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ON ONE CLASS OF ELLIPTIC EQUATIONS CONNECTED WITH THE NONLINEAR WAVES

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Abstract. Nonlinear elliptic equation connected with the nonlinear waves in the infinite area is considered. The non-smooth effective solutions exponentially vanishing at infinity are obtained. By means of such solutions the exact and approximate solutions of different nonlinear elliptic equations are derived. The profiles of nonlinear waves and symmetric solitary waves connected with those solutions are plotted by using "Maple".

1. Introduction

Nonlinear elliptic equations describe wide range of physical phenomena and those equations with the different kind of nonlinearity were considered by numerous authors [4–6, 8, 10, 14, 23, 24, 28–31, 36–45, 47–49, 52, 53].

In this paper we focus on the nonlinear elliptic equation connected with the different nonlinear waves. Particular case of this equation is the cubic nonlinear Schrödinger equation (cNLS).

The equation is considered in the infinite area. The effective solutions exponentially vanishing at infinity and having peaks at some lines are obtained. Non-smooth solitary waves connected with those solutions in a specific class of functions are constructed. Also the bounded solutions are given.

2. Statement of the Problem

In R^3 let us consider the following equation

$$P_1(\psi)\Delta\psi + P_2(\psi)(\nabla\psi)^2 + P_3(\psi) = 0, \tag{1}$$

where $\psi(x,y,z)$ is unknown function, $P_1(\xi)$, $P_2(\xi)$, $P_3(\xi)$ are the polynomials with respect to ξ, $P_1(\xi) = \sum\limits_{i=0}^{n} a_i\xi^i$, $P_2(\xi) = \sum\limits_{i=0}^{n} b_i\xi^i$, $P_3(\xi) = \sum\limits_{i=1}^{n+2} c_i\xi^i$, a_0, b_0, a_i, b_i, c_i, c_{n+1}, c_{n+2}, n, $i = 1,\dots,n$; are some constants.

The particular case of the equation (1) is the following equation

$$\left(1 - \frac{\psi_2^2}{2}\right)\Delta\psi_2 - \psi_2\big(\nabla\psi_2\big)^2 + \lambda_0 R^2\psi_2^3 - A_0\left(\psi_2 - \frac{\psi_2^3}{6}\right) = 0, \tag{2}$$

where λ_0, R, A_0 are the definite constants, ψ_2 is unknown function. When the function ψ_2 has negligible fifth degree value, the equation (2) is the approximation of the cubic nonlinear Schrödinger equation [27]. The solution of this equation was obtained in [27] in the specific class of functions.

Let us consider the following problem

Problem 1. In the space R^3 to find piecewise smooth continuous function ψ vanishing at infinity exponentially, satisfying the equation (1), having second order continuous derivatives $\frac{\partial^2\psi}{\partial x^2}$, $\frac{\partial^2\psi}{\partial y^2}$, $\frac{\partial^2\psi}{\partial z^2}$

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and first order derivatives with the jump planes $x = 0$, $y = 0$, $z = 0$ satisfying the conditions

$$\begin{aligned}\Big(\frac{\partial\psi}{\partial x}\Big)^{-}\Big|_{x=0} &= -\Big(\frac{\partial\psi}{\partial x}\Big)^{+}\Big|_{x=0},\\ \Big(\frac{\partial\psi}{\partial y}\Big)^{-}\Big|_{y=0} &= -\Big(\frac{\partial\psi}{\partial y}\Big)^{+}\Big|_{y=0},\\ \Big(\frac{\partial\psi}{\partial z}\Big)^{-}\Big|_{z=0} &= -\Big(\frac{\partial\psi}{\partial z}\Big)^{+}\Big|_{z=0}.\end{aligned} \tag{3}$$

Note 1. Here $\left(\frac{\partial\psi}{\partial x}\right)^{-}\big|_{x=0}$ and $\left(\frac{\partial\psi}{\partial x}\right)^{+}\big|_{x=0}$ means

$$\lim_{x\longrightarrow 0^-}\frac{\partial\psi}{\partial x}, \quad \lim_{x\longrightarrow 0^+}\frac{\partial\psi}{\partial x}$$

respectively.

