphi^5}+\gamma\,\frac{\partial^4 w}{\partial\varphi^4}+\frac{\rho\,r^2}{E_2}\,\frac{\partial^2}{\partial t^2}\left(\frac{\partial^4 w}{\partial\varphi^4}\right)=0, \tag{3}$$

$\varepsilon=h^2/12\,r^2(1-\nu_1\nu_2)$, $\delta=\delta_0 r/\ell^2$, $\tau_i=T_i^0/E_2h$ $(i=1,2)$, $s^0=S^0/E_2h$, $\gamma=\beta r^2/E_2h$, E_1,E_2,ν_1,ν_2 are, respectively, the E_1,E_2,ν_1,ν_2 moduli of elasticity and Poisson coefficients in the axial and circumferential directions $(E_1\nu_2=E_2\nu_1)$; T_1^0, T_2^0 are meridional and circumferential normal forces of the initial state; S^0 is the shearing stress of the initial state; h the shell thickness; ρ is the material density of the shell; β is the “bed” coefficient of the elastic filler (characterizing elastic rigidity); φ is an angular coordinate; t is time.

The initial state is assumed to be momentless. On the basis of the corresponding solution, taking into account the reaction of the filler and also inequalities (1), we obtain the following approximate expressions

$$\begin{aligned}T_1^0&=P_1\Big[1+\frac{\delta_0}{r}\big(\xi^2(r/\ell)^2-1\big)\Big]+q\delta_0\Big[\xi^2(r/\ell)^2-1\Big],\\ T_2^0&=-2\,P_1\delta_0 r/\ell^2-qr+\beta_0 r w_0,\quad S^0=0,\end{aligned} \tag{4}$$

where w_0 and β_0 are, respectively, deflection and a “bed” coefficient of the filler in the initial state; P_1 is meridional stress; q is external pressure.

Taking into account (2), we get

$$\big|\xi^2(r/\ell)^2-1\big|\,\frac{\partial^2 w}{\partial\xi^2}\ll 2(r/\ell)^2\frac{\partial^2 w}{\partial\varphi^2},\qquad \frac{\delta_0}{2}\,\big|\xi^2(r/\ell)^2-1\big|\,\frac{\partial^2 w}{\partial\xi^2}\ll\frac{\partial^2 w}{\partial\varphi^2}.$$

Therefore expressions (4), after substitution into equation (3), can be simplified and written in the following form:

$$T_1^0=P_1,\quad T_2^0=-2\,P_1\delta_0 r/\ell^2-qr+w_0\beta_0 r,\quad T_i^0=\sigma_i^0 h\quad(i=1,2). \tag{4'}$$

Taking into account the fact that in the initial state the shell deformation ε_φ^0 in the circumferential direction is defined by the equalities

$$\varepsilon_\varphi^0=\frac{\sigma_2^0-\nu_1\sigma_1^0}{E_2}+\alpha_2 T,\quad \varepsilon_\varphi^0=-\frac{w_0}{r}\,,$$

where α_2 is the coefficient of linear extension in the circumferential direction and T is temperature, we have

$$w_0=\big(-\sigma_2^0+\nu_1\sigma_1^0\big)\frac{r}{E_2}-\alpha_2 T_2. \tag{5}$$

Substituting expression (5) into (4′), we obtain

$$\frac{T_2^0}{E_2h}=\frac{\sigma_2^0}{E_2}=-\frac{qr}{E_2h}-2\frac{P_1}{E_2h}\,\delta+\nu_1\frac{\sigma_1^0}{E_2}\,\frac{\beta_0 r^2}{E_2h}-\alpha_2 T\frac{\beta_0 r^2}{E_2h}-\frac{\sigma_2^0}{E_2}\,\frac{\beta_0 r^2}{E_2h}.$$

Introduce the notation

$$E_1=e_1E,\quad E_2=e_2E,$$
$$\frac{qr}{Eh}=\overline{q},\quad \frac{P_1}{Eh}=-p,\quad \frac{\beta_0 r^2}{Eh}=\gamma_0,\quad 1+\gamma_0 e_2^{-1}=g.$$

Then expressions (4′) take the form

$$-\frac{\sigma_1^0}{E_2}=-e_2^{-1}p,\quad -\frac{\sigma_2^0}{E_2}=\big(\overline{q}-2p\delta+\nu_1 p\gamma_0+\alpha_2 T\gamma_0\big)e_2^{-1}g^{-1}. \tag{5'}$$

Note that since R is close to r, in the expressions for stresses (5′) we adopted $R\approx r$.

As a result, equation (3) takes the form

$$\varepsilon\,\frac{\partial^8 w}{\partial\varphi^8}+\frac{e_1}{e_2}\left[\frac{\partial^4 w}{\partial\xi^4}+4\delta\,\frac{\partial^4 w}{\partial\xi^2\,\partial\varphi^2}+4\big(\delta^2+\gamma/4e_1\big)\,\frac{\partial^4 w}{\partial\varphi^4}\right]$$
$$+\big(\overline{q}-2p\delta+\nu_1 p\gamma_0+\alpha_2 T\gamma_0\big)e_2^{-1}g^{-1}\frac{\partial^6 w}{\partial\varphi^6}+p\,\frac{\partial^6 w}{\partial\xi^2\,\partial\varphi^4}e_2^{-1}+\frac{\partial^2}{\partial t^2}\Big(\frac{\partial^4 w}{\partial\varphi^4}\Big)e_2^{-1}=0. \tag{6}$$

We consider the harmonic oscillations. For the given boundary conditions of free support and for equation (6) the solution

$$w=A_{mn}\cos\lambda_m\xi\,\sin n\varphi\,\cos\omega_{mn}t,\quad \lambda_m=m\pi r/2\ell \tag{7}$$
$$(m=2i+1,\quad i=0,1,2,\dots)$$

is satisfied.

Substituting expression (7) into (6), for finding eigenfrequencies, we obtain the following equality (in the sequel, the indices ω_{mn} will be omitted):

$$\omega^2=\frac{E}{\rho r^2}\Big[e_2\varepsilon n^4+e_1\big(\lambda_m^4 n^{-4}+4\,\delta\lambda_m^2 n^{-2}+4\big(\delta^2+\gamma/4e_1\big)$$
$$-\,p\big(\lambda_m^2-2\,\widetilde{\delta}n^2\big)-\big(\overline{q}+d_2T\gamma_0\big)g^{-1}n^2\big)\Big].$$

Introduce the notation

$$\overline{\delta}^{\,2}=\delta^2+\gamma/4e_1\,,\quad \widetilde{\delta}=\Big(\delta-\frac{1}{2}\nu_1\gamma_0\Big)g^{-1},\quad \widetilde{q}=\big(\overline{q}+\alpha T\gamma_0\big)g^{-1},$$
$$\omega^2=\frac{E}{\rho r^2}\Big[e_2\varepsilon n^4+e_1\big(\lambda_m^4 n^{-4}+4\,\delta\lambda_m^2 n^{-2}+4\,\overline{\delta}^{\,2}-p\big(\lambda_m^2-2\,\widetilde{\delta}n^2\big)-\widetilde{q}n^2\big)\Big]. \tag{8}$$

It is not difficult to see that for $p=0$, $\delta>0$, to the least frequency there corresponds $m=1$. It can also be shown that this condition takes place for $\delta<0$, bearing in mind inequalities (1) and the fact that $\omega^2>0$. Therefore, first of all, we consider the forms of oscillations under which there arises one half-wave ($m=1$) over the whole length of the shell and n waves in the circumferential direction. For the compression $p>0$, and for the tension $p<0$; q is a normal pressure which is assumed to be positive if it is external.

To present expression in a dimensionless form, we introduce the dimensionless values

$$\theta=(e_2/e_1)^{1/4}N,\ \ N=n^2/n_0^2,\ \ \overline{P}=P/\sqrt{e_1e_2},\ \ P=p/p_*,$$
$$\widetilde{Q}=\widetilde{q}/\overline{q}_{0*},\ \ \widetilde{q}=(\overline{q}+\alpha T\gamma_0)g^{-1},\ \ n_0^2=\lambda_1\varepsilon^{1/4},\ \ p_*=2\varepsilon^{1/2},$$
$$\overline{q}_{0*}=0,855(1-\nu_1\nu_2)^{-3/4}\Big(\frac{h}{r}\Big)^{3/2}\frac{r}{L}\,,\ \ \delta_*^{\nu}=(e_1/e_2)\delta_*\,,$$
$$\delta_*=\delta\varepsilon_*^{-1/2},\ \ \widetilde{\delta}^{\,\nu}=(e_1/e_2)^{1/4}\Big(\delta-\frac{1}{2}\nu_1\gamma_0\Big)\varepsilon_*^{-1/2}g^{-1},$$
$$\overline{\delta}^{\,\nu^2}=(e_1/e_2)^{1/2}(\delta_*^2+\gamma_*/4e_1)=\overline{\delta}^{\,\nu^2}+e_1e_2)^{-1/2}\frac{\gamma_*}{4}\,,\ \ \gamma_*=\gamma\varepsilon_*^{-1},$$
$$\omega_*^2=2\lambda_1^2\varepsilon^{1/2}E/3r^2\,,\quad \varepsilon=(1-\nu^2)^{-1/2}\frac{h}{2}\Big(\frac{r}{L}\Big)^2, \tag{9}$$

where $p_*,\overline{q}_{0*},\omega_*$ are, respectively, critical loading of compression, critical pressure and the least frequency for the cylindrical isotropic shell of middle length [1, 9]. Thus equality (8) can be written in the following dimensionless form:

$$\omega^2(\theta)/\omega_*^2=0,5\sqrt{e_1e_2}\big(\theta^2+\theta^{-2}+2,37\,\delta_*^{\nu}\theta^{-1}+1,4045\,\overline{\delta}_*^{\,\nu^2}\big)$$
$$-\,1,755\,e_1^{-1/4}e_2^{-3/4}\theta\overline{Q}-2\,\overline{P}\big(1-1,185\,\widetilde{\delta}_*^{\,\nu}\theta\big). \tag{10}$$

The least frequency (for $\omega^2(\theta)>0$) is defined by the condition $[\omega^2(\theta)]'=0$. As a result, we obtain

$$0,8775\,e_1^{-1/4}e_2^{-3/4}\widetilde{Q}-1,185\,\widetilde{\delta}^{\,\nu}\overline{P}=\theta-1,185\,\delta_*^{\nu}\theta^{-2}-\theta^{-3} \tag{11}$$

or

$$\theta^4 - \left(0,8775\, e_1^{-1/4} e_2^{-3/4} \widetilde{Q} - 1,185\, \widetilde{\delta}^{\,\nu} \overline{P}\right)\theta^3 - 1,185\, \delta_*^{\nu} \theta - 1 = 0. \tag{12}$$

This implies that for $\widetilde{Q} = \overline{P} = 0$, we get

$$\theta^4 - 1,185\, \delta_*^{\nu} \theta - 1 = 0. \tag{12$'$}$$

The above equation for an isotropic shell has been considered in [3]. Investigation of the roots of the above equation, similar to that carried out in [3], leads to

$$\begin{aligned} \theta &= \sqrt{1 - 0,0876\, \delta_*^2 (e_1/e_2)^{1/2}} + 0,2962\, \delta_* (e_1/e_2)^{1/4} \quad (\delta_* > 0), \\ \theta &= \sqrt{1 - 0,0876\, \delta_*^2 (e_1/e_2)^{1/2}} - 0,2962\, \delta_* (e_1/e_2)^{1/4} \quad (\delta_* < 0). \end{aligned} \tag{13}$$

In particular, for $\delta_* = 0$, we get the known formula for the cylindrical orthotropic shell of middle length $(n^2 = (e_1/e_2)^{1/4} \lambda_1 \varepsilon^{-1/4})$ [5].