3. Solution of the Problem

Let us consider the function

$$\psi_0 = \exp[-\alpha|x| - \beta|y| - \gamma|z| - D], \ \ \alpha,\, \beta,\, \gamma > 0, \tag{4}$$

where α, β, γ are some non-negative constants, D is an arbitrary parameter.

The function (4) vanishes at infinity exponentially and satisfies the conditions (3). By direct verification we obtain, that it will be the solution of the equation (1) if the constants a_0, b_0, a_i, b_i, c_i, c_{n+1}, c_{n+2}, $i = 1, \dots, n$; satisfy the following conditions

$$\begin{cases} d^2a_0 + c_1 = 0,\\ d^2(a_1 + b_0) + c_2 = 0,\\ d^2(a_2 + b_1) + c_3 = 0,\\ d^2(a_3 + b_2) + c_4 = 0,\\ \dots\dots\dots\dots\dots\dots\dots\\ d^2(a_{n-1} + b_{n-2}) + c_n = 0,\\ d^2(a_n + b_{n-1}) + c_{n+1} = 0,\\ d^2b_n + c_{n+2} = 0,\end{cases} \tag{5}$$

where $\alpha^2 + \beta^2 + \gamma^2 = d^2$.

Hence, we conclude, that the following theorem is valid

Thoerem 1. *If the coefficients a_0, b_0, a_i, b_i, c_i, c_{n+1}, c_{n+2}, $i = 1, \dots, n$; of the equation* (1) *satisfy the system* (5)*, then the function given by the formula* (4) *is the solution of the Problem* 1.

Also, it is easy to see, that the following theorem is true

Thoerem 2. *If the coefficients a_0, b_0, a_i, b_i, c_i, c_{n+1}, c_{n+2}, $i = 1, \dots, n$; of the equation* (1) *satisfy the system* (5) *and $\alpha = 0 \vee \beta = 0, \vee \gamma = 0$ then the function given by the formula* (4) *is the solution of the equation* (1) *bounded in R^3 and satisfying the condition* (3)*.*

Note 2. Here we do not discuss the uniqueness of the solutions of the Problem 1, as the function (4) depends on an arbitrary parameters α, β, γ, D.

In the next chapter we consider some particular cases of (1). By means of the function (4) we will construct exact and approximate solutions of these equations.

4. Examples

Let us consider several cases.

1) In case of $a_0 = c_1 = 0$, $a_1 = 1$, $b_0 = -1$, $b_1 = a_i = b_i = c_i = c_{n+1} = c_{n+2} = 0$, $i = 2, \dots, n$, the equation (1) takes the form

$$\psi\Delta\psi - (\nabla\psi)^2 = 0. \tag{6}$$

According to the Theorem 1 the solution of the equation (6) satisfies the conditions (3) will be given by the formula

$$\psi = R\psi_0, \tag{7}$$

where R is an arbitrary constant and ψ_0 is given by (4).

In Figure 1 the graphic of (7) vanishing at infinity is given for some parameters and in Figure 2 the graphic of (7) bounded at infinity is given. The graphics are constructed by using “Maple”.

Note 3. The Dirichlet problem for the equation (6) was studied in [5, 6].

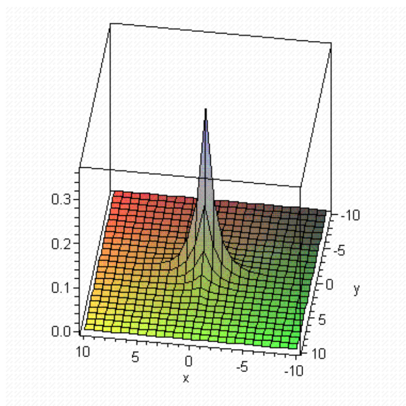

FIGURE 1. The graphic of (7) in case of $D = 1$; $R = 1$; $\alpha = \beta = \gamma = 1$; $z = 0$;