By θ_0 we denote the value of θ which is defined by virtue of (13).

Defining thus the value of θ_0 (for fixed e_1, e_2, δ_*) and substituting it into expression (10) (for $P = \widetilde{Q} = 0$), we obtain the least frequency of a free shell $\omega(\theta_0)$. For clearness, we will now proceed to considering the value $N = \theta(e_1, e_2)^{1/4}$.

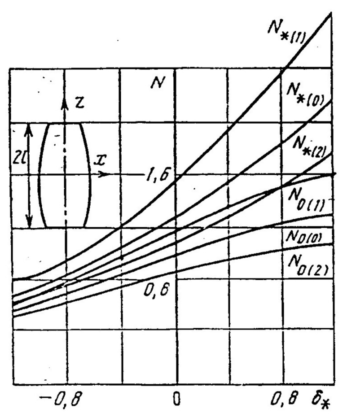

Figure 1

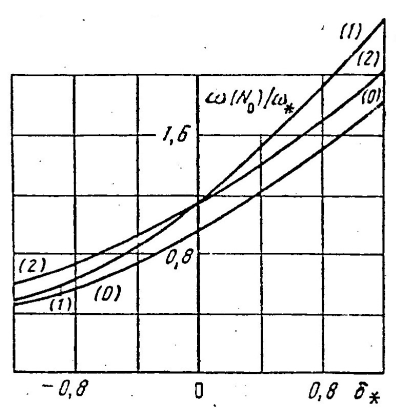

Figure 2

In Figures 1 and 2 we can see the graphs of dependencies $N_0 = n^2/n_0^2$ and $\omega(N_0)/\omega_*$ on the parameter δ_* for the cases $e_1 = e_2 = 1(0)$, $e_1 = 1$, $e_2 = 2(1)$; $e_1 = 2$, $e_2 = 1(2)$; the corresponding curves are denoted by $N_{0(i)}$ and (i) $i = 0, 1, 2$. It can be easily seen that for the convex shells $(\delta > 0)$ the importance of the elastic parameter is greater in the axial direction than in the circumferential one, whereas for the concave shells $(\delta < 0)$, the situation is inverse.

For $\omega = 0$, $P = 0$ from equality (10), we have

$$1,755\, e_1^{-1/4}\, e_2^{-3/4}\, \widetilde{Q} = \theta + \theta^{-3} + 2,37\, \delta_*^{\nu}\theta^{-2} + 1,404\, \overline{\delta}^{\,\nu^2}\theta^{-1}. \tag{14}$$

The least value $\widetilde{Q} > 0$ depending on θ is realized for $\widetilde{Q}'_\theta$. Thus we obtain

$$\theta^4 - 1,404\, \overline{\delta}^{\,\nu^2}\theta^2 - 4,74\, \delta_*^{\nu}\theta - 3 = 0. \tag{15}$$

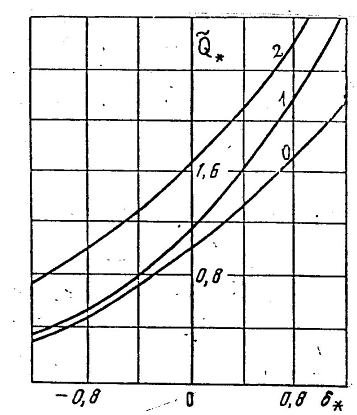

FIGURE 3

The positive root of that equation $\theta = \theta_*$ $(N = N_*)$ corresponds to the number of wave in the transverse direction under which is realized the critical loading of stability loss $\widetilde{Q}_*$. This equation for an isotropic shell is considered in [3], where the expression of the positive root is given explicitly. Generalizing this result to the orthotropic case, we present the roots of dependence of N_* on δ_* for the cases $i = 0, 1, 2$ considered above. In Figure 1, these curves are denoted, respectively, by $N_{*(i)}$. The graphs of dependence of $\widetilde{Q}_*$ on δ_* for those cases are given in Figure 3.

Note that expression (14) for finding the critical loading can be simplified on the basis of (15). From this equation implies that

$$2,37\, \delta_*^{\nu}\theta^{-2} + 1,404\, \overline{\delta}_*^{\,\nu^2}\theta^{-1} = -\big(2,37\, \delta_*^{\nu}\theta^{-2} + 3\,\theta^{-3} - \theta\big). \tag{16}$$

Substituting equality (16) into (14), we get

$$\widetilde{Q}_* = 1,15\, e_1^{1/4} e_2^{3/4} \big(\theta_* - \theta_*^{-3} - 1,185\, \delta_*^{\nu}\theta_*^{-2}\big). \tag{17}$$

From the condition of minimality of frequency (11) for $\overline{P} = 0$, we obtain the following dependence between $\widetilde{Q}$ and θ:

$$\widetilde{Q} = 1,15\, e_1^{1/4} e_2^{3/4} \big(\theta - \theta^{-3} - 1,185\, \delta_*^{\nu}\theta_*^{-2}\big). \tag{18}$$

It is not difficult to notice that from the above equality we have also the relation (17). On the basis of equality (18), for $\widetilde{Q} = 0$, we obtain equation $(12')$, whose root $\theta = \theta_0$ corresponds to the least frequency of the unloaded shell $\omega(\theta_0)$; while for $\widetilde{Q} = \widetilde{Q}_*$, we obtain equation (17), whose root θ_* corresponds to the critical loading, and $\omega = 0$.

Thus, when $\widetilde{Q}$ varies in the interval

$$0 \leq \widetilde{Q} \leq \widetilde{Q}_* \tag{19}$$

the least frequency $\omega(\theta, \widetilde{Q})$ varies in the interval $\omega(\theta_0, \widetilde{Q} = 0) \geq \omega(\theta, \widetilde{Q}) \geq 0$. Relying on the reasoning similar to that cited in [2], we can show that as $\widetilde{Q}$ varies in the interval (19), the value θ, realizing the least frequency $\omega(\theta, \widetilde{Q})$ and connected with $\widetilde{Q}$ by the relation (18), lies in the interval

$$\theta_0 \leq \theta \leq \theta_* \, . \tag{20}$$

Let us pass now to the value $N = \theta(e_1/e_2)^{1/4}$. In particular, for $\delta = \gamma = 0$, inequalities (19) and (20) take the form

$$0 \leq \widetilde{Q} \leq e_1^{1/4}\, e_2^{3/4}\,, \quad (e_1/e_2)^{1/4} \leq N \leq 1,315\, (e_1/e_2)^{1/4}. \tag{21}$$

For an isotropic case, inequalities (21) coincide with those presented in [2], $0 \leq \widetilde{Q} \leq 1$, $1 \leq N \leq 1,315$.

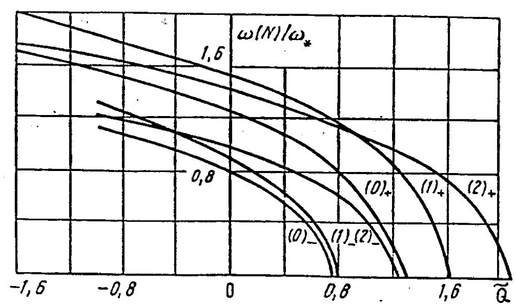

FIGURE 4

By virtue of equality (18) it is not difficult to construct the curves $N(\widetilde{Q})$ realizing the least frequency for different values $e_1, e_2, \delta_*\,, \gamma_*\,, T$. Towards this end, we fix these parameters and having the value θ, from the interval (20), we define the corresponding value $\widetilde{Q}$ by formula (18). Substituting these values in formula (10), we obtain (for the case $P = 0$ under considertion) the corresponding value of the least frequency. In Figure 4, we can see the curves of dependence of the least frequency ω/ω_* on $\widetilde{Q}$ (for $\gamma = 0$) for $\delta_* = 0,4$ and $\delta_* = -0,4$ for the cases $i = 0, 1, 2$. The curves are denoted by $(0)_+$, $(1)_+$, $(2)_+$, and $(0)_-$, $(1)_-$, $(2)_-$, respectively.

On the basis of the given curves and the results obtained in [1], it is easy to notice that if for the cylindrical shell in the absence of prestress the influence of orthotropy parameters is practically the same, then for the convex shells this effect occurs only for $\widetilde{Q} \approx 0,9$ and, in addition, on the interval $0 \leq \widetilde{Q} \leq 0,9$, the leading role belongs to the elastic parameter in the axial direction as compared with the circumferential one, whereas on the interval $0,9 \leq \widetilde{Q} \leq 1,6$ the situation is inverse.

Consider now the case $\overline{P} \neq 0$, $\widetilde{Q} = 0$ ($q = 0$, $\gamma = 0$) with $\overline{\delta}_*^{\nu^2} = \delta_*^{\nu^2}$, $\widetilde{\delta}_*^{\nu} = \delta_*^{\nu}$. On the basis of (10) and (11), we have

$$\omega^2/\omega_*^2 = 0,5\,\sqrt{e_1 e_2}\,\big[\theta^2 + \theta^{-2} + 2,375\,\delta_*^{\nu}\theta^{-1} + 1,404\,\delta_*^{\nu^2} - 2\,\overline{P}(1 - 1,185\,\delta_*^{\nu}\theta)\big], \tag{22}$$

$$-\,1,185\,\delta_*^{\nu}\overline{P} = Q - 1,185\,\delta_*^{\nu}\theta^{-2} - \theta^{-3} \tag{23}$$

or

$$\theta^4 + 1,185\,\delta_*^{\nu}\overline{P}\theta^3 - 1,185\,\delta_*^{\nu}\theta - 1 = 0. \tag{24}$$

From equation (24), for $\delta_* = 0$, we obtain the equation $\theta^4 - 1 = 0$ whose positive root $\theta = 1$ ($N = (e_1/e_2)^{1/4}$). Consequently, for the orthotropic cylindrical shell of middle length the least frequency is realized for $N = (e_1/e_2)^{1/4}$, independently of $\overline{P}$. For the isotropic case, all the above-said is in a full agreement with [6]. Moreover, from (24), for $\overline{P} = 1$, we find that the positive root of that equation does not depend on δ_*^ν.

For $\omega = 0$, equation (22) takes the form

$$\overline{P} = \frac{\theta^2 + \theta^{-2} + 2,37\,\delta_*^\nu \theta^{-1} + 1,404\,\delta_*^{\nu^2}}{2(1 - 1,185\,\delta_*^\nu \theta)}\,. \tag{25}$$

As is known, the least value of $\overline{P}$ is called a critical loading. In particular, for $\delta_* = 0$, $\theta = 1$, from (25), we get the known formula of the critical contracting force for the cylindrical orthotropic shell $\overline{P} = 1$ [9]. The least value $\overline{P}$ ($\overline{P} > 0$), depending on θ, realizes for $\overline{P}'_\theta = 0$. Thus we get

$$\begin{aligned} &2\big(\theta - \theta^{-3} - 1,185\,\delta_*^\nu \theta^{-2}\big)\big(1 - 1,185\,\delta_*^\nu \theta\big) \\ &\quad = -1,185\,\delta_*^\nu \big(\theta^2 + \theta^{-2} + 2,37\,\delta_*^\nu \theta^{-1} + 1,404\,\delta_*^{\nu^2}\big). \end{aligned} \tag{26}$$

In a simpler form, (26) is the fifth degree equation, so it is impossible to define its roots exactly. Therefore we have suggested somewhat different way of finding the positive root of that equation. We denote the positive root of that equation by θ_{*p}. The value $\theta = \theta_{*p}$ corresponds to a number of waves in the transversal direction under which is realized the critical loading of the stability loss $\overline{P}_*$. Substituting equality (26) into (25), we obtain

$$-1,185\,\delta_*^\nu \overline{P}_* = \theta_{*p} - 1,185\,\delta_*^\nu \theta_{*p}^{-2} - \theta_{*p}^{-3}. \tag{27}$$

It is not difficult to notice that equality (27) is likewise follows from equality (23) for $\omega = 0$.