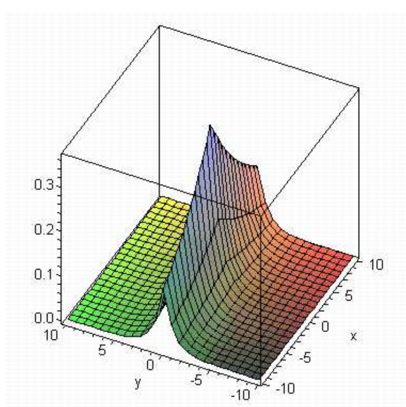

FIGURE 2. The graphic of (7) in case of $D = 1$; $R = 1$; $\alpha = 0.1$; $\beta = 1$; $\gamma = 0$;

2) In case of $a_0 = 1$, $c_1 = -A_0 + \lambda_0$, $a_1 = b_1 = a_i = b_i = c_i = c_{n+1} = c_{n+2} = 0$, $i = 2, \dots, n$, the equation (1) represents the well-known Helmholtz equation

$$\Delta\psi - (A_0 - \lambda_0)\psi = 0, \tag{8}$$

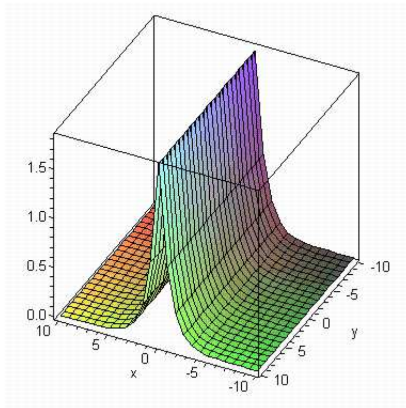

FIGURE 3. The linear wave. The graphic of (9) in case of $m = 1$; $D_1 = 4$; $R_1 = 100$; $\alpha = 1$; $\beta = \gamma = 0$; $A_0 - \lambda_0 = 1$.

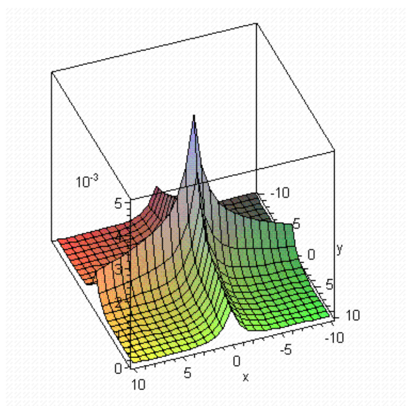

FIGURE 4. Superposition of linear waves. The graphic of (9) in case of $m = 2$; $D_1 = D_2 = 5$; $R_1 = R_2 = 1$; $z = 1$; $\alpha_1 = \beta_2 = 0.1$; $A_0 - \lambda_0 = 2.01$; $\beta_1 = \alpha_2 = \gamma_1 = \gamma_2 = 1$.

By the Theorem 1, the solution of the equation (8) for the Problem 1 is given by (7), where R is an arbitrary constant, α, β, γ satisfy the conditions

$$\alpha^2 + \beta^2 + \gamma^2 = A_0 - \lambda_0, \quad A_0 > \lambda_0.$$

The function (7) represents some class of stationary non-smooth linear waves. Their superposition is also the solution of (8) and is given by the sum

$$\psi = \sum_1^m R_k \exp\big[- \alpha_k|x| - \beta_k|y| - \gamma_k|z| - D_k\big], \quad D_k > 0, \tag{9}$$

where R_k are an arbitrary constants and $\alpha_k^2 + \beta_k^2 + \gamma_k^2 = A_0 - \lambda_0$, $A_0 > \lambda_0$, m is an arbitrary natural number.

The graphics of (9) are given in Figures 3, 4 for the different parameters.

In case of $A_0 = \lambda_0$ the equation (8) will be reduced to the Laplace equation

$$\Delta\psi = 0. \tag{10}$$

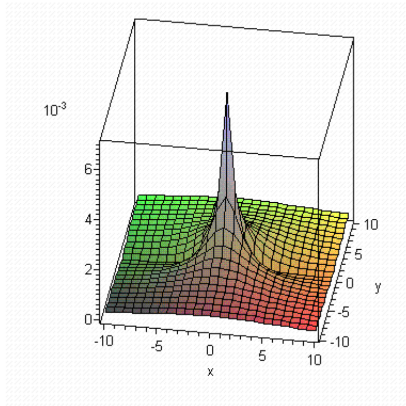

FIGURE 5. The linear wave. The graphic of (11) in case of $D = 5$; $R_1 = 1$; $a = 1$; $R_2 = R_3 = R_4 = R_5 = R_6 = 0$.

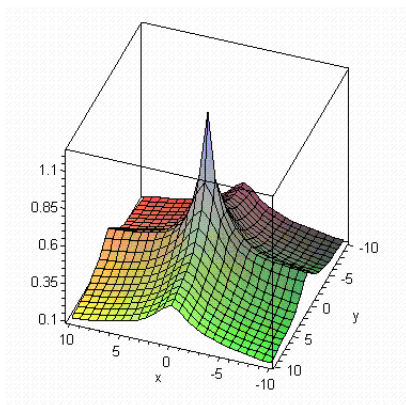

FIGURE 6. Superposition of linear waves. The graphic of (11) in case of $D = 5$; $a = 1$; $z = 1$; $R_1 = R_2 = R_3 = 100$; $R_4 = R_5 = R_6 = 0$.

Using well-known Poisson formula [5, 6, 8, 33], we obtain the non-smooth solution of the Problem 1 for the equation (10)

$$\begin{aligned}\psi_0 = {} & R_1 \frac{|y|}{\pi} \int_{-\infty}^{\infty} \frac{f(t)dt}{(t-x)^2+y^2} + R_2 \frac{|y|}{\pi} \int_{-\infty}^{\infty} \frac{f(t)dt}{(t-z)^2+y^2} \\ & + R_3 \frac{|x|}{\pi} \int_{-\infty}^{\infty} \frac{f(t)dt}{(t-z)^2+x^2} + R_4 \frac{|x|}{\pi} \int_{-\infty}^{\infty} \frac{f(t)dt}{(t-y)^2+x^2} \\ & + R_5 \frac{|z|}{\pi} \int_{-\infty}^{\infty} \frac{f(t)dt}{(t-x)^2+z^2} + R_6 \frac{|z|}{\pi} \int_{-\infty}^{\infty} \frac{f(t)dt}{(t-y)^2+z^2}, \end{aligned} \tag{11}$$

$f(t)$ $(-\infty < t < +\infty)$, is the function vanishing at infinity, having second order continues derivatives and first order continues derivatives except the point $t = 0$, where the following conditions are satisfied

$$(f')^+(0) = -(f')^-(0), \quad (f'')^+(0) = (f'')^-(0), \quad |f(t)| \leq e^{-D}, \quad D \geq 5,$$

R_1, R_2, R_3, R_4, R_5, R_6, $|R_1| + |R_2| + |R_3| + |R_4| + |R_5| + |R_6| \neq 0$ are non-negative constants.

The graphics of (11) are given in Figures 5, 6 in the case $f(t) = e^{-a|t|-D}$, $a > 0$.

3) Now, let us consider the equation

$$\Delta\psi + c_1\psi + c_2\psi^2 = 0. \tag{12}$$

The equation (12) represents (1) in case of $a_0 = 1$, $a_1 = a_2 = b_0 = b_1 = b_2 = a_i = b_i = c_i = c_{n+1} = c_{n+2} = 0$, $i = 3, \dots, n$;

The function (4) will be the solution of (12) only in the case $c_2 = 0$ (see Example 2), but by means of the function (4) we can construct approximate solutions of the equation (12) vanishing at infinity for which $c_2 \neq 0$.

Let us introduce the notation

$$\psi = R\sin^2\psi_1, \tag{13}$$

where ψ_1 is a function having negligible fifth degree value, R is some parameter.