Consequently, the values $\overline{P}, \theta$ satisfying equality (23) for which expression (22) vanishes, are the critical values of $\overline{P}_*$, θ_{*p}.

By virtue of equality (24), for $\overline{P} = 0$, we obtain equation (12′) whose positive root is denoted, as above, by θ_0 and corresponds to the least frequency of the unloaded shell, whereas for $\overline{P} = \overline{P}_*$, we obtain equation (27) whose positive root $\theta = \theta_{*p}$ corresponds to $\omega = 0$.

Thus, for $\overline{P}$, varying in the interval

$$0 \le \overline{P} \le \overline{P}_* \tag{28}$$

the least frequency varies in the interval $[\omega(\theta_0, \overline{P} = 0), 0]$.

Analogously to the investigation carried out in [2], we can show that when $\overline{P}$ varies in the interval (28) for $\delta_* \ge 0$, the value of θ realizing the least frequency $\omega(\theta, \overline{P})$ lies in the interval

$$\theta_0 \le \theta \le \theta_*. \tag{29}$$

In particular for $\delta_* = 0$ inequalities (28) and (29) take the form $0 \le \overline{P} \le 1$, $\theta_0 = \theta_* = 1$ (or $0 \le P \le e_1^{1/2} e_2^{1/2}$, $N_0 = N_* = (e_1/e_2)^{1/4}$).

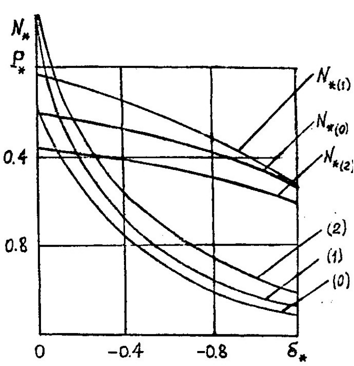

Figure 5

Dependencies $N_* = n_*^2/n_0^2$ and $P = p_*/p_0^2$ on the parameter $\delta \le 0$ for the cases $i = 0, 1, 2$ are given in Figure 5. The corresponding curves are denoted by $N_{*(i)}$ and (i). It is not difficult to see that for the concave shells of importance is the elastic parameter in the circumferential direction as compared with the axial one.

By virtue of equation (27), we can construct the dependence $N(\overline{P})$ which realizes the least frequency of the prestressed shell for various values of δ_*. To this end, we fix the parameters e_1, e_2, δ_* and having the value of θ from the interval (29), we find $\overline{P}_*$ by formula (27).

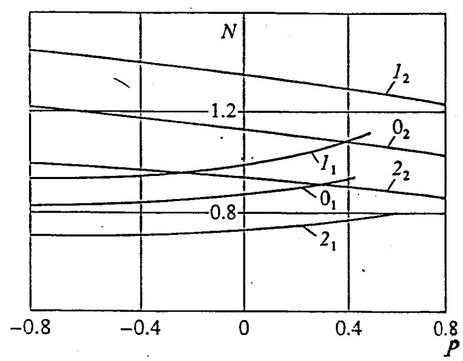

Figure 6

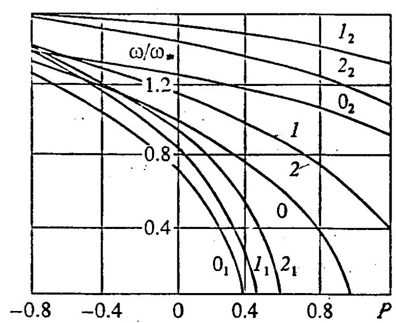

Figure 7

In Figure 6, we can see the values $N(P)$ for the cases $i = 0, 1, 2$ (for $\delta_* = 0,4$ and $\delta_* = -0,4$) which are denoted by i_1 and i_2. Figure 7, gives the curves of dependence of dimensionless least frequencies $\omega(N, P)/\omega_*$ on P for the above-mentioned cases which are likewise denoted by i_1 and i_2. Moreover, in Figure 7, we see the graph of dependence of ω/ω_* on P for the cylindrical shell ($\delta_* = 0$) in the cases $i = 0, 1, 2$ denoted, respectively, by $0, 1, 2$. On the basis of these graphs, it is not difficult to notice that if the influence of the ortotropy parameters for the cylindrical shell is practically the same, then for the concave shell, the influence of an elastic parameter in the circumferential direction is much more greater as compared with the axial elastic parameter, whereas the situation is opposite for the convex shells.

In the case of tensile forces $\overline{P} < 0$, equations (22) and (23) take the form

$$\omega^2/\omega_*^2 = 0,5\, e_1^{1/2} e_2^{1/2} \big[\theta^2 + \theta^{-2} + 2,37\, \delta_*^{\nu} \theta^{-1} + 1,404\, \delta_*^{\nu^2} + 2|\overline{P}|(1 - 1,185\, \delta_*^{\nu}\theta)\big], \tag{30}$$

$$1,185\, \delta_*^{\nu} |\overline{P}| = \theta - 1,185\, \delta_*^{\nu} \theta^{-2} - \theta^{-3}. \tag{31}$$

Analogously to the above-said, on the basis of formulas (30) and (31), we can construct the corresponding dependencies. In Figure 7, on the left of the Oy-axis we can see the graphs of dependence of ω/ω_* on $\overline{P}<0$ for the cases $i=0,1,2$ (for $\delta_*=0,4$ and $\delta_*=-0,4$).