Taking into account

$$\sin\psi_1 \approx \psi_1 - \psi_1^3/6; \quad \sin^2\psi_1 \approx \psi_1^2 - \psi_1^4/3; \tag{14}$$

and putting (13) into (12) we obtain the following equation

$$\begin{aligned} 2\left(\psi_1 - \frac{2}{3}\psi_1^3\right)\Delta\psi_1 + 2\left(1 - 2\psi_1^2 + \frac{2}{3}\psi_1^4\right)(\nabla\psi_1)^2 \\ + c_1\left(\psi_1^2 - \frac{1}{3}\psi_1^4\right) + c_2R\psi_1^4 = 0, \end{aligned} \tag{15}$$

The equation (15) is the approximation of the equation (12) with the accuracy $\frac{8|R|d^2}{3}\psi_1^6$.

If

$$c_1 = -4d^2 = -c_2R \tag{16}$$

the function given by (4) will be the approximate solution of the equation (15) with the accuracy $\frac{8|R|d^2}{3}\exp(-6D)$, i. e. this function is the exact solution of the equation

$$2\left(\psi_1 - \frac{2}{3}\psi_1^3\right)\Delta\psi_1 + 2\left(1 - 2\psi_1^2\right)(\nabla\psi_1)^2 + c_1\left(\psi_1^2 - \frac{1}{3}\psi_1^4\right) + c_2R\psi_1^4 = 0.$$

According to (13), (14), (15), (16) the approximate solution of the equation (12) will be given by

$$\psi = R\sin^2\left\{\exp[-\alpha|x| - \beta|y| - \gamma|z| - D]\right\}, \tag{17}$$

where

$$4(\alpha^2 + \beta^2 + \gamma^2) = c_2R = -c_1, \quad c_1 < 0,$$

and the parameter D is chosen accordingly for the desired accuracy in such a way, that the quantity e^{-5D} is negligible (for example for $D = 3$, $e^{-15} \approx 10^{-7}$).

It is obvious

$$|\psi| \leq R\exp(-2D).$$

The graphics of (17) for some parameters are given in Figures 7, 8 in case of $D = 4$.

Note 4. The equation (12) is connected with the crystal growth [23, 24].

4) Now, let us consider the case $a_0 = 1$; $c_1 = -A_0$; $c_3 = \lambda_0$; $c_2 = a_1 = a_2 = a_3 = b_0 = b_1 = b_2 = b_3 = a_i = b_i = c_i = c_{n+1} = c_{n+2} = 0$, $i = 4, \dots, n$; then the equation (1) takes the form

$$\Delta\psi + \lambda_0\psi^3 - A_0\psi = 0, \tag{18}$$

The function (4) will be the solution of (18) only in the case $\lambda_0 = 0$.

By means of the function (4) we will construct approximate solutions of the equation (18) vanishing at infinity for which $\lambda_0 \neq 0$.

Let us introduce the following notation

$$\psi = R\sin\psi_2, \tag{19}$$

where ψ_2 is a function having negligible fifth degree value, $R > 0$ is some parameter.

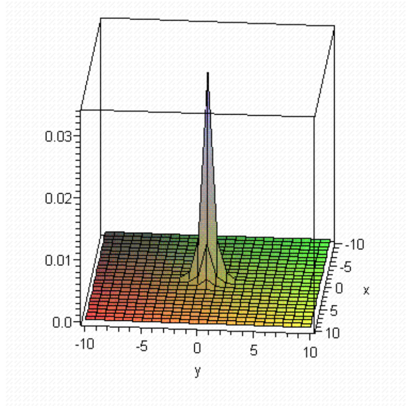

FIGURE 7. The graphic of (17) in case of $R = 100$; $\alpha = \beta = \gamma = 1$; $z = 0$;