Consider now a general case $\overline{P}\neq 0$, $\widetilde{Q}\neq 0$. Just as above, the frequency is defined by equality (10). For $\omega=0$, by virtue of (10), we obtain

$$1,755\, e_1^{-1/4} e_2^{-3/4}\widetilde{Q}=\theta+\theta^{-3}+2,37\,\delta_*^{\nu}\theta^{-2}+1,404\,\overline{\delta}_*^{\nu^2}\theta^{-1}-2\,\overline{P}(\theta^{-1}-1,185\,\overline{\delta}_*^{\nu}). \tag{32}$$

The least value $\widetilde{Q}>0$ depending on θ is realized for $Q'_\theta=0$. Thus we have

$$\begin{gathered}\theta^4+c\theta^2+d\theta+e=0,\quad c=2\overline{P}-1,404\,\overline{\delta}_*^{\nu^2},\\ d=-4,74\,\delta_*^{\nu},\quad e=-3.\end{gathered} \tag{33}$$

The roots of the last equation coincide with those of the two square equations

$$\begin{gathered}\theta^2+\frac{A_{1,2}}{2}\,\theta+\Big(y-\frac{d}{A_{1,2}}\Big)=0,\quad A_{1,2}=\pm\sqrt{8\,\alpha},\\ \theta_{1,2}=-\sqrt{\frac{\alpha}{2}}\pm\sqrt{\frac{d}{\sqrt{8\,\alpha}}-\frac{\alpha_1}{2}},\quad \theta_{3,4}=-\sqrt{\frac{\alpha}{2}}\pm\sqrt{-\frac{d}{\sqrt{8\,\alpha}}-\frac{\alpha_1}{2}},\\ \alpha=y_1-c/2,\quad \alpha_1=y_1+c/2,\end{gathered} \tag{34}$$

where y_1 is any real root of the cubic equation

$$y^3-\frac{c}{2}\,y^2-ey+\Big(\frac{ce}{2}-\frac{d^2}{8}\Big)=0 \tag{35}$$

or

$$z^3+3\,pz+2\,q=0\quad (z=y-c/6), \tag{36}$$

$$\begin{aligned}p&=1-\Big(2\,\overline{P}-1,404\,\overline{\delta}_*^{\nu^2}\Big)^2\Big/36,\\ q&=-\frac{1}{2}\Big(2\,\overline{P}+1,404\,\overline{\delta}_*^{\nu}\Big)^2\left[1-\frac{(2\,\overline{P}-1,404\,\overline{\delta}_*^{\nu^2})^3}{108\,(2\,\overline{P}+1,404\,\overline{\delta}_*^2)}\right].\end{aligned} \tag{37}$$

If we assume that

$$\Big(2\,\overline{P}-1,404\,\overline{\delta}_*^{\nu^2}\Big)^2\Big/36\ll 1,$$

then expressions (37) take the form $p=1$, $q=-\frac{1}{2}(2\,\overline{P}+1,404\,\overline{\delta}_*^{\nu^2})$. Since the discriminant of equation (36) is $D=q^2+p^3>0$, we have only one real root

$$z=\Big(-q+\sqrt{q^2+p^3}\Big)^{1/3}+\Big(-q-\sqrt{q^2+p^3}\Big)^{1/3} \tag{38}$$

If we assume that

$$\big(2\,\overline{P}+1,404\,\overline{\delta}_*^{\nu^2}\big)\big/36\ll 1 \tag{38$'$}$$

and expand the expressions appearing in (38) in series, omitting all values of the second order of smallness, we arrive at $z=\big[2\,\overline{P}+1,404\,(\overline{\delta}_*^{\nu^2}-\gamma_*/4)\big]/3$. Then on the basis of (34), (36) and (33), we obtain

$$\begin{aligned}\alpha&=z-c/3=2\cdot 1,404\,\delta_*^{\nu^2},\\ \alpha_1&=z+\frac{2}{3}\,c=2\overline{P}-1,404\Big(\delta_*^{\nu^2}+\frac{3}{4}\,\gamma_*\Big)\Big/3.\end{aligned} \tag{39}$$

Taking into account that y_1 is the root of equation (35), we have

$$\frac{d^2}{8(y_1-c/2)}=y_1^2-e,$$

whence we get

$$\frac{|d|}{\sqrt{8\,\alpha}} = \sqrt{y_1^2 - e} > y_1 = \frac{y_1}{2} + \frac{y_1}{2} + \frac{c}{4} - \frac{c}{4} = \frac{1}{2}\Big(y_1 - \frac{c}{2}\Big) + \frac{1}{2}\Big(y_1 + \frac{c}{2}\Big).$$

Consequently,

$$\frac{|d|}{\sqrt{8\,\alpha}} - \frac{\alpha_1}{2} > \frac{\alpha}{2}\,. \tag{40}$$