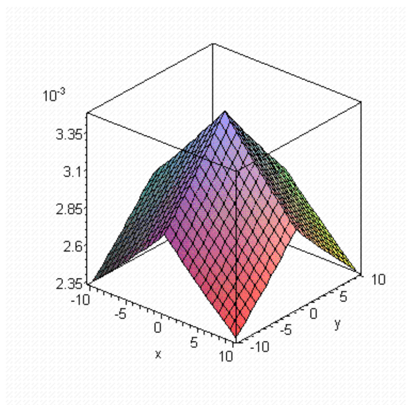

FIGURE 8. The graphic of (17) in case of $R = 1$; $\alpha = \beta = \gamma = 0.01$; $z = 0$;

Putting (19) into the left hand side of (18) and taking into the account (14) one obtains

$$\left(1 - \frac{\psi_2^2}{2} + \frac{\psi_2^4}{24}\right) \Delta\psi_2 - \left(\psi_2 - \frac{\psi_2^3}{6}\right) (\nabla\psi_2)^2 + \lambda_0 R^2 \left(\psi_2 - \frac{\psi_2^3}{6}\right)^3 - A_0 \left(\psi_2 - \frac{\psi_2^3}{6}\right) = 0. \tag{20}$$

As ψ_2^5 is negligible,the function (4) will be the solution of the equation (20) with the accuracy $A_0 \frac{\exp(-5D)}{2}$ and the exact solution of the equation (2). Hence, the function ψ given by the formula (19) is the solution of the equation (18) with the accuracy $A_0 \frac{\exp(-5D)}{2}$.

According to (4), (5), (19) the approximate solution of (18) will be given by the formula

$$\psi = R \sin \{\exp[-\alpha|x| - \beta|y| - \gamma|z| - D]\}, \tag{21}$$

where

$$\alpha^2 + \beta^2 + \gamma^2 = A_0, \quad \lambda_0 R^2 = 4A_0/3; \quad A_0 > 0, \tag{22}$$

and the constant D is chosen for the desired accuracy in such a way, that e^{-5D} is negligible (for example for $D = 4$, $e^{-20} \approx 2 \times 10^{-9}$).

The equation (18) is the cubic nonlinear Schrödinger type equation (cNLS). By the formulaes (21), (22) the modulus r of some class of solitary waves is given [25, 27]. The different classes of solitary waves are obtained in [1–3, 7, 9–13, 15–21, 26, 28–36, 40, 42–53].

In Figures 9, 10 the graphics of (21) are plotted for different parameters for the case $R = 10$ and $D = 4$ by using "Maple".

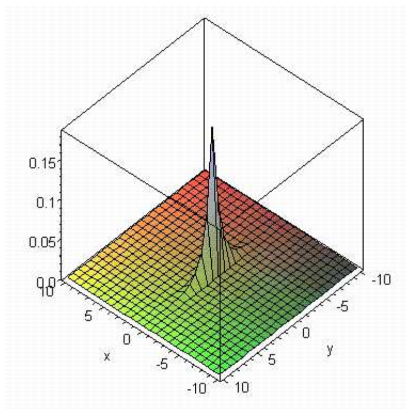

FIGURE 9. The modulus of the solitary wave. The graphic of (21) in case of $\alpha = 10$; $\beta = \gamma = 1$; $z = 0$; $A_0 = 102$; $\lambda_0 = 1.36$.

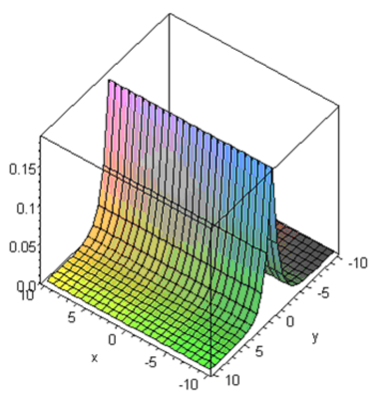

FIGURE 10. The modulus of the solitary wave. The graphic of (21) in case of $\alpha = 1$; $\beta = \gamma = 0$; $A_0 = 1$; $\lambda_0 = 0.013333$.

Note 5. In the works [21, 22] the equation (18) is equivalently reduced to the nonlinear integral equation.

Note 6. The equation (1) was considered in 3D, but the results of the current paper are valid in any dimensions.

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