Since $N^2 = n^2/n_0^2$, of our interest are only positive roots of equation (33). Taking into account inequality (40), we find that for $\delta_* < 0$ ($d > 0$), positive is only the root θ_1, and for $\delta_* > 0$ ($d < 0$), positive is the root θ_3. Substituting the values d, α, α_1, according to equalities (33) and (39), into (34), we obtain

$$\theta_{1,2} = \sqrt{\sqrt{3} + 0,234\Big(\delta_*^{\nu^2} + \frac{3}{4\,e_1}\,\gamma_*^\nu\Big) - \overline{P}} \pm 0,684\,|\delta_*^\nu|, \tag{41}$$

where the indices "1" and "2" correspond to $\delta_* > 0$ and $\delta_* < 0$, respectively. It should be noted that the above formula is, according to inequality (38′), valid for comparatively not large values of rigidity of the elastic filler γ_*^ν. Taking into account that θ in an expanded form is $\theta = (e_1/e_2)^{1/4} n^2/\lambda_1 \varepsilon^{-1} 4$, we have

$$\begin{aligned} n_{1,2}^2 = (e_1/e_2)^{1/4}\bigg\{&\bigg(\sqrt{3} + 0,270(e_1/e_2)^{1/2}\varepsilon^{-1/2}\Big[\Big(\frac{\delta_0}{\ell}\Big)^2 \\ &+ \frac{3}{4}\,\frac{\gamma}{e_1}\Big(\frac{\ell}{r}\Big)^2\Big] - \overline{P}\bigg)^{1/2} \pm 0,735\,\Big(\frac{e_1}{e_2}\Big)^{1/4}\varphi^{-1/4}\,\frac{|\delta_0|}{\ell}\bigg\}\lambda_1\varepsilon^{-1/4}. \end{aligned} \tag{42}$$

In particular, for $\delta_0 = \gamma = p = 0$, we obtain the well-known formula for a critical number of waves of the cylindrical shell of middle length: $n^2 = (e_1/e_2)^{1/4}\sqrt{3}\,\lambda_1\varepsilon^{-1/4}$ [5].

From formula (42), it is not difficult to notice that under the action of contracting forces a number of critical circumferential waves decreases, while under the action of tensile forces this number increases.

Formula (39), as it has been mentioned above, takes place if condition (38′) is fulfilled. In the case if this condition is not fulfilled we have to proceed from full expressions (37). Defining thus the values θ_* (for fixed $\delta_*^\nu, \gamma_*^\nu, \overline{P}, e_1, e_2$) and substituting into (32), we obtain the corresponding critical value of $\widetilde{Q}_*$. In an expanded form, formula (32) for a critical pressure has the form

$$\begin{aligned} \overline{q}_{kp} = 0,570\,e_1^{1/4}e_2^{3/4}g\Big[&\theta_* + \theta_*^{-3} + 2,37\,\delta_*^\nu\theta_*^{-2} \\ &+ 1,404\big(\delta_*^{\nu^2} + \gamma_*^\nu/4e_1\big)\theta_*^{-1} - 2\,\overline{P}\big(\theta_*^{-1} - 1,185\,\widetilde{\delta}_*^\nu\big)\Big]\overline{q}_{0*} - \gamma_0\alpha_2 T. \end{aligned}$$

Note that the obtained value of $\widetilde{Q}_*$ on the basis of formula (32) for the isotropic cylindrical shell coincides for ($\delta_* = 0$, $\gamma_* = 0$) $P > 0$ practically with the results obtained in [4].

Consider now equation (12) and write it in the form

$$\begin{gathered} \theta^4 + b\theta^3 + d\theta + e = 0, \quad b = 1,185\,\delta_*^\nu\overline{P} - 0,8775\,\widetilde{Q}^{\,\nu}, \\ \widetilde{Q}^{\,\nu} = e_1^{-1/4}e_2^{3/4}\widetilde{Q}, \quad d = -1,185\,\delta_*^\nu, \quad e = -1. \end{gathered} \tag{43}$$

The roots of this equation coincide with those of the following two equations

$$\theta^2 + (b + B_{1,2})\frac{\theta}{2} + \Big(y_1 + \frac{by_1 - d}{B_{1,2}}\Big) = 0, \quad B_{1,2} = \pm\sqrt{8\big(y_1 - b^2/8\big)}. \tag{44}$$

Introduce the notation

$$\gamma_1 = y_1 + b^2/8, \quad \gamma_2 = y_1 - b^2/4. \tag{45}$$

Then the roots of these equations will take the form

$$\theta_{1,2} = -\frac{\sqrt{8\,\gamma_1 + b}}{4} \pm \sqrt{-\frac{by_1 - d}{\sqrt{8\,\gamma_1}} + \frac{b\sqrt{8\,\gamma_1} - 4\,\gamma_2}{8}}, \tag{46}$$

$$\theta_{3,4} = \frac{\sqrt{8\,\gamma_1 + b}}{4} \pm \sqrt{\frac{by_1 - d}{\sqrt{8\,\gamma_1}} - \frac{b\sqrt{8\,\gamma_1} + 4\,\gamma_2}{8}}, \tag{47}$$

where y_1 is any real root of the cubic equation

$$y^3 + 3\,py + 2q = 0, \quad 3\,p = 1 - \frac{1,185^2\widetilde{\delta}_*^{\nu^2}\overline{P}M}{4}\,,$$

$$2\,q = -\frac{1,185^2\widetilde{\delta}_*^{\nu^2}(1 - \overline{P}^2M^2)}{8}\,, \quad M = 1 - 0,7405\,\widetilde{Q}/\widetilde{\delta}_*^{\nu}\overline{P}$$

for

$$\begin{gathered}\frac{1,185^2\widetilde{\delta}^{\nu^2}|\overline{P}M|}{4} \ll 1 \quad \big(\widetilde{\delta}_*^{\nu} \le 0,5, \ \ |\overline{P}M| \le 0,5\big),\\ p = \frac{1}{3}, \quad q = -1,185^2\,\widetilde{\delta}_*^{\nu^2}\big(1 - \overline{P}^{\,\nu}M^2\big)/16.\end{gathered} \tag{48}$$

Since the discriminant of this equation $D > 0$, we have one real root

$$y_1 = \big(-q + \sqrt{D}\big)^{1/3} + \big(-q - \sqrt{D}\big)^{1/3},$$

$$\sqrt{D} = \sqrt{1 + 0,208\,\widetilde{\delta}_*^{\nu^4}\big(1 - \overline{P}^2M^2\big)/3^{3/2}}.$$

If we assume

$$0,208\,\widetilde{\delta}_*^{\nu^4}\big(1 - \overline{P}^2M^2\big) \ll 1 \tag{49}$$

then in a full analogy with the above-said, we obtain $y_1 = 0,1755\,\widetilde{\delta}_*^{\nu^2}\big(1 - \overline{P}^2M^2\big)$. Under the restrictions (48), inequality (49) is all the more fulfilled. Substituting the values y_1, b, d, e_1, e_2 into expressions (46) and (47) and also taking into account inequality (48), we find that for $d > 0$ ($\delta_*^{\nu} < 0$), positive is only the root θ_1, whereas for $d < 0$ ($\delta_*^{\nu} > 0$), positive is the root θ_3. As a result, we have

$$\begin{aligned}\theta_1 = \Big[&1 + 0,1755\,\widetilde{\delta}_*^{\nu^2}\overline{P}M_1\big(1 - \overline{P}^2M_1^2\big) - 0,0877\,\widetilde{\delta}_*^{\nu^2}\big(1 + 2\,\overline{P}M_1\\ &- 2\,\overline{P}^2M_1^2\big)\Big]^{1/2} + 0,2962\,\widetilde{\delta}_*^{\nu}\big(1 - \overline{P}M_1\big) \ \ (\delta_*^{\nu} > 0),\end{aligned} \tag{50}$$

$$\begin{aligned}\theta_2 = \Big[&1 + 0,1755\,\widetilde{\delta}_*^{\nu^2}PM_2\big(1 - \overline{P}^2M_2^2\big) - 0,0877\,\widetilde{\delta}_*^{\nu^2}\big(1 + 2\,\overline{P}M_2\\ &- 2\,\overline{P}M_2\big)\Big]^{1/2} - 0,2962\,\widetilde{\delta}_*^{\nu}\big(1 - \overline{P}M_1\big) \ \ (\delta_*^{\nu} < 0)\end{aligned} \tag{51}$$

$$M_1 = 1 - 0,7405\,\widetilde{Q}^{\nu}/\delta^{\nu}\overline{P}, \quad M_1 = 1 + 0,7405\,\widetilde{Q}^{\nu}/|\delta^{\nu}|\overline{P}.$$

For $\widetilde{\delta}_*^{\nu} > 0$, $\overline{P}/\widetilde{Q} > 0$ the value $M_1 = 0$, if $\delta_*^{\nu} = 0,7405\,\overline{P}/\widetilde{Q}^{\nu}$; for $\delta_*^{\nu} < 0$, $\overline{P}/\widetilde{Q}^{\,\nu} < 0$, the value $M_2 = 0$, if $|\delta_*^{\nu}| = -0,7405\,\overline{P}/\widetilde{Q}^{\,\nu}$, and formulas (50), (51) take the form

$$\theta = \sqrt{1 - 0,0877\,\widetilde{\delta}_*^{\nu^2}} + 0,2962\,\widetilde{\delta}_*^{\nu} \quad (\delta_*^{\nu} > 0),$$

$$\theta = \sqrt{1 - 0,0877\,\widetilde{\delta}_*^{\nu^2}} - 0,2962\,\big|\widetilde{\delta}_*^{\nu}\big| \quad (\delta_*^{\nu} < 0).$$

Note that this case of the certain values $\widetilde{\delta}_*^{\nu}$ corresponds to the cases for which the normal circumferential stresses due to meridional loading, external pressure and also temperature effect neutralise mutually each other.

For $\gamma_0 = 0$, $e_1 = e_2 = 1$ we have $\widetilde{\delta}_*^{\nu} = \delta_*$, $\widetilde{q} = \overline{q}$ and for θ, we obtain the formula given in [3].

Substituting the obtained expression for θ (for fixed δ_*^{ν}, $\overline{P}$, $\widetilde{Q}$, γ^{ν}) into formula (10), we obtain the corresponding least value of the dimensionless frequency ω/ω_*.

The above obtained formulas and graphs show how much substantially vary critical loading, the least frequency and the forms of wave formation depending on the orthotropy parameters, shell shape and external effects.

References

1. A. L. Gol'denveĭzer, V. B. Lidskiĭ, P. E. Tovstik, Free Oscillations of Thin Elastic Shells. (Russian) *Nauka, Moscow*, 1979.
2. S. Kukudzhanov, On influence of normal pressure on the frequencies of natural oscillations of cylindrical shells. (Russian) *Inzh. Zh. Mekh. Tverd. Tela*, **3** (1968), 140–144.
3. S. Kukudzhanov, On the effect of normal pressure on frequencies of proper oscillations of shells of revolution, close by their form to cylindrical ones. (Russian) *Izvestiya RAN, MTT* **6** (1996), 121–126.
4. H. M. Mushtari, A. V. Sachenkov, On stability of cylindrical and conical shells of circular section under the simultaneous action of axial compression and external normal pressure. (Russian) *Prikl. Mat. Meh.* **18** (1954), 667–674.
5. H. M. Mushtari, K. Z. Vlasov, Nonlinear Theory of Elastic Shells. (Russian) *Tatknigoizdat, Kazan*, 1957.
6. M. V. Nikulin, Influence of Axial Forces on Proper Oscillation Frequencies of a Cylindrical Shell. (Russian) *Oborongiz, Moscow*, 1959.
7. P. M. Ogibalov, V. F. Gribanov, Thermal Stability of Plates and Shells. (Russian) *Moskow State Univ.* 1968.
8. V. Z. Vlasov, General Theory of Shells and Its Applications in Technology. (Russian) *Gosudarstvennoe Izdatel'stvo Tehniko-Teoretičeskoĭ Literatury, Moscow-Leningrad*, 1949.
9. A. S. Vol'mir, Stability of Deformation Systems. (Russian) *Fizmatlit, Nauka, Moscow*, 1967.

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